Duality and pricing

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Section 6.1.1-6.1.2, 6.2.1 in *Convex Optimization of Power Systems.*

1 The power industry

- Initially: competitive, no regulation (Edison vs. Westinghouse, later others)
- 1935: Samuel Insull, PUHCA
- 1935-1978: Vertically integrated, gov't regulated monopoly
- 1978: PURPA - utilities must buy from IPPs
- 1978 - present: deregulation

Current format:

- Generators (OPG) sell power to:
  - large loads (steel mills) and utilities (HydroOne) via bilateral contracts
  - system operators (IESO) in spot markets
- System operators sell power to large loads and utilities in spot markets
- Utilities sell power to smaller loads.

Today: Spot markets

- Add liquidity, increase competition, lower barrier to entry for new generators
- Typically 3 markets:
  - Day-ahead: bulk quantities

  - Hour-ahead: moderate adjustments
  - Real-time (5 min. ahead): balancing

- Basic operation:
  - Generators sell power to ISO, ISO sells to loads
  - Transactions occur at nodal prices AKA LMPs.
  - Prices are dual variables of OPF.

2 Duality

Primal problem

$$\begin{aligned} P \ = \ \min_{x} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0 \\ & h_i(x) = 0 \end{aligned}$$

Lagrangian:

$$L(x, \alpha, \beta) = f(x) + \sum_i \alpha_i g_i(x) + \sum_i \beta_i h_i(x), \quad \alpha_i \geq 0$$

Dual function

$$\mathcal{L}(\alpha, \beta) = \min_x \ L(x, \alpha, \beta).$$

Always concave because minimum over family of affine functions (draw). If x is primal feasible ($g_i(x) \leq 0$, $h_i(x) = 0$), then since $\alpha \geq 0$,

$$L(x, \alpha, \beta) \leq f(x).$$

True for all x, so

$$\mathcal{L}(\alpha, \beta) \leq P,$$

Weak duality. Want largest lower bound ... *dual problem*:

$$\begin{aligned} D \ = \ \max_{\alpha,\beta} \quad & \mathcal{L}(\alpha, \beta) \\ \text{subject to} \quad & \alpha \geq 0 \end{aligned}$$

Concave maximization regardless of primal. *Strong duality*:

$$D = P.$$

Holds when primal is convex (+ constraint qualification).

2.1 Example: LP duality

Standard form LP:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & l^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

Dual function:

$$\begin{aligned} \mathcal{L}(\alpha, \beta) &= \min_x \; l^T x + \beta^T (b - Ax) - \alpha^T x \\ &= \beta^T b + \min_x \; (l - A^T \beta - \alpha)^T x \end{aligned}$$

Equals $-\infty$ unless if $A^T \beta + \alpha = l$ ... a constraint in the maximization. Standard form LP dual:

$$\begin{array}{ll} \underset{\alpha, \beta}{\text{maximize}} & \beta^T b \\ \text{subject to} & A^T \beta + \alpha = l \\ & \alpha \geq 0 \end{array}$$

Note that the dual variable α can be eliminated by consolidating the two constraints into $A^T \beta \leq l$. Strong duality guaranteed by convexity.

2.2 Complementary slackness

Strong duality, $D = P$, implies at optimal solution (x^*, α^*, β^*),

$$\begin{aligned} f\left(x^*\right) &= \mathcal{L}\left(\alpha^*, \beta^*\right) \\ &= \min_x L(x, \alpha^*, \beta^*) \\ &\leq L(x^*, \alpha^*, \beta^*) \\ &= f(x^*) + \sum_i \alpha_i^* g_i\left(x^*\right) + \sum_i \beta_i^* h_i\left(x^*\right) \\ &\leq f(x^*) \end{aligned}$$

because the latter terms must be negative due to feasibility. Therefore

$$\alpha_i^* g_i\left(x^*\right) = 0.$$

Central in transmission economics.

