

Name_____

337-F2007

Write your name above. Materials not needed for the exam, such as books and backpacks, must be placed at the front of the room during the exam. Electronic devices such as pagers and cell phones must be turned off and put away for the duration of the exam.

Thank you for taking Math 337! Good Luck!

Math 337 – Fall 2007 Final Exam

Instructions Show your work and mark your answers clearly. All work must be done in the examination booklets provided. No books, notes, calculators or scratch paper are allowed. This question sheet must be submitted with your exam booklet. Put your name on all exam booklets. Sign the honor code pledge. Check your work; partial credits will be limited. You must remain in the classroom until the exam has ended.

Problem 1 (16 points) True or false:

- (i) If Q has orthonormal columns, then $QQ^T = I$.
- (ii) If A is a 3 by 3 skew-symmetric matrix (i.e., $A^T = -A$), then $\det(A) = 0$.
- (iii) A and A^T always have the same eigenvalues.
- (iv) The projection matrix is always diagonalizable.
- (v) The determinant of $AB - BA$ is zero.
- (vi) If P is the projection matrix onto the plane $x + y - z = 0$, then three eigenvalues of P are 0, 0 and 1.
- (vii) If A and B are positive definite, then $A + B$ is positive definite.
- (viii) $5x^2 + 12xy + 7y^2$ is always positive for $(x, y) \neq (0, 0)$ since all coefficients are positive.

Problem 2 (12 points) Consider the matrix A and the vector b :

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

- (a) (4 pts) Find the LU factorization of A .
- (b) (4 pts) Find the inverse of A .
- (c) (4 pts) Find the complete solution of $Ax = b$.

Problem 3 (12 points) Consider the matrix A and the vector b :

$$A = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 1 & 1 & 3 & 6 \\ 2 & -1 & 3 & 6 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

- (a) (8 pts) Find bases for the four fundamental subspaces of A .
- (b) (4 pts) Find the conditions on b_1 , b_2 and b_3 so that $Ax = b$ has a solution.

Problem 4 (14 points) This matrix A has column 1 + column 2 = column 3:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

- (a) (4 pts) Find an orthonormal basis for the column space $C(A)$.
- (b) (4 pts) The projection matrix P onto the column space does not come from the usual formula $A(A^T A)^{-1} A^T$. Why not - what goes wrong with this formula?
- (c) (6 pts) Find the projection matrix P onto the column space of A .

Problem 5 (10 points)

- (a) (6 pts) Compute the determinants of the following matrices. Show your work.

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, \quad B = \begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ d & 0 & 0 & 0 \end{bmatrix}.$$

- (b) (4 pts) If you know that $\det A = 1$, what is the determinant of B ? Justify your answer.

$$A = \begin{bmatrix} \text{row 1} \\ \text{row 2} \\ \text{row 3} \end{bmatrix}, \quad B = \begin{bmatrix} \text{row 3} + \text{row 2} + \text{row 1} \\ 2 \text{ row 2} + \text{row 1} \\ 3 \text{ row 1} \end{bmatrix}.$$

Problem 6 (14 points) Compute the eigenvalues and eigenvectors of the following matrices. Show your work.

(a) (4 pts) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

(b) (4 pts) $B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$,

(c) (6 pts) $C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

Problem 7 (12 points)

(a) (6 pts) If $a \neq c$, find the eigenvalue matrix Λ and eigenvector matrix S in

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = S\Lambda S^{-1}.$$

(b) (6 pts) Find explicitly the *four entries* in the matrix A^{100} . Simplify your answers.

Problem 8 (10 points)

(a) (6 pts) Find the best line $C + Dt$ to fit $b = 3, 3, 0, -1, 0$ at times $t = -2, -1, 0, 1, 2$.

(b) (4 pts) For which numbers c is the following matrix positive definite?

$$A = \begin{bmatrix} c-1 & \sqrt{2} \\ \sqrt{2} & 4-c \end{bmatrix}.$$

Carefully check all your answers.