Math 473/573
Fall 2016
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Homework 5

1. Analyze the stability of the following so-called λ-ω system

$$\frac{dx}{dt} = \lambda\, x - \omega\, y, \tag{1}$$

$$\frac{dy}{dt} = \omega\, x + \lambda\, y. \tag{2}$$

Check your answers by simulating the model.

2. Consider the following nonlinear equation for the evolution of a complex quantity u

$$\frac{du}{dt} = i\,(\omega + a|u|^2)\,u + (\lambda - b|u|^2)\,u \tag{3}$$

where u is a complex quantity $u = x + i\,y$. The function u can be interpreted as comprising the components (x, y) of the magnetization vector that oscillates pointing towards the z-direction. Equation (3) can be viewed as the nonlinear Schrödinger equation describing the magnetic moment precession in the so-called spin torque nano-oscillators (STNOs) and in magnetic thin films becomes the equation of a general oscillator if turning the diffusion term off.

Equation (3) can be decomposed into its real and imaginary parts by substituting $u = x+i\,y$ into (3), rearranging terms, and collecting the terms with real and imaginary parts in separate equations. This yields

$$\frac{dx}{dt} = (\lambda - b\,r^2)\,x - (\omega + a\,r^2)\,y, \tag{4}$$

$$\frac{dy}{dt} = (\omega + a\,r^2)\,x + (\lambda - b\,r^2)\,y, \tag{5}$$

where

$$r^2 = x^2 + y^2. \tag{6}$$

Clearly, when $a = b = 0$ system (4)-(5) reduces to the linear λ-ω system (1)-(2) whose fixed point is either a stable or an unstable focus, except for one particular value of λ for which the fixed-point is a center. Therefore, except for this particular value of λ, the system system does not exhibit persistent oscillations. Persistent oscillations can be obtained for other combinations of values of a and b. These oscillations result from the presence of a stable limit cycle. However, the system can also exhibit an unstable limit cycle.

(a) Find an expression for the amplitude r_c of the limit cycle in terms of the model parameters.

(b) Analyze the stability of the limit cycle.

(c) Find an expression for the evolution of the angular variable θ in terms of the model parameters.

Hint: change variables to poloar coordinates $x = r\cos\theta$, $y = r\,sin\theta$ as in Example 6.3.2 of the textbook.

Check your answers by simulating the model.