

Math 337 — Complex Numbers

Background and definitions. The well-known quadratic formula tells us that the solution of the quadratic equation $az^2 + bz + c = 0$ is

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

When $b^2 - 4ac < 0$ this equation lacks a solution in the real numbers \mathbf{R} . However, expanding the set of numbers considered to be valid solutions to include $i = \sqrt{-1}$ and all real linear combination of i and 1 makes it possible to find solutions. The expanded set of numbers is called the complex numbers and denoted as \mathbf{C} . Every complex number z can be written in the form $\alpha + i\beta$ where α and β are real numbers; in this form, α is called the *real part* of z and β is called *the imaginary part* of z . If solutions of the quadratic equation $az^2 + bz + c = 0$ are sought in \mathbf{C} they can always be found except in one uninteresting case: $a = 0, b = 0, c \neq 0$.

The complex numbers \mathbf{C} correspond to the plane \mathbf{R}^2 in a natural way.

$$\mathbf{C} = \{\alpha + i\beta : (\alpha, \beta) \in \mathbf{R}^2\}$$

This correspondence indicates how complex numbers should be plotted. The complex number $z = \alpha + \beta i$ is plotted in the plane as the point with coordinates (α, β) . When plotting complex numbers the “ x -axis” is called the “real axis” and the “ y -axis” is called the “imaginary axis.”

The *length* of a complex number (also called its *modulus* or *absolute value*) is defined naturally as

$$|\alpha + i\beta| = \sqrt{\alpha^2 + \beta^2}.$$

Of course, the length of every complex number is nonnegative.

The *conjugate* of a complex number $z = a + bi$ is $\bar{z} = a - bi$. That is, the conjugate is denoted with a bar, and the value of \bar{z} is that of z with the imaginary part replaced by its negative. Graphically, \bar{z} is the reflection of z about the real axis.

Arithmetic operations. Adding and subtracting complex numbers involves nothing more than adding and subtracting the real and imaginary parts:

$$(x+iy) + (u+iv) = (x+u) + i(y+v) \quad \text{and} \quad (x+iy) - (u+iv) = (x-u) + i(y-v).$$

That is, adding and subtracting complex numbers corresponds to the same operations in \mathbf{R}^2 . The definitions of multiplication and division are natural symbolically, but their geometric interpretation is best seen using polar coordinates. For multiplication, we have

$$(x + iy)(u + iv) = xu + ixv + iyu + i^2yv = \boxed{(xu - yv) + i(xv + yu)}.$$

Simplification of division makes use of the conjugate and the property that

$$|z|^2 = z\bar{z}.$$

In particular, we multiply the quotient by one written as the conjugate of the denominator divided by itself:

$$\frac{x + iy}{u + iv} = \frac{x + iy}{u + iv} \cdot \frac{u - iv}{u - iv} = \frac{(xu + yv) + i(uy - xv)}{u^2 + v^2} = \boxed{\frac{xu + yv}{u^2 + v^2} + i \frac{uy - xv}{u^2 + v^2}}.$$

A very significant fact about complex arithmetic is that it obeys the same commutative, distributive and associative laws as real arithmetic. It is also useful to note that taking the complex conjugate commutes with the arithmetic operations; that is, the conjugate of a sum is the sum of the conjugates, the conjugate of a product is the product of the conjugates, and so forth.

Problem 1. Compute the LU factorization of each of the following complex matrix:

$$\begin{pmatrix} 1 + i & 1 - i \\ 2 - i & i \end{pmatrix}.$$

Fully simplify all entries of L and U .

Problem 2. Suppose that z , u and w are complex numbers. Show that $z(u + w) = zu + zw$.

Problem 3. Show that every complex number $\alpha + i\beta$ has a square root. Hint: Solve the equation $(x + iy)^2 = \alpha + i\beta$ for x and y .

The complex numbers embedded in the 2×2 real matrices. While matrix multiplication is generally not commutative and complex multiplication is always commutative, there is a set of 2×2 real matrices that are *isomorphic* to the complex numbers.

$$\alpha + i\beta \quad \leftrightarrow \quad \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

With this correspondence complex arithmetic becomes matrix arithmetic, the conjugate becomes the transpose, and the square length becomes the determinant.

