Convex relaxations of OPF

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1 SDP relaxation of OPF

A generic QCP:

$$\begin{aligned} \min_{x} \quad & x^* C x \\ \text{s.t.} \quad & x^* A_i x \leq b_i \end{aligned}$$

Equivalent to:

$$\begin{aligned} \min_{x} \quad & \text{trace}(xx^* C) \\ \text{s.t.} \quad & \text{trace}(xx^* A_i) \leq b_i \end{aligned}$$

and

$$\begin{aligned} \min_{x,X} \quad & \text{trace}(XC) \\ \text{s.t.} \quad & \text{trace}(XA_i) \leq b_i \\ & X = xx^* \iff X \succeq 0,\ \text{rank}(X) = 1 \end{aligned}$$

Only the rank constraint is nonconvex. Dropping it yields the Shor relaxation, an SDP (actually its dual).

- Note: Hermitian PSD implies Cholesky, rank one guarantees vector form.
- Cholesky: $A = BB^*$, $O(n^3)$ computation time.

Apply to OPF:

$$\begin{aligned}
\min_{P,Q,V} \quad & \sum_i f_i(P_i) \\
\text{s.t.} \quad & P_{ij} + jQ_{ij} = V_i(V_i - V_j)^* y_{ij}^* \\
& P_i + jQ_i = \sum_j P_{ij} + jQ_{ij} \\
& \underline{P}_i \leq P_i \leq \overline{P}_i \\
& \underline{Q}_i \leq Q_i \leq \overline{Q}_i \\
& \underline{V}_i \leq |V_i| \leq \overline{V}_i \iff \underline{V}_i^2 \leq |V_i|^2 \leq \overline{V}_i^2 \\
& P_{ij}^2 + Q_{ij}^2 \leq \overline{S}_{ij}^2
\end{aligned}$$

Equivalent to:

$$\begin{aligned}
\min_{P,Q,V} \quad & \sum_i f_i(P_i) \\
\text{s.t.} \quad & P_{ij} + jQ_{ij} = (W_{ii} - W_{ij}) y_{ij}^* \\
& P_i + jQ_i = \sum_j P_{ij} + jQ_{ij} \\
& \underline{P}_i \leq P_i \leq \overline{P}_i \\
& \underline{Q}_i \leq Q_i \leq \overline{Q}_i \\
& \underline{V}_i \leq |V_i| \leq \overline{V}_i \iff \underline{V}_i^2 \leq W_{ii} \leq \overline{V}_i^2 \\
& P_{ij}^2 + Q_{ij}^2 \leq \overline{S}_{ij}^2 \\
& W \succeq 0, \ \text{rank}(W) = 1
\end{aligned}$$

Drop rank constraint. Originally introduced in [1]. Technical survey in [4].

- Solve in PT
- SDP not that fast in practice

2 Second-order cone programming

- Generalizes LP
- Generalized by SDP

- Maturity closer to LP - featured in CPLEX, Gurobi, Mosek, etc., including MISOCP

Standard form SOCP:

$$\begin{array}{ll} \min\limits_{x} & f^* x \\ \text{s.t.} & \|A_i x + b_i\| \leq c_i^* x + d_i \end{array}$$

2.1 Example: hyperbolic constraints

Hyperbolic constraint:

$$x_3^2 \leq x_1 x_2, \quad x_1 \geq 0, \quad x_2 \geq 0$$

Set

$$\begin{aligned} A_i &= \begin{bmatrix} 0 & 0 & 2 \\ 1 & -1 & 0 \end{bmatrix} \\ b_i &= 0 \\ c_i^T &= [1\ 1\ 0] \\ d_i &= 0 \end{aligned}$$

Then SOCP takes on the form:

$$\left\| \begin{bmatrix} 2x_3 \\ x_1 - x_2 \end{bmatrix} \right\| \leq x_1 + x_2$$

Square both sides gives hyperbolic constraint.

2.2 Relation to LP

Standard form LP:

$$\begin{array}{ll} \min\limits_{x} & f^* x \\ \text{s.t.} & Gx = h \\ & x \geq 0 \end{array}$$

We get this by setting A_i and b_i to zero in SOCP. Any LP can be written as SOCP.

2.3 Relation to SDP

Can we write $\|A_i x + b_i\| \leq c_i^* x + d_i$ as an SDP constraint? The Shur complement of

$$\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$$

is $D = C - B^* A^{-1} B$. We know: $A \succeq 0$ and $D \succeq 0$ iff whole matrix is PSD.

