Frequency Response of FIR Filters

Lecture #10

Chapter 6

Properties of the Frequency Response

- Relationship of the Frequency Response to the Difference Equation and Impulse Response

$$\text{Difference Equation} \Leftrightarrow \text{Impulse Response}$$

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] \Leftrightarrow h[n] = \sum_{k=0}^{M} b_k \delta[n-k]$$

$$\text{Difference Equation} \Leftrightarrow \text{Frequency Response}$$

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] \Leftrightarrow y[n] = \left(\sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}\right) A e^{j\phi} e^{j\hat{\omega}n} = H(e^{j\hat{\omega}}) A e^{j\phi} e^{j\hat{\omega}n}$$

$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k] \Leftrightarrow H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k} = \sum_{k=0}^{M} h[k] e^{-j\hat{\omega}k}$$

$$\text{Time Domain} \Leftrightarrow \text{Frequency Domain}$$

Go between the difference equation, impulse response and the frequency response by knowing the b_k's

Example

$$h[n] = -\delta[n] + 3\delta[n-1] - \delta[n-2]$$

$$\{b_k\} = \{-1, 3, -1\}$$

$$y[n] = -x[n] + 3x[n-1] - x[n-2]$$

$$H(e^{j\hat{\omega}}) = -1 + 3e^{-j\hat{\omega}} - e^{-j2\hat{\omega}}$$

Periodicity of the Frequency Response

- The Frequency Response is a periodic function of 2π

$$H(e^{j(\hat{\omega}+2\pi)}) = \sum_{k=0}^{M} b_k e^{-j(\hat{\omega}+2\pi)k}$$

$$= \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k} e^{-j2\pi k}$$

$$= H(e^{j\hat{\omega}})$$

since $e^{-j2\pi k} = 1$ when $k = 1$

Therefore, we always express $H(e^{j\hat{\omega}})$

over one period, e.g., $-\pi < \hat{\omega} < \pi$

Conjugate Symmetry

- If the filter coefficients are real (i.e., $b_k = b_k^*$), then the frequency response has conjugate symmetry and

$$H(e^{-j\hat{\omega}}) = H^*(e^{j\hat{\omega}})$$

- As a result,
  - In polar form, the magnitude is an even function and the phase is an odd function
  - In Cartesian form, the real part is an even function and the imaginary part is an odd function
- Therefore, we only have to show the frequency for one half of a period, (e.g., between 0 and π)

Proof of Conjugate Symmetry

$$H^*(e^{j\hat{\omega}}) = \left(\sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}\right)^*$$

$$= \sum_{k=0}^{M} b_k{}^* e^{+j\hat{\omega}k}$$

$$= \sum_{k=0}^{M} b_k e^{-j(-\hat{\omega})k}$$

$$= H(e^{-j\hat{\omega}})$$

Proof of Conjugate Symmetry

$$H(e^{j\hat{\omega}}) = \left|H(e^{j\hat{\omega}})\right| e^{j\angle H(e^{j\hat{\omega}})}$$

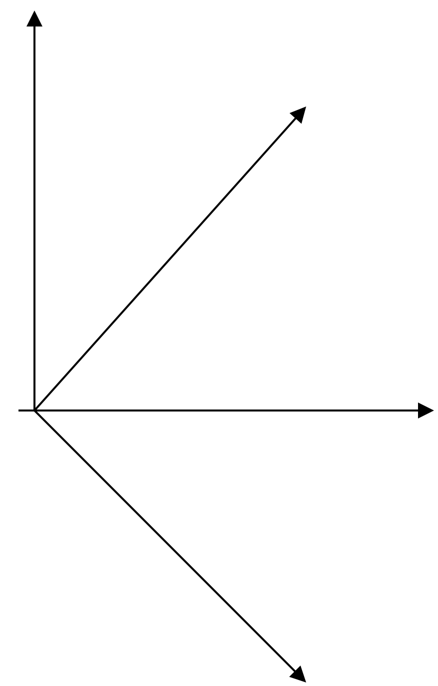
$$H(e^{-j\hat{\omega}}) = \left|H(e^{-j\hat{\omega}})\right| e^{j\angle H(e^{-j\hat{\omega}})}$$

$$= \left|H^*(e^{j\hat{\omega}})\right| e^{j\angle H^*(e^{j\hat{\omega}})}$$

$$= \left|H(e^{j\hat{\omega}})\right| e^{-j\angle H(e^{j\hat{\omega}})}$$

$$\left|H(e^{-j\hat{\omega}})\right| = \left|H(e^{j\hat{\omega}})\right| \text{ even function}$$

$$\angle H(e^{-j\hat{\omega}}) = -\angle H(e^{j\hat{\omega}}) \text{ odd function}$$



