Math 712 Homework Assignment 2
Due: Monday Sep. 22

Instructions: *1. Everything must be returned in report form and must be type-written.*
2. You must submit your code.
3. Late assignments are NOT accepted.

Problem 1.
Determine the coefficients a_0, a_1, a_{-1} so that the scheme

$$v_j^{n+1} = a_{-1}v_{j-1}^n + a_0 v_j^n + a_1 v_{j+1}^n,$$

for the solution of the advection equation $u_t = u_x$ agrees with the Taylor series expansion of $u(x_j, t_{n+1})$ to as high order as possible. Verify that the result is the Lax-Wendroff scheme.

Problem 2.
Consider the following initial value problem

$$\begin{aligned} &u_t + u_x + u_{xxx} = 0, \quad -\infty < x < \infty, \quad t > 0, \\ &u(x,0) = \delta(x). \end{aligned} \tag{1}$$

(a) Show that $\int_{-\infty}^{\infty} u(x,t)dx$ is a conserved quantity and give its value, assuming u and its derivatives vanish at infinity.

(b) Use Fourier transform methods to show that the solution can be written as

$$u(x,t) = \frac{\sqrt{2\pi}}{(3t)^{1/3}} Ai\left(\frac{x-t}{(3t)^{1/3}}\right),$$

where $Ai(\cdot)$ represents the Airy function. (You may look up different representations of the Airy function, in particular one involving a Fourier type integral - one place to look is Abramowitz and Stegun, available online with a link from my course website).

(c) Produce graphs of $u(x,t)$ for $t = 0.1, 0.5, 1, 2.5, 5$ and discuss the behavior of the solution. Perform a numerical quadrature of the solutions at these times and compare the results. Explain.