This correspondence is interesting and suggestive mathematically; however, it is not generally useful for computational purposes.

Problem 4. If α , β , γ and δ are real, show that

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} \gamma & \delta \\ -\delta & \gamma \end{pmatrix} = \begin{pmatrix} \gamma & \delta \\ -\delta & \gamma \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

Problem 5. Simplify the two expressions below:

$$(a) (1 + 3i) \frac{(2 - 3i) - \overline{(2 + i)}}{3 + i}$$

$$(b) \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix} \left[\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}^T \right] \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}^{-1}$$

Polynomials. Analysis of polynomials often benefits from use of complex numbers. Two closely related problems involving polynomials are factorization and root finding. The Fundamental Theorem of Algebra tells us the relationship between these two problems. In particular,

$$p(z) = \alpha_n z^n + \alpha_{n-1} z^{n-1} + \cdots + \alpha_1 z + \alpha_0 = \alpha_n \prod_{\ell=1}^n (z - \lambda_\ell)$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are roots of $p(z)$.

Problem 6. Factor the following polynomials:

$$(a) 6z^2 - 20z + 7, \quad (b) \lambda^3 - 8, \quad (c) z^4 - 3z^2 + 2.$$

To every $n \times n$ matrix A we associate a polynomial $p_A(z) = \det(A - zI)$ called the *characteristic polynomial* of A . The roots of $p_A(z)$ are the *eigenvalues* of A .

Problem 7. Find the characteristic polynomials and eigenvalues of the following matrices.

$$(a) \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \quad (b) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (c) \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

The Hamilton-Cayley theorem states that $p_A(A) = 0$. For example, if

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad \text{then} \quad p_A(z) = z^2 - 2z - 3 \quad \text{and}$$

$$p_A(A) = A^2 - 2A - 3A^0 = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} - \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Problem 8. For each matrix in problem 7, verify that $p_A(A) = 0$.

Polar representation and m th roots. Euler's formula relates the exponential and sinusoidal functions:

$$e^{i\theta} = \cos \theta + i \sin \theta.$$

Notice that $|e^{i\theta}| = \cos^2 \theta + \sin^2 \theta = 1$. Every complex number $z = \alpha + i\beta$ can be written in *polar form* $R \exp(i\theta)$ where $R = |z| = \sqrt{\alpha^2 + \beta^2}$, $\cos \theta = \alpha/|z|$ and $\sin \theta = \beta/|z|$. The nonnegative real number R is the modulus of z , the real number θ is called the *argument* of z .

Problem 9. Find the polar form for each of the following complex numbers.

$$(a) 1 \qquad (b) i \qquad (c) -1 \qquad (d) -3 - 4i$$

Notice that if $z_1 = R_1 e^{i\theta_1}$ and $z_2 = R_2 e^{i\theta_2}$ then $z_1 z_2 = R_3 e^{i\theta_3}$ where $R_3 = R_1 R_2$ and $\theta_3 = \theta_1 + \theta_2$.

***m*th roots.** The solutions of $z^m = 1$ are called the *m*th roots of unity. There are exactly *m* distinct *m*th roots of unity. These are

$$z = \left\{ 1, e^{2\pi i/m}, e^{4\pi i/m}, e^{6\pi i/m}, \dots, e^{2(m-1)\pi i/m} \right\}$$

Problem 10. Compute and plot the square roots of unity, the third roots of unity and the fourth roots of unity.

To find the *m*th roots of a general complex number z , it is generally convenient to first write it in polar form: $z = R e^{i\theta}$. Then the solutions of $w^m = z$ are

$$w = \left\{ R^{1/m} e^{i\theta/m}, R^{1/m} e^{i(\theta+2\pi)/m}, R^{1/m} e^{i(\theta+4\pi)/m}, \dots, R^{1/m} e^{i((\theta+2(m-1))\pi)/m} \right\}.$$

Problem 11. Verify that

$$\left(R^{1/m} e^{i(\theta+2k\pi)/m} \right)^m = R e^{i\theta} \quad \text{for } k = 0, 1, \dots, m-1.$$

Problem 12. Compute and plot the 4th roots of $z = -1$.