Proof of Conjugate Symmetry

$$H(e^{j\hat{\omega}}) = \Re e[H(e^{j\hat{\omega}})] + j\Im m[H(e^{j\hat{\omega}})]$$

$$H(e^{-j\hat{\omega}}) = \Re e[H(e^{-j\hat{\omega}})] + j\Im m[H(e^{-j\hat{\omega}})]$$

$$= \Re e[H^*(e^{j\hat{\omega}})] + j\Im m[H^*(e^{j\hat{\omega}})]$$

$$= \Re e[H(e^{j\hat{\omega}})] - j\Im m[H(e^{j\hat{\omega}})]$$

$$\Re e[H(e^{-j\hat{\omega}})] = \Re e[H(e^{j\hat{\omega}})] \text{ even function}$$

$$\Im m[H(e^{-j\hat{\omega}})] = -\Im m[H(e^{j\hat{\omega}})] \text{ odd function}$$

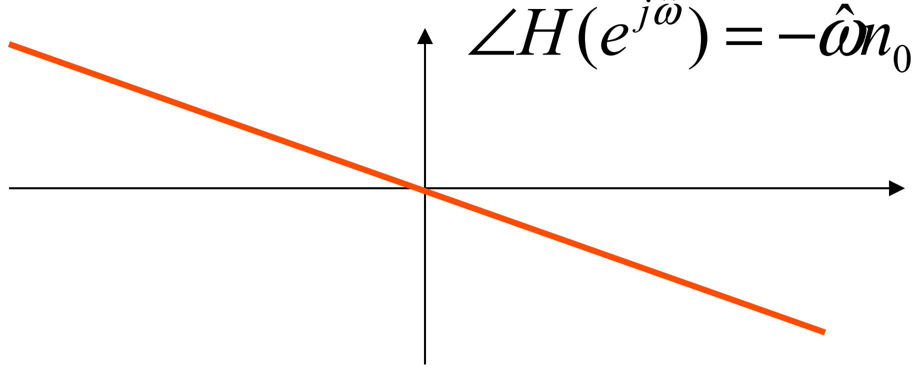
Graphical Representation of the Frequency Response

- The frequency response varies with frequency
- By choosing the coefficients of the difference equation, the shape of the frequency response vs frequency can be developed.
- Examples are:
  - filters which only pass low frequencies
  - filters which only pass high frequencies
  - filters which only alter the phase
- Therefore, we usually plot the amplitude and phase of the frequency response vs. frequency
  - This is sometimes called the Bode Plot

Delay System

- A simple FIR filter: $y[n]=x[n-n_0]$
- Therefore, from the difference equation, $k = n_0$ and $b_{n_0}=1$ and the Frequency Response becomes:

$$H(e^{j\hat{\omega}}) = e^{-j\hat{\omega}n_0}$$



First-Difference System
High Pass Filter

$$y[n] = x[n] - x[n-1]$$

$$H(e^{j\hat{\omega}}) = 1 - e^{-j\hat{\omega}} = e^{-j\hat{\omega}/2}(e^{j\hat{\omega}/2} - e^{-j\hat{\omega}/2})$$

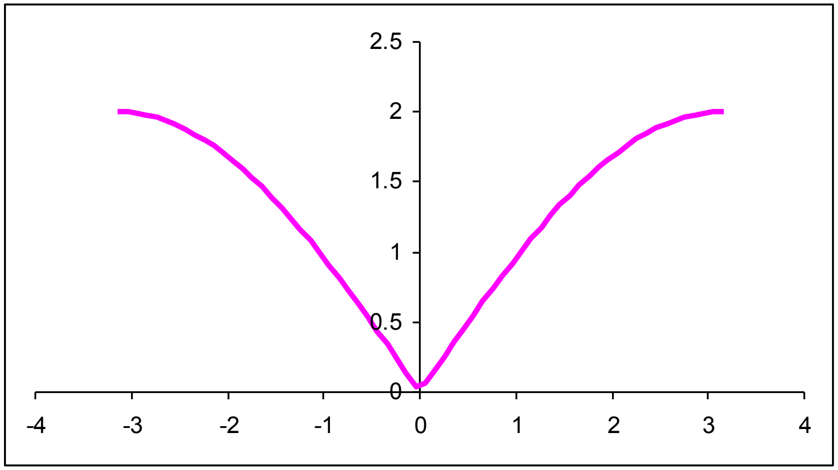
$$= 2je^{-j\hat{\omega}/2}\sin(\hat{\omega}/2)$$