Consider SD constraint:

$$\begin{bmatrix} (c_i^* x + d_i) I & A_i x + b_i \\ (A_i x + b_i)^* & c_i^* x + d_i \end{bmatrix} \succeq 0$$

Recall the diagonal must be real. Shur complement implies $I \succeq 0$ and $(c_i^T x + d_i)^2 - (A_i x + b_i)^* I (A_i x + b_i) \geq 0$, which is the same as $\|A_i x + b_i\| \leq c_i^* x + d_i$.

- Any SOCP can be written as an SDP, not vice versa.

2.4 SOCP relaxation of SDP

- In theory, SDP great, sometimes in practice.
- SOCP is fast in practice
- Relax SDP to SOCP?

Observation from [3]. Recall,

- $X \succeq 0 \iff$ all principal minors of X nonnegative.
- Just the 2×2 minors is then a relaxation:

$$\begin{vmatrix} X_{ii} & X_{ij} \\ X_{ji} & X_{jj} \end{vmatrix} = X_{ii} X_{jj} - X_{ij} X_{ji} = X_{ii} X_{jj} - |X_{ij}|^2 \geq 0$$

- A hyperbolic constraint:

$$\left\| \begin{bmatrix} 2X_{ij} \\ X_{ii} - X_{jj} \end{bmatrix} \right\| \leq X_{ii} + X_{jj}$$

- Applied directly to power in [2].

3 Exactness of relaxations

Exact OPF:

$$
\begin{aligned}
\min_{P,Q,V} \quad & \sum_i f_i(P_i) \\
\text{s.t.} \quad & P_{ij} + jQ_{ij} = V_i(V_i - V_j)^* y_{ij}^* \\
& P_i + jQ_i = \sum_j P_{ij} + jQ_{ij} \\
& \underline{P}_i \leq P_i \leq \overline{P}_i \\
& \underline{Q}_i \leq Q_i \leq \overline{Q}_i \\
& \underline{V}_i^2 \leq |V_i|^2 \leq \overline{V}_i^2 \\
& P_{ij}^2 + Q_{ij}^2 \leq \overline{S}_{ij}^2
\end{aligned}
$$

SDP relaxation:

$$
\begin{aligned}
\min_{P,Q,V} \quad & \sum_i f_i(P_i) \\
\text{s.t.} \quad & P_{ij} + jQ_{ij} = (W_{ii} - W_{ij}) y_{ij}^* \\
& P_i + jQ_i = \sum_j P_{ij} + jQ_{ij} \\
& \underline{P}_i \leq P_i \leq \overline{P}_i \\
& \underline{Q}_i \leq Q_i \leq \overline{Q}_i \\
& \underline{V}_i^2 \leq W_{ii} \leq \overline{V}_i^2 \\
& P_{ij}^2 + Q_{ij}^2 \leq \overline{S}_{ij}^2 \\
& W \succeq 0
\end{aligned}
$$

SOCP relaxation:

$$
\begin{aligned}
\min_{P,Q,V} \quad & \sum_i f_i(P_i) \\
\text{s.t.} \quad & P_{ij} + jQ_{ij} = (W_{ii} - W_{ij}) y_{ij}^* \\
& P_i + jQ_i = \sum_j P_{ij} + jQ_{ij} \\
& \underline{P}_i \leq P_i \leq \overline{P}_i \\
& \underline{Q}_i \leq Q_i \leq \overline{Q}_i \\
& \underline{V}_i^2 \leq W_{ii} \leq \overline{V}_i^2 \\
& P_{ij}^2 + Q_{ij}^2 \leq \overline{S}_{ij}^2 \\
& \left\| \begin{bmatrix} 2W_{ij} \\ W_{ii} - W_{jj} \end{bmatrix} \right\| \leq W_{ii} + W_{jj} \quad \forall ij
\end{aligned}
$$

Exactness:

- Suppose $V \in \mathbb{C}^n$ solves exact. Then $W = VV^*$ is feasible for SDP and SOCP relaxations.
- Suppose $W \in \mathbb{C}^{n \times n}$ solves SDP. Then W is feasible for SOCP relaxations.
- Suppose W solves SDP relaxation and $\text{rank}(W) = 1$. $\exists\, V$ s.t. $VV^* = W$ and V is feasible for exact OPF. Relaxation is "exact".
- Given rank one W, voltages obtained from Cholesky.

3.1 Radial networks

Theorem: Suppose no line or voltage limits, $\underline{P}_i = \underline{Q}_i = -\infty$, and network is radial. Then SOC and SD relaxations are exact.

$$
\begin{aligned}
\min_{P,Q,V} \quad & \sum_i f_i(P_i) \\
\text{s.t.} \quad & P_{ij} + jQ_{ij} = (W_{ii} - W_{ij})y_{ij}^* \\
& P_i + jQ_i = \sum_j P_{ij} + jQ_{ij} \\
& P_i \le \overline{P}_i \\
& Q_i \le \overline{Q}_i \\
& |W_{ij}|^2 = W_{ij}W_{ij}^* \le W_{ii}W_{jj}
\end{aligned}
$$

Proof sketch: General approach:

- Consider a solution to SOC
- Show that SOC solution is feasible for SDP and exact.
- Then relaxation implies optimality.

