

Name_____

337-F2006

Write your name above. Materials not needed for the exam, such as books and backpacks, must be placed at the front of the room during the exam. Electronic devices such as pagers and cell phones must be turned off and put away for the duration of the exam.

Thank you for taking Math 337! Good Luck!

Math 337 – Fall 2006 Final Exam

Instructions *Show your work and mark your answers clearly. All work must be done in the examination booklets provided. No books, notes, calculators or scratch paper are allowed. This question sheet must be submitted with your exam booklet. Put your name on all exam booklets. Sign the honor code pledge. Check your work; partial credits will be limited. You must remain in the classroom until the exam has ended.*

Problem 1 (16 pts) True or false:

- (i) If a 4 by 4 matrix has $\det A = -1$, then $\det(4A) = -4$.
- (ii) If A is not invertible then AB is not invertible.
- (iii) AB and BA have the same determinant.
- (iv) If a 3 by 3 matrix A has eigenvalues 1, 2, 3, then A is not invertible.
- (v) A matrix with real eigenvalues and eigenvectors is symmetric.
- (vi) If $A^T Ax = \mathbf{0}$, then $Ax = \mathbf{0}$.
- (vii) $3x^2 + 8xy + 5y^2$ is always positive for $(x, y) \neq (0, 0)$.
- (viii) If A is positive definite then A^{-1} is also positive definite.

Problem 2 (12 points) Suppose that a and b are real numbers and $a \neq 0$.

- (a) Find the LU factorization of the matrix A where

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

- (b) Find the inverse of A .

Problem 3 (16 points) Consider the matrix A .

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 5 & 7 & 6 \\ 2 & 7 & 13 & 14 \end{bmatrix}.$$

- (a) Find the rank of A .
- (b) Find a basis for the row space of A .
- (c) Find a basis for the column space of A .
- (d) Find a basis for the nullspace of A .
- (e) Find a basis for the left nullspace of A .
- (f) Find the complete solution of $A\mathbf{x} = \mathbf{b}$ for $\mathbf{b} = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}$.

Problem 4 (12 points)

- (a) Find the orthogonal projection of the vector $(1, 2, 1)$ onto the plane $x + y - z = 0$.
- (b) Determine the shortest distance from the vector $(1, 2, 1)$ to the plane $x + y - z = 0$.

Problem 5 (10 points) Compute the determinants of the following matrices. Show your work.

$$A = \begin{bmatrix} 0 & 0 & a \\ b & 0 & 0 \\ 0 & c & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 101 & 201 & 301 \\ 102 & 202 & 302 \\ 103 & 203 & 303 \end{bmatrix}.$$

Problem 6 (12 points)

- (a) Compute the eigenvalues and eigenvectors of the following matrix:

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

- (b) Diagonalize the matrix $A^T A$ using an orthogonal matrix Q , where the matrix $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$. Be sure to give your answer in terms of matrix products (i.e., $Q\Lambda Q^T$).

Problem 7 (12 points) Starting with $p_0 = 1$ and $q_0 = 1$ consider the following recursion:

$$\begin{aligned} p_n &= 0.7p_{n-1} + 0.4q_{n-1} \\ q_n &= 0.3p_{n-1} + 0.6q_{n-1} \end{aligned}$$

- (a) Find a matrix A such that:

$$\begin{bmatrix} p_n \\ q_n \end{bmatrix} = A \begin{bmatrix} p_{n-1} \\ q_{n-1} \end{bmatrix}$$

- (b) By diagonalizing A , find a formula for $\begin{bmatrix} p_n \\ q_n \end{bmatrix}$. (*Hint: To diagonalize a matrix A means to find an invertible matrix S and a diagonal matrix Λ such that $A = S\Lambda S^{-1}$.)*

- (c) What is the limit of $\begin{bmatrix} p_n \\ q_n \end{bmatrix}$ as $n \rightarrow \infty$?

Problem 8 (10 points) Determine whether or not the following matrices are positive definite. Show your work and clearly state your reasons for your answers.

$$A = \begin{bmatrix} 3 & -3 \\ -3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 10 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 2 & 0 & b \\ 0 & 0 & 3 & c \\ a & b & c & 0 \end{bmatrix}$$

where a, b, c are real numbers.

Carefully check all your answers.