3 Pricing the linearized power flow

Recall linearized power flow:

$$\begin{aligned} \underset{p,\theta}{\text{minimize}} \quad & \sum_i f_i(p_i) \\ \text{subject to} \quad & \\ \lambda_i : \quad & p_i = \sum_j b_{ij}(\theta_i - \theta_j) \\ \chi_{ij} \geq 0 : \quad & b_{ij}(\theta_i - \theta_j) \leq \overline{s}_{ij} \end{aligned}$$

Lagrangian:

$$L(p, \theta, \lambda, \chi) \;=\; \sum_i f_i(p_i) + \lambda_i \left(-p_i + \sum_j b_{ij}(\theta_i - \theta_j) \right) + \sum_j \chi_{ij} \left(b_{ij}(\theta_i - \theta_j) - \overline{s}_{ij} \right)$$

Stationarity conditions:

$$\begin{aligned} \frac{\partial f_i(p_i)}{\partial p_i} - \lambda_i &= 0 \quad \forall i \\ \sum_{j:j\sim i} b_{ij}(\lambda_i - \lambda_j + \chi_{ij} - \chi_{ji}) &= 0 \quad \forall i \end{aligned}$$

λ_i: the price at node i. Agent i solves:

$$\underset{p_i}{\text{minimize}} \quad f_i(p_i) - \lambda_i p_i$$

- Strong duality means the solution to this matches centralized if λ_i is optimal multiplier.
- Nodal or locational marginal pricing
- An instance of the Fund. Theorems of Welfare Econ. - also used to price comm. channels, Kelly mechanism.

$$
\begin{aligned}
\sum_i \lambda_i p_i &= \sum_i \lambda_i \sum_{j:j\sim i} b_{ij}(\theta_i - \theta_j) \\
&= \sum_i \theta_i \sum_{j:j\sim i} b_{ij}(\lambda_i - \lambda_j) \\
&= \sum_i \theta_i \sum_{j:j\sim i} b_{ij}(\chi_{ji} - \chi_{ij}) \\
&= -\sum_i \sum_{j:j\sim i} \chi_{ij} b_{ij}(\theta_i - \theta_j) \\
&= -\sum_i \sum_{j:j\sim i} \chi_{ij} \overline{s}_{ij} \quad \text{by comp. slackness}
\end{aligned}
$$

Therefore

$$
\sum_i \lambda_i p_i + \sum_{i\sim j} \chi_{ij} \overline{s}_{ij} = 0.
$$

Intuition:

- 1st term: SO's budget balance
- 2nd term: always positive
- $\Rightarrow$ SO always makes money

3.1 Example: 2 bus network

$$
\begin{aligned}
\underset{p,\theta}{\text{minimize}} \quad & f_1(p_1) + f_2(p_2) \\
\text{subject to} \quad & \\
\lambda_1 : \quad & p_1 = b(\theta_1 - \theta_2) \\
\lambda_2 : \quad & p_2 = b(\theta_2 - \theta_1) \\
\chi_{12} \geq 0 : \quad & b_{12}(\theta_1 - \theta_2) \leq \overline{s}_{12} \\
\chi_{21} \geq 0 : \quad & b_{12}(\theta_2 - \theta_1) \leq \overline{s}_{12}
\end{aligned}
$$

Stationarity:

$$
\begin{aligned}
\frac{\partial f_i(p_i)}{\partial p_i} - \lambda_i &= 0 \\
b_{12}(\lambda_1 - \lambda_2 + \chi_{12} - \chi_{21}) &= 0
\end{aligned}
$$

Suppose line is uncongested (define). Then comp. slack. implies

- $\chi_{12} = \chi_{21} = 0$
- which implies $\lambda_1 = \lambda_2$
- SO budget is $\lambda_1 p_1 + \lambda_2 p_2 = \lambda_1(p_1 - p_1) = 0$, balanced.

Now suppose congested (define). Then comp. slack. implies

- $\chi_{12} > 0$ (if power flow from 1 to 2),

$$\lambda_1 - \lambda_2 = -\chi_{12} < 0$$

- which implies $\lambda_2 > \lambda_1$
- SO budget is $\lambda_1 p_1 + \lambda_2 p_2 = p_1(\lambda_1 - \lambda_2) < 0$, extra money.

3.2 More discussion

- In radial networks, $\lambda_1 - \lambda_2 = \chi_{21} - \chi_{12}$, nodal price differences equal sum of shadow prices along line
- Not true in general networks!!!
- χ helps define financial transmission rights (in two weeks)
- [1] gives good discussion

Convex relaxations

- Also have strong duality, can define real + reactive power prices
- Reactive power pricing is dubious though
- Congestion is present, but above relationships don't hold

Limitations

- Strong duality is required. Exact power flow not convex.
- Unit commitment - many costs there, not captured. Lead to generators being underpaid.
- Uplift or make-whole payments are made to generators in this case.

References

[1] Felix Wu, Pravin Varaiya, Pablo Spiller, and Shmuel Oren. Folk theorems on transmission access: Proofs and counterexamples. *Journal of Regulatory Economics*, 10:5–23, 1996.