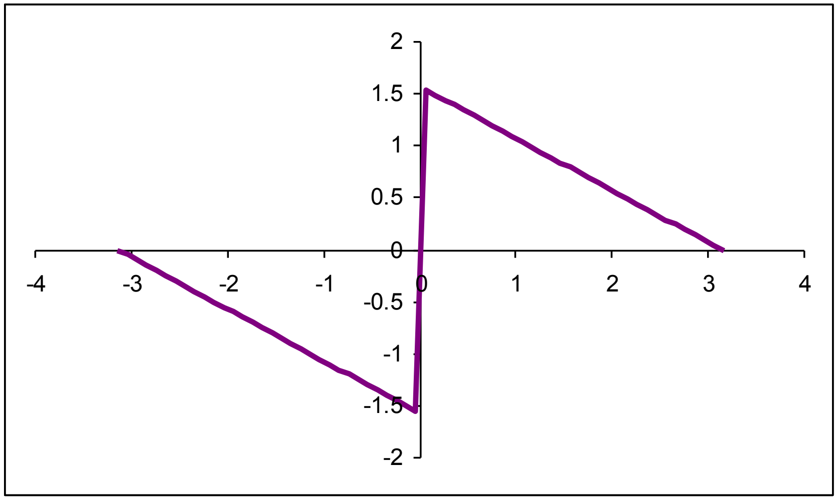
$$= 2e^{j\pi/2}e^{-j\hat{\omega}/2}\sin(\hat{\omega}/2)$$

$$= 2e^{-j(\hat{\omega}-\pi)/2}\sin(\hat{\omega}/2)$$

$$\left|H(e^{j\hat{\omega}})\right| = 2\left|\sin\frac{\hat{\omega}}{2}\right|$$

$$\angle H(e^{j\hat{\omega}}) = -(\hat{\omega}-\pi)/2$$





Simple Low Pass Filter

$$y[n] = x[n] + 2x[n-1] + x[n-2]$$

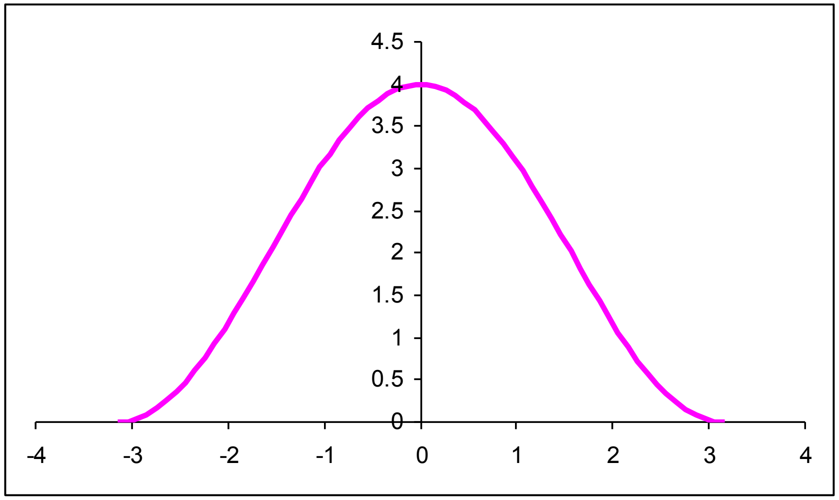
$$H(e^{j\hat{\omega}}) = 1 + 2e^{-j\hat{\omega}} + e^{-j2\hat{\omega}}$$

