Name____ 337-F2006

Write your name above. Materials not needed for the exam, such as books and backpacks, must be placed at the front of the room during the exam. Electronic devices such as pagers and cell phones must be turned off and put away for the duration of the exam.

Math 337 – Fall 2006 Second Common Exam

Instructions *Show your work and mark your answers clearly. All work must be done in the examination booklets provided. No books, notes, calculators or scratch paper are allowed. This question sheet must be submitted with your exam booklet. Put your name on all exam booklets. Sign the honor code pledge. Check your work; partial credits will be limited. You must remain in the classroom until the exam has ended.*

Problem 1 (26 points) (a) (14 pts) True or false:

(i) If $\mathbf{u}$ and $\mathbf{v}$ are nonzero column vectors and $A = \mathbf{u}\mathbf{v}^T$, then the rank of A is 2.

(ii) If A has full row rank, then $Ax = \mathbf{b}$ has at least one solution.

(iii) Four vectors in $\mathbf{R}^3$ can not be linearly independent.

(iv) The columns of a matrix form a basis for its column space.

(v) If columns of A are orthonormal, then AA^T is a projection matrix.

(vi) $(1,1,1)$ is perpendicular to $(1,1,-2)$ so the planes $x+y+z=0$ and $x+y-2z=0$ are orthogonal subspaces.

(vii) If P is a projection matrix, then $P^2 = P$.

(b) (12 pts) Construct a matrix with the required property or explain why that is impossible:

(i) Column space contains $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, nullspace contains $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

(ii) Row space contains $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, nullspace contains $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

(iii) $Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ has a solution and $A^T \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(CONTINUED ON THE BACK)

Problem 2 (24 points) Consider the matrix A and the vectors $\mathbf{b}$ and $\mathbf{c}$.

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 5 & 7 & 6 \\ 4 & 6 & 10 & 4 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} 2 \\ 3 \\ 10 \end{bmatrix}.$$

(a) (6 pts) Reduce A to its reduced row echelon form R by row transformations.

(b) (6 pts) Find the solvability condition on b_1, b_2, b_3 for $A\mathbf{x} = \mathbf{b}$ to be solvable.

(c) (12 pts) Find the complete solution (also called the general solution) to $A\mathbf{x} = \mathbf{c}$.

Problem 3 (24 points) Consider the following matrix A.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 17 & 1 & 0 \\ 29 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

(a) (4 pts) What is the rank of A? Explain.

(b) (4 pts) What are the dimensions of four fundamental subspaces (i.e., column space, nullspace, row space, and left nullspace) of A?

(c) (16 pts) Find a basis for each of the four fundamental subspaces of A. (*Hint*: Best if you don't work too hard!)

Problem 4 (26 points) Consider the matrix A and the vector $\mathbf{b}$.

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 1 \\ 2 & -4 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ -7 \end{bmatrix}.$$

(a) (10 pts) Use Gram-Schmidt to find orthonormal vectors $\mathbf{q_1}$, $\mathbf{q_2}$ such that $\mathbf{q_1}$, $\mathbf{q_2}$ span the column space of A.

(b) (6 pts) Find a third vector $\mathbf{q_3}$ such that $\mathbf{q_1}$, $\mathbf{q_2}$, $\mathbf{q_3}$ form an orthonormal basis for $\mathbf{R}^3$.

(c) (10 pts) Find the least squares solution to $A\mathbf{x} = \mathbf{b}$.

END OF QUESTION SHEET

Carefully check all your answers.