Optimal power flow and friends

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Section 3.2-3.3, 4.3, 4.4, 5.1, 5.2 (summarily) in *Convex Optimization of Power Systems*.

1 Optimal power flow

Complex power flow through a line:

$$S_{12} = V_1(V_1 - V_2)^* y_{12}^*$$

Network: buses, $\mathcal{N}$, lines $\mathcal{E}$. Nodal power balance:

$$S_i = \sum_{j:ij\in\mathcal{E}} V_i(V_i - V_j)^* y_{ij}^*$$

A nonlinear system of equations.

1.1 Load flow

Solve the NL system with boundary conditions:

- S_i fixed if i is a load
- P_i, $|V_i|$ if i is a generator
- V_i fixed if i is the slack bus

This is actually also NP-hard. Classical tools: Jacobi, Gauss-Seidel, Newton-Raphson.

1.2 Real coordinates

Recall:

$$V_i = w_i + jx_i.$$

In rectangular coordinates:

$$\begin{aligned} P_{ij} &= g_{ij}(w_i^2 + x_i^2) - g_{ij}(w_i w_j + x_i x_j) + b_{ij}(w_j x_i - w_i x_j) \\ Q_{ij} &= b_{ij}(w_i^2 + x_i^2) - g_{ij}(w_j x_i - w_i x_j) - b_{ij}(w_i w_j + x_i x_j) \end{aligned}$$

Also:

$$V_i = |V_i| e^{j\theta_i}.$$

Real and imaginary parts of power flow are (polar coordinates):

$$\begin{aligned} P_{ij} &= g_{ij}|V_i|^2 - |V_i||V_j|(g_{ij}\cos(\theta_i - \theta_j) - b_{ij}\sin(\theta_i - \theta_j)) \\ Q_{ij} &= b_{ij}|V_i|^2 - |V_i||V_j|(g_{ij}\sin(\theta_i - \theta_j) + b_{ij}\cos(\theta_i - \theta_j)) \end{aligned}$$

Linearization assumptions:

- $|V_i| = 1$
- $b_{ij} >> g_{ij} = 0$
- $\sin(\theta_i - \theta_j) \approx \theta_i - \theta_j$
- Neglect Q

Then

$$\begin{aligned} P_{ij} &= b_{ij}(\theta_i - \theta_j) \\ P_i &= \sum_j P_{ij} \end{aligned}$$

Matrix form:

$$B_{ij} = \begin{cases} -b_{ij} & i \neq j \\ \sum_j b_{ij} & i = j \end{cases}$$

Then

$$P = B\theta.$$

- For load flow, solve easily with pseudo-inverse ($\text{rank}(B) = n - 1$)

  $$B^\dagger = \sum_{i:\lambda_i \neq 0} \frac{1}{\lambda_i} \gamma_i \gamma_i^T.$$

  Then $\theta = B^\dagger P$ ... minimizes $\sum_i \theta_i^2$, alternative methods as well.
- Also called DC power flow ... terrible name, don't use.

1.3 OPF

What we have so far:

- Exact power flow model - a physical constraint
- Its linearization

We need

- Objective
- Operational constraints

Objective is usually the cost of producing power.

- $f_i(P_i)$: cost of producing P_i at bus i, e.g. $f_i(P_i) = a_i P_i^2 + b_i P_i$
- $\sum_i f_i(P_i)$: OPF objective
- $\sum_{i \in \mathcal{G}} P_i$ - minimize losses, also $\sum_{ij} P_{ij}$

Operational constraints:

- Power limits: $\underline{P}_i \leq P_i \leq \overline{P}_i$, $\underline{Q}_i \leq Q_i \leq \overline{Q}_i$ ... typically equality for loads.
- Voltage limits: $\underline{V}_i \leq |V_i| \leq \overline{V}_i$
- Transmission limits: $|S_{ij}| = \sqrt{P_{ij}^2 + Q_{ij}^2} \leq \overline{S}_{ij} \Longleftrightarrow P_{ij}^2 + Q_{ij}^2 \leq \overline{S}_{ij}^2$

All together:

$$
\begin{aligned}
\min_{P,Q,V} \quad & \sum_i f_i(P_i) \\
\text{s.t.} \quad & S_{ij} = V_i (V_i - V_j)^* y_{ij}^* \\
& S_i = \sum_j S_{ij} \\
& \underline{P}_i \leq \text{Real}[S_i] \leq \overline{P}_i \\
& \underline{Q}_i \leq \text{Imag}[S_i] \leq \overline{Q}_i \\
& \underline{V}_i \leq |V_i| \leq \overline{V}_i \\
& P_{ij}^2 + Q_{ij}^2 \leq \overline{S}_{ij}^2
\end{aligned}
$$

Properties of OPF

- A QCP in complex coordinates
- Nonconvex because of power flow and voltage limit $\Rightarrow$ NP-hard
- Real coordinates via: $V_i = X_i + jW_i$ or $V_i = |V_i|e^{j\theta} = |V_i|(\cos(\theta) + j\sin(\theta))$, e.g.