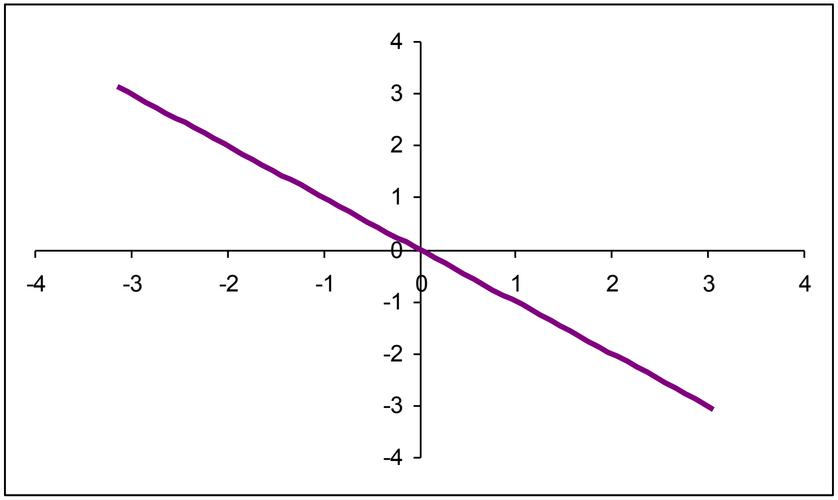
$$= e^{-j\hat{\omega}}(e^{j\hat{\omega}} + 2 + e^{-j\hat{\omega}})$$

$$= (2 + 2\cos\hat{\omega})e^{-j\hat{\omega}}$$

$$\left|H(e^{j\hat{\omega}})\right| = (2 + 2\cos\hat{\omega})$$

$$\angle H(e^{j\hat{\omega}}) = -\hat{\omega}$$

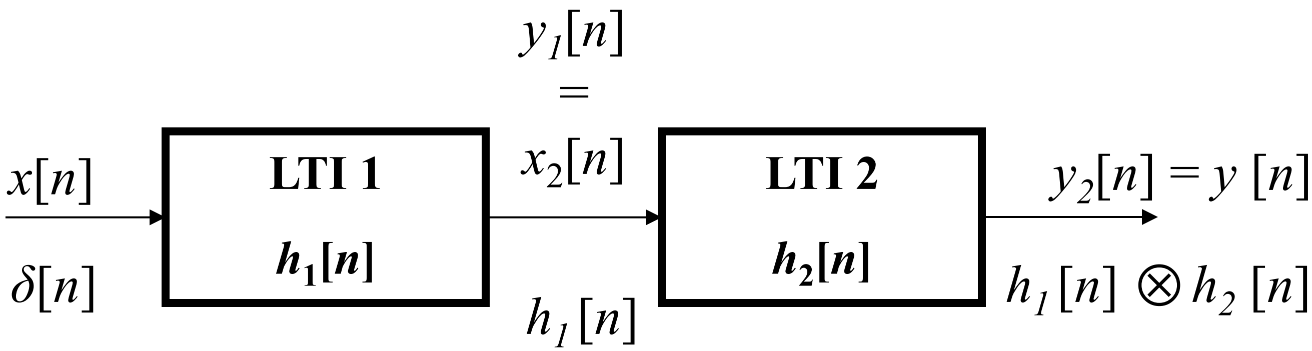




Cascaded LTI Systems

- Recall that two systems cascaded together, then the overall impulse response is the convolution of the two individual impulse responses.
- It turns out the the frequency response of a cascaded system is the product of the individual frequency responses.

Proof of the Frequency Response of Cascaded Systems



$$y_1[n] = H_1(e^{j\hat{\omega}})e^{j\hat{\omega}n}$$

$$y[n] = y_2[n] = H_2(e^{j\hat{\omega}})y_1[n] = H_2(e^{j\hat{\omega}})H_1(e^{j\hat{\omega}})e^{j\hat{\omega}n}$$

$$H_T(e^{j\hat{\omega}}) = H_2(e^{j\hat{\omega}})H_1(e^{j\hat{\omega}})$$

Therefore, these processes are related

$$h_1[n] \otimes h_2[n] \Leftrightarrow H_2(e^{j\hat{\omega}})H_1(e^{j\hat{\omega}})$$

