Name____ 337-F2004-3.14159265358979

Write your name above. Materials not needed for the exam, such as books and backpacks, must be placed at the front of the room during the exam. Electronic devices such as pagers and cell phones must be turned off for the duration of the exam.

Math 337 – Fall 2004 Final Examination

Instructions. *Show your work. All work must be done in the examination booklets provided. No books, notes, calculators or scratch paper are allowed. This question sheet must be submitted with your exam booklet. Put your name on all exam booklets. Sign the honor code pledge. Check your work. You must remain in the classroom until the exam has ended.*

Problem 1. (14 points) Find the general solution of the following system of equations:

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

Problem 2. (12 points) For the matrix A below find (a) the LU factorization, (b) the determinant, and (c) the inverse.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

Problem 3. (16 points)

(a) For the 1×3 matrix A below, find an orthonormal basis for its null space.

$$A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

(b) Use projection to find the point in the plane $x + y + z = 0$ that is closest to $(0, 1, 0)$. *Hint: The plane $x + y + z = 0$ is the same as the null space in part (a).*

Problem 4. (12 points) Find the eigenvalues for each of the following matrices:

$$(a) \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \quad \text{and} \quad (b) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

(CONTINUED ON THE BACK)

Problem 5. (10 points) Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Determine whether or not each of the following vectors is an eigenvector of A. If the vector is an eigenvector give the corresponding eigenvalue.

$$(a) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad (b) \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (c) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (d) \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} \quad (e) \begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix}$$

Problem 6. (16 points) Diagonalize the following two matrices:

$$(a) \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \qquad \text{and} \qquad (b) \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}.$$

Hint: To diagonalize a matrix A means to find an invertible matrix S and a diagonal matrix Λ such that $A = S\Lambda S^{-1}$.

Problem 7. (8 points) Which of the following transformations are linear?

$(a)\ T(v_1, v_2) = (v_2, v_1)$ $\qquad$ $(b)\ T(v_1, v_2) = (v_1, v_1)$

$(c)\ T(v_1, v_2) = (0, v_1)$ $\qquad$ $(d)\ T(v_1, v_2) = (0, 1)$

Problem 8. (6 points) The sequence x_0, x_1, x_2, $\ldots$ has the property that

$$x_{n+2} = \frac{x_{n+1} + x_n}{2} \qquad \text{for } n = 0, 1, 2, \ldots.$$

Notice that

$$\begin{bmatrix} x_{n+2} \\ x_{n+1} \end{bmatrix} = A \begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} \qquad \text{with } A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}.$$

(a) Diagonalize A.
(b) Find a formula for (x_{n+1}, x_n) in terms of (x_1, x_0) and powers of A.
(c) Suppose $x_0 = 0$ and $x_1 = 1$. Give an explicit formula for x_n.

Problem 9. (6 points) Suppose that a and b are real numbers satisfying $a^2 + b^2 = 1$. Consider the symmetric matrix

$$A = \begin{bmatrix} 1 + a^2 & -ab \\ -ab & 1 + b^2 \end{bmatrix}.$$

(a) Find the eigenvalues of A. *(Hint: Computing the determinant and trace of A may be helpful. Remember $a^2 + b^2 = 1$.)*
(b) Find the eigenvectors of A.
(c) Diagonalize A with an orthogonal matrix of eigenvectors.

END OF QUESTION SHEET