Part 1: show hyperbolic constraints must be tight.

- Consider an optimal solution to SOCP, W. Observe W_{ij} is free if ij not a line.
- Suppose for contradiction that $|W_{ij}|^2 < W_{ii}W_{jj}$ (not equality).
- $\exists\ \epsilon > 0$ s.t. $|W_{ij} + \epsilon|^2 \le W_{ii}W_{jj}$. Make substitution.
- For feasibility, also substitute

  $$
  \begin{aligned}
  P_{ij} + jQ_{ij} - \epsilon y_{ij}^* &\longleftrightarrow P_{ij} + jQ_{ij} \\
  P_i + jQ_i - \epsilon y_{ij}^* &\longleftrightarrow P_i + jQ_i
  \end{aligned}
  $$

  (Valid because $\underline{P}_i = \underline{Q}_i = -\infty$.)
- Since $\mathrm{Re}(y_{ij}) > 0$ (positive resistance), $f_i(P_i)$ decreases to $f_i(P_i - \epsilon g_{ij})$, reducing the objective ... CONTRADICTION!
- Therefore, $|W_{ij}|^2 = W_{ii}W_{jj}$ for all lines ij at an optimal solution.

Part 2: construct voltage vector.

- $|W_{ij}|^2 = W_{ii}W_{jj}$ implies low rank, i.e.,

$$\det \begin{bmatrix} W_{ii} & W_{ij} \\ W_{ij}^* & W_{jj} \end{bmatrix} = W_{ii}W_{jj} - |W_{ij}|^2 = 0.$$

- Therefore, Cholesky factorization gives:

$$\begin{bmatrix} W_{ii} & W_{ij} \\ W_{ij}^* & W_{jj} \end{bmatrix} = \begin{bmatrix} V_i \\ V_j \end{bmatrix} \begin{bmatrix} V_i \\ V_j \end{bmatrix}^*$$

  Can we construct a consistent voltage vector?

- Yes, because radial ... use induction on path $\{n_1, ..., n_m\}$.
  - Base case: Set $|V_{n_1}| = \sqrt{W_{n_1 n_1}}$, $\angle V_{n_1} = 0$.
  - Inductive step: Suppose $V_{n_1}, ..., V_{n_k}$ are known. Then we can solve

$$\begin{bmatrix} W_{n_k n_k} & W_{n_k n_{k+1}} \\ W_{n_k n_{k+1}}^* & W_{n_{k+1} n_{k+1}} \end{bmatrix} = \begin{bmatrix} V_{n_k} \\ V_{n_{k+1}} \end{bmatrix} \begin{bmatrix} V_{n_k} \\ V_{n_{k+1}} \end{bmatrix}^*$$

    for $V_{n_{k+1}}$. By radiality, $n_{k+1} \notin \{n_1, ..., n_k\}$ ... no possible contradiction with prior factorization. Specifically, $V_{n_{k+1}} = W_{n_k n_{k+1}}^* / V_{n_k}^*$.

- Choose root node n_r with $V_{n_r} = \sqrt{W_{n_r}} \angle 0$, do Cholesky factorization along unique paths from n_r to all other nodes ... yields voltage vector V.

- By construction, $W' = VV^*$ is feasible and hence optimal for SDP relaxation. V is feasible and hence optimal for exact problem.

- **QED.**

Reflections:

- Distribution systems are radial, transmission are sparse
- Not many actual networks satisfy assumptions, many are close
- Don't forget - balanced steady-state model is already a huge approximation.
- Relaxations perform extremely well as approximations, better than linearizing.

References

[1] Xiaoqing Bai, Hua Wei, Katsuki Fujisawa, and Yong Wang. Semidefinite programming for optimal power flow problems. *International Journal of Electrical Power and Energy Systems*, 30(6-7):383 – 392, 2008.

[2] R.A. Jabr. Radial distribution load flow using conic programming. *Power Systems, IEEE Transactions on*, 21(3):1458 –1459, Aug. 2006.

[3] Sunyoung Kim and Masakazu Kojima. Exact solutions of some nonconvex quadratic optimization problems via SDP and SOCP relaxations. *Computational Optimization and Applications*, 26:143–154, 2003.

[4] S.H. Low. Convex relaxation of optimal power flow – Part I: Formulations and equivalence. *Control of Network Systems, IEEE Transactions on*, 1(1):15–27, March 2014.