Running-Average Filtering

- A simple LTI system defined as the L-point running average

$$y[n] = \frac{1}{L}\sum_{k=0}^{L-1} x[n-k]$$

$$= \frac{1}{L}(x[n] + x[n-1] + \cdots x[n-(L-1)])$$

- The frequency response is then

$$H(e^{j\hat{\omega}}) = \frac{1}{L}\sum_{k=0}^{L-1} e^{-j\hat{\omega}k}$$

The Frequency Response of the Running Average Filter

$$H(e^{j\hat{\omega}}) = \frac{1}{L}\sum_{k=0}^{L-1} e^{-j\hat{\omega}k}$$

Using the formula for the partial sums of a geometric series

$$\sum_{k=0}^{L-1} \alpha^k = \frac{1-\alpha^L}{1-\alpha}$$

$$H(e^{j\hat{\omega}}) = \frac{1}{L}\sum_{k=0}^{L-1} e^{-j\hat{\omega}k} = (\frac{1}{L})(\frac{1-e^{-j\hat{\omega}L}}{1-e^{-j\hat{\omega}}})$$

$$= (\frac{1}{L})(\frac{e^{-j\hat{\omega}L/2}(e^{+j\hat{\omega}L/2} - e^{-j\hat{\omega}L/2})}{e^{-j\hat{\omega}/2}(e^{+j\hat{\omega}/2} - e^{-j\hat{\omega}/2})})$$

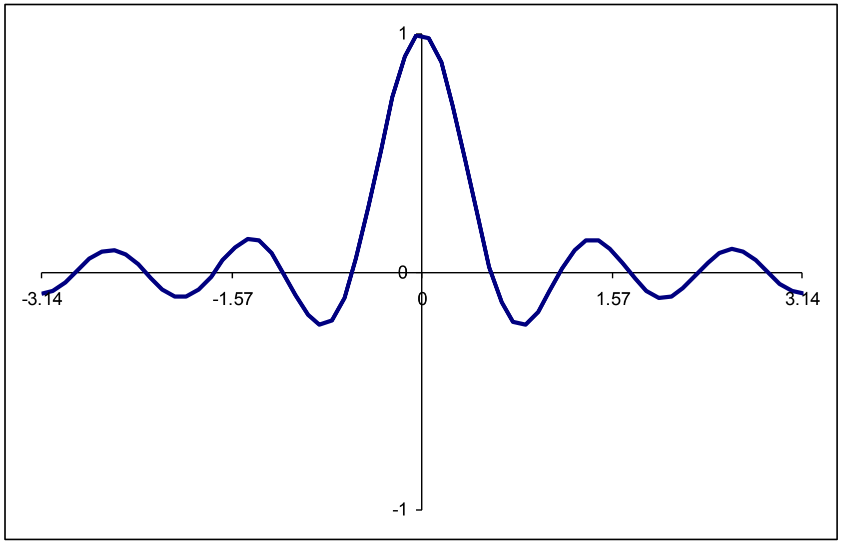
$$= (\frac{1}{L})(\frac{\sin \hat{\omega}L/2}{\sin \hat{\omega}/2})e^{-j\hat{\omega}(L-1)/2}$$

Therefore,

$$H(e^{j\hat{\omega}}) = D_L(e^{j\hat{\omega}})e^{-j\hat{\omega}(L-1)/2}$$

$$D_L(e^{j\hat{\omega}}) = \frac{\sin \hat{\omega}L/2}{L \sin \hat{\omega}/2} \Leftarrow \text{Dirichlet Function}$$

Plot of the Dirichlet Function



What happens when $\hat{\omega} = 0$?

$$D_L(e^{j0}) = \frac{\sin 0}{L \sin 0} = \frac{0}{0}$$

Using L'Hopital's rule

$$\lim_{\hat{\omega}\to 0} D_L(e^{j\hat{\omega}}) = \lim_{\hat{\omega}\to 0} \frac{\sin \hat{\omega}L/2}{L \sin \hat{\omega}/2} = \frac{\lim_{\hat{\omega}\to 0} \frac{d \sin \hat{\omega}L/2}{d\hat{\omega}}}{L \lim_{\hat{\omega}\to 0} \frac{d \sin \hat{\omega}/2}{d\hat{\omega}}}$$

$$= \frac{\lim_{\hat{\omega}\to 0} (L/2)(\cos \hat{\omega}L/2)}{L \lim_{\hat{\omega}\to 0} (1/2)(\cos \hat{\omega}/2)} = \frac{(L/2)}{(L/2)} = 1$$

- Properties of $D_L(e^{j\hat{\omega}})$:
  - Even Function and periodic in 2π
  - Maximum at 0
  - Has zeroes at integer multiples of $2\pi / L$
- Low Pass Filter

Smoothing an Image

- See Figures 6-11 through 6-15 for example of the application of the running average filter.

