Review: power systems and optimization

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Chaper 2 in *Convex Optimization of Power Systems*.

1 Basic electrical quantities

Three-phase, balanced alternating current:

$$\begin{aligned} v_k(t) &= \bar{V}\cos(\omega t + \phi_k) \\ i_k(t) &= \bar{I}\cos(\omega t + \phi_k - \psi) \\ \phi_k &= \frac{2\pi k}{3}, \quad k = 0, 1, 2 \end{aligned}$$

(Remember, $v_k(t)$ is voltage difference to ground.)

Instantaneous power (in a phase):

$$\begin{aligned} p_k(t) &= v_k(t)i_k(t) \\ &= \frac{\bar{V}\bar{I}}{2}\cos(\psi)(1 + \cos(2(\omega t + \phi_k))) + \frac{\bar{V}\bar{I}}{2}\sin(\psi)\sin(2(\omega t + \phi_k)) \end{aligned}$$

Why three phases?

- Zero return current: $\sum_k i_k(t) = 0$
- Constant instantaneous power: $\sum_k p_k(t) = \frac{3\bar{V}\bar{I}\cos(\phi)}{2}$.

Phasors: all parameters are constant (steady-state).

$$\begin{aligned} V_k &= \frac{\bar{V}}{\sqrt{2}}e^{j\phi_k} \\ I_k &= \frac{\bar{I}}{\sqrt{2}}e^{j(\phi_k - \psi)} \end{aligned}$$

Phasor power quantities:

- Complex power (in one phase)

$$S = V_k I_k^* = \frac{\bar{V}\bar{I}}{2}(\cos(\psi) + j\sin(\psi))$$

- Real power

$$P = \text{real}[S] = \frac{\bar{V}\bar{I}}{2}\cos(\psi) = \text{mean}[\mathrm{p_k(t)}]$$

- Reactive power

$$Q = \text{imag}[S] = \frac{\bar{V}\bar{I}}{2}\sin(\psi)$$

What are real and reactive power? Observe:

$$p_k(t) = P(1 + \cos(2(\omega t + \phi_k))) + Q\sin(2(\omega t + \phi_k))$$

P is the coefficient of the non-zero mean part. Q is the coefficient of the zero mean part.
Per phase analysis: symmetry between lines enables us to just analyze one phase, drop k index.

2 Power flow

Consider a power line with phasor impedance Z_{12} and admittance $Y_{12} = g_{12} - jb_{12}$. Ohm's law:

$$I_{12} = (V_1 - V_2)Y_{12}.$$

(a linear relationship)
The power flow is:

$$S_{12} = V_1 I_{12}^* = V_1(V_1 - V_2)^* Y_{12}^*.$$

(a quadratic relationship)
Why not

$$L_{12} = (V_1 - V_2)(V_1 - V_2)^* Y_{12}^* = |V_1 - V_2|^2 Y_{12}^*?$$

This is the loss in the line. Observe:

$$\begin{aligned} S_{12} + S_{21} &= V_1(V_1 - V_2)^* Y_{12}^* + V_2(V_2 - V_1)^* Y_{12}^* \\ &= (V_1 - V_2)(V_1 - V_2)^* Y_{12}^* \\ &= L_{12} \end{aligned}$$

3 Optimization

$f(x)$, $x \in \mathbb{R}^n$.

- x_0 is a global minimum of $f(x)$ if $f(x_0) \leq f(x)$ for all x.
- x_0 is a local minimum of $f(x)$ if $f(x_0) \leq f(x)$ for all x s.t. $\|x - x_0\| \leq \epsilon$, $\epsilon > 0$.

Convexity:

- Function: $f(x)$ is convex if:

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$$

  for all $0 \leq \alpha \leq 1$.

- Set: $\mathcal{X}$ is convex if $x, y \in \mathcal{X}$ implies $\alpha x + (1-\alpha)y \in \mathcal{X}$.
- If $g(x)$ is convex, then $\mathcal{X} = \{x \mid g(x) \leq 0\}$ is convex.

Optimization problem:

$$\begin{array}{cl} \min\limits_{x \in \mathbb{R}^n} & f(x) \\ \text{s.t.} & g_i(x) \leq 0, \; i = 1, ..., m \end{array}$$

If $f(x)$ and all $g_i(x)$ are convex, then any local minimum is a global minimum ... a convex optimization problem.

Computational tractability:

- Convex optimization: easy, often polynomial-time
- Nonconvex optimization: hard, often NP-hard

(NP-hard: no polynomial-time (efficient) algorithm can exist)

3.1 Linear programming

(slight misnomer, affine is more accurate)

- $f(x) = c^T x$

- Affine constraints: $g_i(x) = a_i^T x - b_i$ (usually as vector: $Ax \leq b$)
- Easiest type of optimization
- Solvable in PT with an IP method, or very fast with simplex method.
- Quadratic programming with $f(x) = x^T C x$ is also easy if $C \succeq 0$ (psd).
- Is it convex? Check definition, yes.

3.2 Mixed-integer programming

$$\begin{aligned} \min_{x,y} \quad & f(x,y) \\ \text{s.t.} \quad & g_i(x,y) \leq 0,\ i = 1, ..., m \\ & y_i \in \mathbb{Z} \quad \text{(the integers)} \end{aligned}$$

- NP-hard even when f and g_i are all linear!
- Branch-and-bound and cutting planes are powerful heuristics
- $y_i \in \mathbb{Z}$ is nononvex!
- As we'll see, common in power systems.

