Math 712 Homework Assignment 3

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Problem 1.

$$\begin{aligned} u_t &= u_x, \quad 0 < x < 2\pi, \quad t > 0, \\ u(x,0) &= sin(x), \end{aligned} \tag{1}$$

(a)

Notation : $v_j^n = u(x_j, t_n), \quad h = \triangle x, \quad k = \triangle t.$

Leap frog scheme

$$v_j^{n+1} = v_j^{n-1} + \frac{k}{h}\left(v_{j+1}^n - v_{j-1}^n\right). \tag{2}$$

Upwind scheme

$$v_j^{n+1} = v_j^n + \frac{k}{h}\left(v_{j+1}^n - v_j^n\right). \tag{3}$$

Lax-Wendroff scheme

$$v_j^{n+1} = v_j^n + k\frac{v_{j+1}^n - v_{j-1}^n}{2h} + \frac{k^2}{2}\frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{h^2}. \tag{4}$$

Note : For simplicity I just choose the boundary condition to be exact solution. So it's $u(2\pi,t) = sin(2\pi + t)$ for Upwind scheme and one more condition $u(0,t) = sin(t)$ for the others. In fact, the better way to do this problem is to choose a larger domain in the beginning and shrink computational domain in each time step. For example, if $h = k = 0.1$, in order to obtain the solution on $[0, 2\pi]$ at time $t = 1$, initially the computational domain should be chosen as $[0, 2\pi + 1]$ for Upwind scheme and $[-1, 2\pi + 1]$ for Lax-Wendroff scheme. Besides, the initial two time steps for Leap-frog scheme is chosen to be exact for simplicity.

(b)

In Fig 1∼3, the upper level shows the numerical(dotted) and exact(solid) solution while the lower level shows the infinity norm of the difference between numerical and exact solution. As we can see from Fig 1, the numerical solution is unstable for $\lambda = 1.1$ and is exact for $\lambda = 1$ in Fig 2. It also justifies that $\lambda = 1$ is necessary for stability.

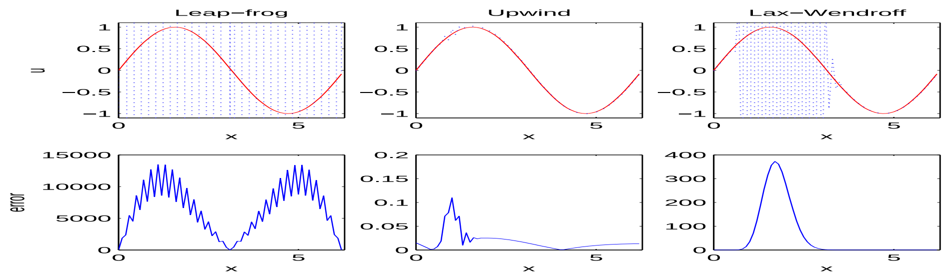

Figure 1: $\lambda = 1.1$

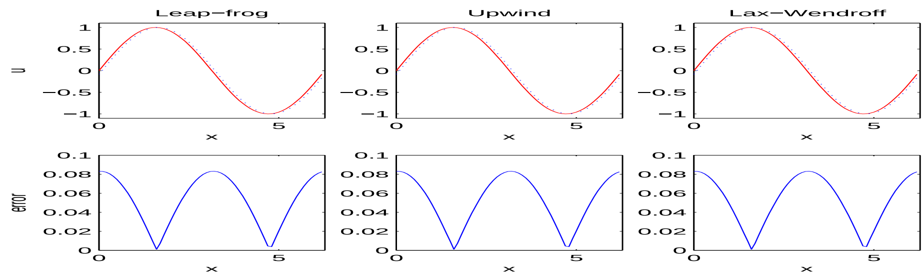

Figure 2: $\lambda = 1$

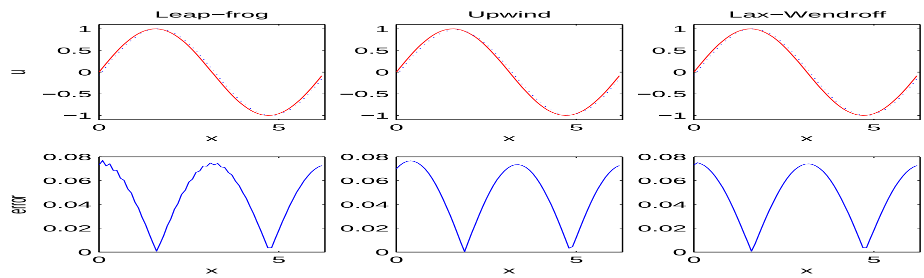

Figure 3: $\lambda = 0.9$

(c)

The upper level shows exact(solid) and numerical(dotted) solution in different time while the lower level shows the error(infinity norm). Actually the exact solution is $sin(x+t)$, which means it moves to the left hand side as time goes. We can see from Fig 4 to Fig 7, truly the solution moves toward the left hand side. Besides, the error is increasing as time goes.