Reconstruction of a Continuous-time signal

- Recall:
  - The sampling theorem suggests that a process exists for reconstructing a continuous-time signal from its samples.
  - If we know the sampling rate and know its spectrum then we can reconstruct the continuous-time signal by scaling the principal alias of the discrete-time signal to the frequency of the continuous signal.
  - The principal alias will always be in the range between $0 \sim \pi$ if the sampling rate is greater than the Nyquist rate.

Continued

- If continuous-time signal has a frequency of ω, then the discrete-time signal will have a principal alias of

$$\hat{\omega} = \omega T_s = \frac{\omega}{f_s}$$

- So we can use this equation to determine the frequency of the continuous-time signal from the principal alias:

$$\omega = \hat{\omega} f_s = \frac{\hat{\omega}}{T_s}$$

- Note that the principal alias must be less than *p* if the Nyquist rate is used

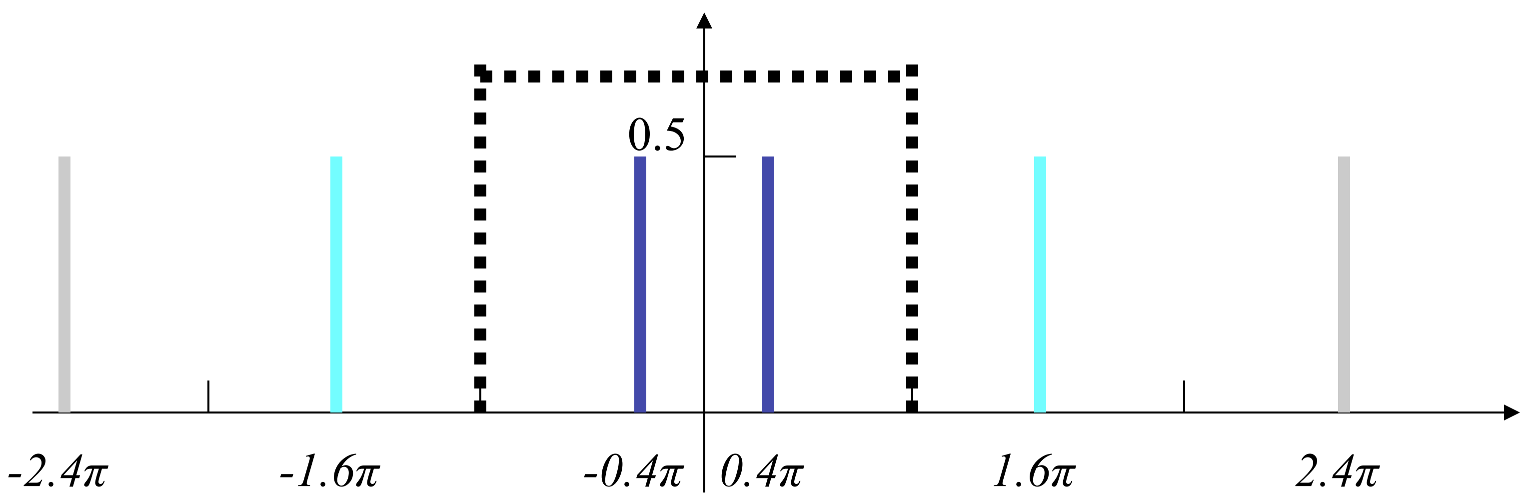
$$\hat{\omega} = \omega_{MAX} T_s = 2\pi f_{MAX} T_s = \frac{2\pi f_{MAX}}{f_s} = \frac{2\pi f_{MAX}}{f_s (\geq 2 f_{MAX})} \leq \pi$$

- And the reconstructed continuous-time frequency must be

$$\omega = 2\pi f = \hat{\omega} f_s \Rightarrow f = \frac{\hat{\omega} f_s}{2\pi} \leq \frac{\pi f_s}{2\pi} = \frac{f_s}{2}$$