$$
\begin{aligned}
\min_{P,Q,V} \quad & \sum_i f_i(P_i) \\
\text{s.t.} \quad & P_{ij} = g_{ij}|V_i|^2 - |V_i||V_j|(g_{ij}\cos(\theta_i - \theta_j) - b_{ij}\sin(\theta_i - \theta_j)) \\
& Q_{ij} = b_{ij}|V_i|^2 - |V_i||V_j|(g_{ij}\sin(\theta_i - \theta_j) + b_{ij}\cos(\theta_i - \theta_j)) \\
& P_i = \sum_j P_{ij} \\
& Q_i = \sum_j Q_{ij} \\
& \underline{P}_i \leq P_i \leq \overline{P}_i \\
& \underline{Q}_i \leq Q_i \leq \overline{Q}_i \\
& \underline{V}_i \leq |V_i| \leq \overline{V}_i \\
& P_{ij}^2 + Q_{ij}^2 \leq \overline{S}_{ij}^2
\end{aligned}
$$

Use

- Run every 5 minutes to dispatch system
- Inside of many other problems (today)

Linearized OPF:

$$
\begin{aligned}
\min_{P,\theta} \quad & \sum_i f_i(P_i) \\
\text{s.t.} \quad & P_{ij} = b_{ij}(\theta_i - \theta_j) \\
& P_i = \sum_j P_{ij} \\
& \underline{P}_i \leq P_i \leq \overline{P}_i \\
& |P_{ij}| \leq \overline{S}_{ij}
\end{aligned}
$$

Properties

- Less accurate, LP or QP depending on objective
- Solvable in PT with IP method, or fast with simplex

2 Unit commitment

- Generators take days to turn on/off
- Online/offline time scheduled days in advance
- Also reserves
- On/off status must be modeled discretely

Single node model:

- N generators
- T time periods
- D^t: demand at time t
- P_n^t: power from gen. n at t
- $x_n^t \in \{0,1\}$: generator n's on/off status at time t.

$$\begin{aligned}
\min_{x,P} \quad & \sum_{n,t} f_n(P_n^t) + g_n(x_n^t, x_n^{t-1}) \\
\text{s.t.} \quad & \sum_{n=1}^{N} P_n^t = D^t \\
& 0 \le P_n^t \le \overline{P}_n x_n^t \\
& x_n^t \in \{0,1\}
\end{aligned}$$

Other types of constraints:

- Min. up time, U_n:
$$\sum_{k=t}^{t+U_n-1} x_n^k \ge U_n(x_n^t - x_n^{t-1})$$
- Min. down time, ramping constraints, etc.

3 Generation and transmission planning

Generation:

$$\begin{aligned} \min_{P,\theta,x} \quad & \sum_i c_i x_i \\ \text{s.t.} \quad & P_{ij} = b_{ij}(\theta_i - \theta_j) \\ & P_i = \sum_j P_{ij} \\ & \underline{P}_i \leq P_i \leq \overline{P}_i(x_i^0 + x_i) \\ & |P_{ij}| \leq \overline{S}_{ij} \\ & 0 \leq x_i \leq \bar{x}_i,\ x_i \in \mathbb{Z} \end{aligned}$$

- An MILP - hard but tractable up to a few hundred variables
- Can be formulated with AC models as well
- Solution is $x = 0$ if already feasible.

Transmission

$$\begin{aligned} \min_{P,\theta,x} \quad & \sum_{ij} c_{ij} x_{ij} \\ \text{s.t.} \quad & P_{ij} = b_{ij}(x_{ij}^0 + x_{ij})(\theta_i - \theta_j) \\ & P_i = \sum_j P_{ij} \\ & \underline{P}_i \leq P_i \leq \overline{P}_i \\ & |P_{ij}| \leq \overline{S}_{ij}(x_{ij}^0 + x_{ij}) \\ & 0 \leq x_{ij} \leq \bar{x}_{ij},\ x_{ij} \in \mathbb{Z} \end{aligned}$$

- Formerly linear power flow constraint nonconvex - bad news
- Relaxations work well (tomorrow)

4 Reconfiguration and transmission switching

The disjunctive constraint:

$$g_1(x) \leq 0 \quad \text{or} \quad g_2(x) \leq 0$$

Use integer variables to write:

$$g_1(x) \leq My, \quad g_2(x) \leq M(1-y), \quad y \in \{0,1\},$$

where M is large enough.

Transmission switching

$$\begin{aligned}
\min_{P,\theta,x} \quad & \sum_i f_i(P_i) \\
\text{s.t.} \quad & P_i = \sum_j P_{ij} \\
& \underline{P}_i \leq P_i \leq \overline{P}_i \\
& |P_{ij} - b_{ij}(\theta_i - \theta_j)| \leq M(1 - x_{ij}) \\
& |P_{ij}| \leq \overline{S}_{ij} x_{ij} \\
& x_{ij} \in \{0,1\}
\end{aligned}$$

- Similar to Braess's paradox: creating a new route can reduce performance
- Closing/opening lines can change performance
- Reconfiguration: same, but additional constraint that network is a tree - can be done with more linear constraints.

References

[1] Emmanuel J. Candès, Thomas Strohmer, and Vladislav Voroninski. PhaseLift: Exact and stable signal recovery from magnitude measurements via convex programming. *Communications on Pure and Applied Mathematics*, 66(8):1241–1274, 2013.