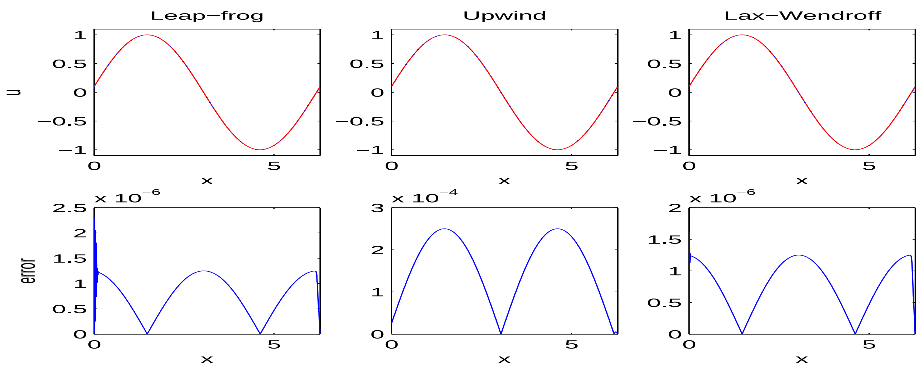

Figure 4: $T = 0.1$

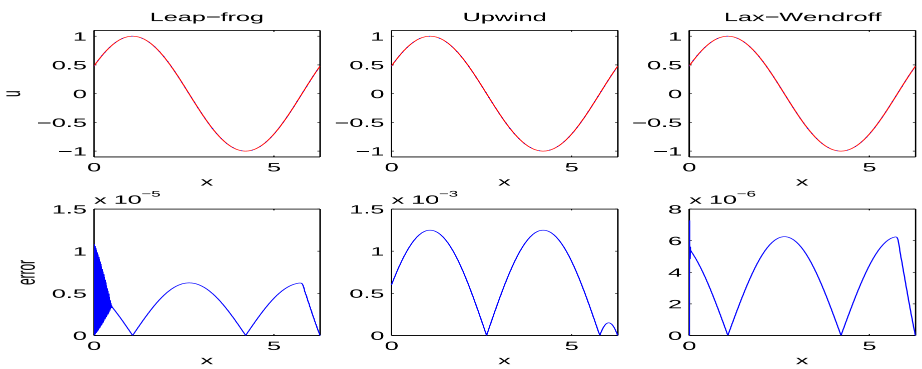

Figure 5: $T = 0.5$

(d)

In this section I fix $\lambda = \frac{\Delta t}{\Delta x} = 0.5$, $T = 1$. It's easily to see from following table that the Upwind scheme is 1st order and Leap-frog and Lax-Wendroff scheme are 2nd order methods.

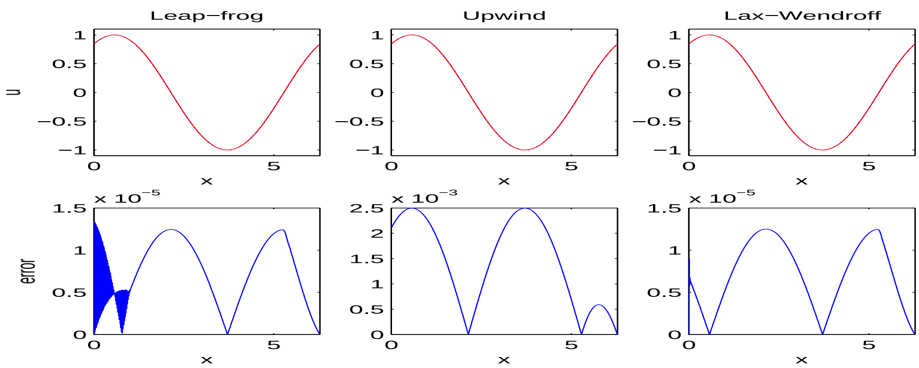

Figure 6: $T = 1$

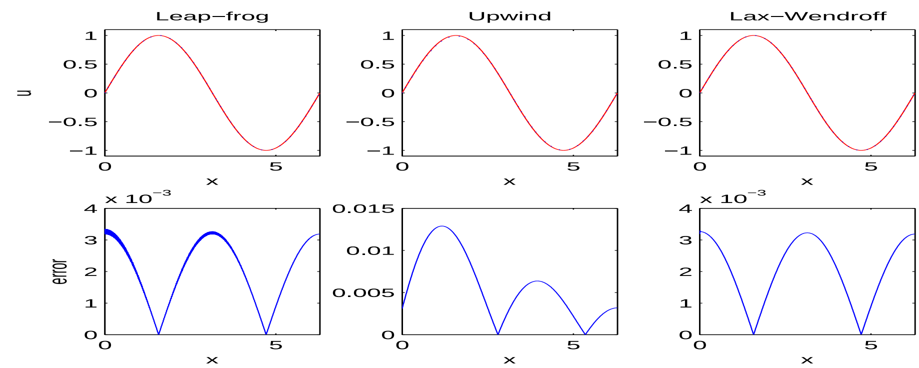

Figure 7: $T = 2\pi$

	$k = 0.01$	$k = 0.005$	$k = 0.0025$	$k = 0.00125$
Leap-frog	8.4412e-005	2.1178e-005	5.3038e-006	1.3276e-006
Convergence rate		1.9949	1.9975	1.9982
Upwind	0.0085	0.0043	0.0021	0.0011
Convergence rate		0.9831	1.0339	0.9329
Lax-Wendroff	8.2324e-005	2.0588e-005	5.1479e-006	1.2876e-006
Convergence rate		1.9995	1.9997	1.9993

Table 1: Convergence rate