Low Pass Filter

- Since we are within the Nyquist rate, the principal alias is $< \pi$
- Best reconstruction is Low Pass Filter or what the text calls: Ideal Bandlimited Interpolation



Reconstruction of a Continuous-time signal in terms of the Frequency Response

$x(t) = Xe^{j\omega t}$ $\Leftarrow$ Continuous-Time Signal

$x[n] = Xe^{j\omega n T_s} = Xe^{j\hat{\omega} n}$ $\Leftarrow$ Sampled Continuous-Time Signal

$\Leftarrow$ Ideal C-to-D conversion

$$\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s}$$

$y[n] = H(e^{j\hat{\omega}})\, Xe^{j\hat{\omega} n}$ $\Leftarrow$ Applying the a filter to recover the signal

$= H(e^{j\omega T_s})\, Xe^{j\omega T_s n}$

$y(t) = H(e^{j\omega T_s})\, Xe^{j\omega t}$ $\Leftarrow$ Ideal D-to-C conversion

$\Leftarrow$ only good for $-\pi/T_s < \omega < \pi/T_s$

$\Leftarrow$ since for $-\pi < \hat{\omega} < \pi$ to obtain the principal alias

Example

$$y[n]=\frac{1}{11}\sum_{k=0}^{10}x[n-k] \qquad \Leftarrow 11\text{-point filter}$$

$$H(e^{j\hat{\omega}})=\frac{\sin\hat{\omega}11/2}{11\sin\hat{\omega}/2}e^{-j\hat{\omega}5} \qquad \Leftarrow \text{Its Frequency Response}$$

$$x(t)=\cos(2\pi(25)t)+\cos(2\pi(250)t) \qquad \Leftarrow \text{Analog Signal } f_s=1000$$

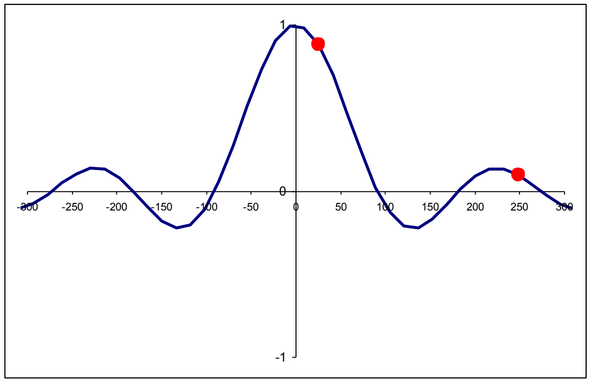
$$H(e^{j\hat{\omega}})=H(e^{j\omega T_s})=H(e^{j\omega/1000}) \qquad \Leftarrow \text{FR for } f_s=1000$$

$$=H(e^{j2\pi f/1000})$$

$$H(e^{j2\pi(25)/1000})=\frac{\sin(2\pi(25)/1000\times 11/2)}{11\sin(2\pi(25)/1000/2)}e^{-j2\pi(25)/1000\times 5}$$

$$=\frac{\sin(\pi(25)(11)/1000)}{11\sin(\pi(25)/1000)}e^{-j\pi(25)/1000}$$

$$=0.8811e^{-j\pi/4} \qquad \Leftarrow \text{FR at } f=25$$



$$H(e^{j2\pi(250)/1000})=\frac{\sin(2\pi(250)/1000\times 11/2)}{11\sin(2\pi(250)/1000/2)}e^{-j2\pi(250)/1000\times 5}$$

$$=\frac{\sin(\pi(250)(11)/1000)}{11\sin(\pi(250)/1000)}e^{-j\pi(250)/100}$$

$$=0.0909e^{-j(2\pi+\pi/2)}=0.0909e^{-j\pi/2} \qquad \Leftarrow \text{FR at } f=250$$

$$y(t)=.8811\cos(2\pi(25)t-\pi/4)$$

$$+0.0909\cos(2\pi(250)t-\pi/2) \qquad \Leftarrow \text{Reconstructed Signal}$$

Homework

- Exercises:
  - 6.2-6.6
- Problems:
  - 6.14 Use Matlab to plot the Frequency Response; show your code
  - 6.15
  - 6.17, 6.19, Use Matlab to plot the Frequency Response; show your code
  - 6.20, 6.21