3.3 Semidefinite programming

Positive semidefinite:

- $X \in \mathbb{C}^{n \times n}$, Hermitian: $X = X^*$ (conjugate transpose)
- Definition: $z^* X z \geq 0$ for all $z \in \mathbb{C}^n$
- Equivalent: all eigs. of X nonnegative, all principal minors nonnegative
- Notation: $X \succeq 0$

$X \succeq 0$ is convex constraint.
Proof: Suppose $X, Y \succeq 0$. Then

$$z^*(\alpha X + (1-\alpha)Y)z = \alpha z^* X z + (1-\alpha) z^* Y z \geq 0.$$

Done!

A semidefinite program (SDP):

$$\begin{aligned} \min_{X} \quad & \text{trace}(CX) \\ \text{s.t.} \quad & \text{trace}(A_i X) = b_i \\ & X \succeq 0 \end{aligned}$$

Feautres of SDP:

- Convex, 1 minimum
- Generalization of LP (don't solve LP as SDP)
- SDP's can be solved in polynomial-time using interior point method.

3.3.1 Example: eigenvalue optimization

Suppose $A(x) \in \mathbb{C}^{n\times n}$ is a linear function of x. Consider:

$$\begin{aligned} \min_{x,\lambda} \quad & \lambda \\ \text{s.t.} \quad & \lambda \text{ is the largest eig. of } A(x) \end{aligned}$$

Eigenvalue definition (informal):

$$\begin{aligned} A(x)v = \lambda v \quad & \Rightarrow \quad v^* A(x) v = \lambda v^* v \\ & \Rightarrow \quad \frac{v^* A(x) v}{v^* v} = \lambda \\ & \Rightarrow \quad \max_{v \in \mathbb{C}^n} \frac{v^* A(x) v}{v^* v} = \lambda_{\max} \end{aligned}$$

...Rayleigh quotient. This implies:

$$\lambda_{\max} v^* I v \geq v^* A(x) v \quad \forall v \in \mathbb{C}^n$$

Equivalent to

$$\begin{aligned} \min_{\lambda,x} \quad & \lambda \\ \text{s.t.} \quad & v^*(\lambda I - A(x))v \geq 0 \quad \forall v \in \mathbb{C}^n \end{aligned}$$

which, by definition of PSD, is equivalent to

$$\begin{aligned} \min_{\lambda,x} \quad & \lambda \\ \text{s.t.} \quad & \lambda I - A(x) \succeq 0 \end{aligned}$$

3.4 Quadratically constrained programming

$$\begin{aligned} \min_x \quad & x^* C x \\ \text{s.t.} \quad & x^* A_i x \leq b_i \end{aligned}$$

How difficult?

- If $C \succeq 0$ and $A_i \succeq 0$, solvable in PT.
- If any are not PSD, NP-hard.

How general?

- Binary constraints: $x \in \{0, 1\} \Leftrightarrow x^2 = x$
- Power flow: $v_1(v_1 - v_2)^* y_{12}^* = ...$
- Both nonconvex!

3.5 Relaxations

Hard problem:

$$F_1 : \min_{x \in \mathcal{X}} f(x)$$

Relaxation:

$$F_2 : \min_{x \in \mathcal{Y}} f(x), \quad X \subset Y$$

Facts

- Obj. of $F_2 \leq$ Obj. of F_1.
- If x is optimal for relaxation and feasible for exact, x is optimal for exact. **Proof:** Suppose x is relaxed optimal and feasible suboptimal for exact problem. Then $\exists\ y$ s.t. $f(y) < f(x)$, $y \in X$. But by relaxation, $y \in Y$, and therefore x is not relaxed optimal, a contradiction. QED.

3.5.1 SD relaxation

Trace is invariant under cyclic permutations. QCP can be equivalently written:

$$\begin{aligned} \min_x \quad & \text{trace}(x x^* C) \\ \text{s.t.} \quad & \text{trace}(x x^* A_i) \leq b_i \end{aligned}$$

Identical to:

$$\begin{aligned} \min_{x,X} \quad & \text{trace}(XC) \\ \text{s.t.} \quad & \text{trace}(XA_i) \le b_i \\ & X = xx^* \iff X \succeq 0,\ \text{rank}(X) = 1 \end{aligned}$$

$X = xx^*$ by itself is equivalent to $X \succeq 0$ (unique Cholesky decomposition), $\text{rank}(X) = 1$.
Removing a constraint enlarges the feasible set, i.e. relaxation:

$$\begin{aligned} \min_{X} \quad & \text{trace}(XC) \\ \text{s.t.} \quad & \text{trace}(XA_i) \le b_i \\ & X \succeq 0 \end{aligned}$$

If solution, X, has rank 1, then relaxation is tight. Feasible, optimal exact solution is Cholesky: $X = xx^*$.

3.5.2 Example: Max-cut

Adjacency matrix

$$A_{ij} = \begin{cases} 1 & \text{if } i \sim j \\ 0 & \text{otherwise} \end{cases}$$

Find the biggest cut (draw)... NP-complete. Mathematically:

$$\begin{aligned} \max_{x} \quad & \frac{1}{2} \sum_{ij} A_{ij}(1 - x_i x_j) \\ \text{s.t.} \quad & x_i \in \{-1, 1\} \end{aligned}$$

Equivalence:

$$x_i \in \{-1, 1\} \Leftrightarrow x_i^2 = 1.$$

Convex relaxation: $X = xx^T$ Equivalent formulation:

$$\begin{aligned} \max_{x} \quad & \frac{1}{2} \sum_{ij} A_{ij}(1 - X_{ij}) \\ \text{s.t.} \quad & X_{ii} = 1,\ X \succeq 0 \end{aligned}$$