Frequency Response

Lecture #12

Chapter 10

Ideal Filters

- We want to study $H(j\omega)$ functions which provide frequency selectivity such as:
  - Low Pass
  - High Pass
  - Band Pass
- However, we will look a ideal filtering, that is, filter which have ideal performance but are very difficult to construct.

A simple Filter – Ideal Delay

- Ideal Delay Filter => $y(t) = x(t - t_d)$: the output is same as the input except shifted in time by an amount t_d seconds.
- The impulse response is just $h(t) = \delta(t - t_d)$
- The frequency response is then

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} \delta(t - t_d)e^{-j\omega t}dt$$

$$= e^{-j\omega t_d}$$

Alternatively, if $x(t) = e^{j(\omega t + \phi)}$, then

$$y(t) = x(t - t_d) = e^{j[\omega(t - t_d) + \phi]} = e^{-j\omega t_d} e^{j(\omega t + \phi)}$$

But $y(t) = H(j\omega)e^{j(\omega t + \phi)}$

And therefore,

$$H(j\omega) = e^{-j\omega t_d}$$

- The Frequency Response of an Ideal Delay filter has a constant magnitude with a phase that is linear with frequency
- Therefore, it does not affect the magnitude of the input. It only effects the phase by an amount of $-\omega t_d$

Example

A signal of the form $x(t) = 10e^{j\pi/4}e^{j200\pi t}$ is input to an ideal delay filter with delay of 0.001 sec.

The frequency response is: $H(j\omega) = e^{-j\omega 0.001}$

$H(j200\pi) = e^{-j200\pi(0.001)}$

Then the output signal becomes:

$$y(t) = H(j\omega)x(t) = H(j200\pi)10e^{j\pi/4}e^{j200\pi t}$$

$$= e^{-j200\pi(0.001)}10e^{j\pi/4}e^{j200\pi t} = e^{-j0.2\pi}10e^{j\pi/4}e^{j200\pi t}$$

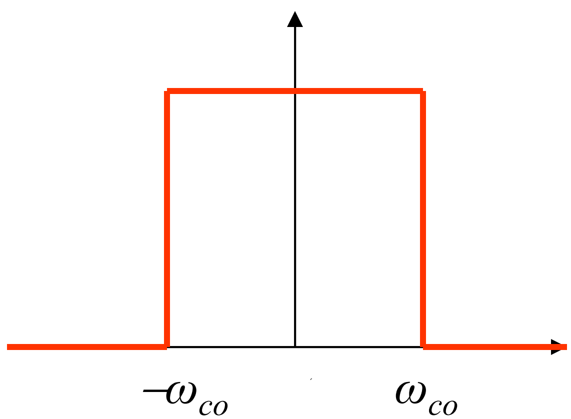
$$= 10e^{j(200\pi t + \frac{\pi}{4} - 0.2\pi)} = 10e^{j(200\pi t + 0.05\pi)}$$

Or rewritten as: $y(t) = 10e^{j[200\pi(t-0.001)+\frac{\pi}{4}]}$

Ideal Low Pass Filter

- This filter only passes frequencies below a value ω_{co} and attenuates all frequencies above ω_{co}.
- We call ω_{co} the cutoff frequency.
- Therefore, the frequency response of a low pass filter is:

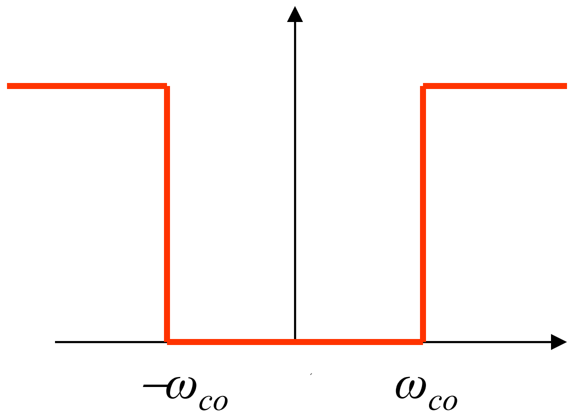
$$H_{lp}(j\omega) = \begin{cases} 1 & |\omega| \leq \omega_{co} \\ 0 & |\omega| > \omega_{co} \end{cases}$$



Ideal High Pass Filter

- This filter only passes frequencies above a value ω_{co} and attenuates all frequencies below ω_{co}.
- We call ω_{co} the cutoff frequency.
- Therefore, the frequency response of a high pass filter is:

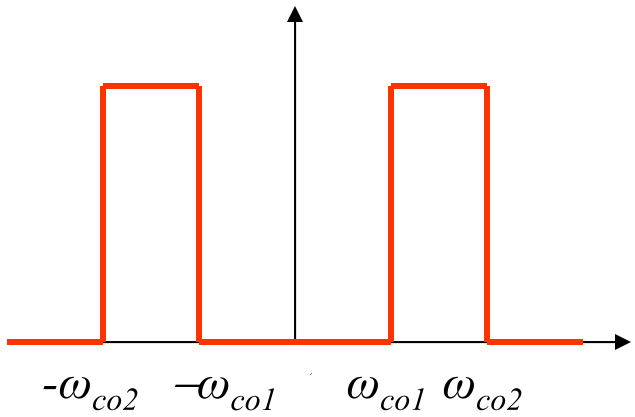
$$H_{hp}(j\omega) = \begin{cases} 1 & |\omega| \geq \omega_{co} \\ 0 & |\omega| < \omega_{co} \end{cases}$$



Ideal Band Pass Filter

- This filter only passes frequencies above a value ω_{co1} and below a value ω_{co2} and attenuates all other frequencies outside this range.
- We call ω_{co1} the lower (or low) cutoff frequency and ω_{co2} the upper (or high) cutoff frequency.
- Therefore, the frequency response of a bandpass filter is:

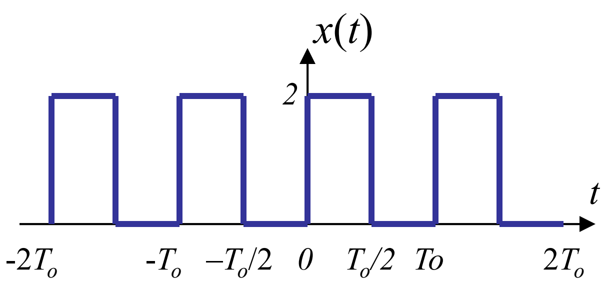
$$H_{bp}(j\omega) = \begin{cases} 0 & |\omega| < \omega_{co1} \\ 1 & \omega_{co1} \le |\omega| \le \omega_{co2} \\ 0 & |\omega| > \omega_{co2} \end{cases}$$



Application of Ideal Filters

- We will apply a band pass filter to a periodic square wave filter out its fundamental frequency.
- Let our input signal have a period of $T_o = 500\mu$s or f_o=2kHz => ω_o=2π(2000) rad/sec and its form over one period is:

$$x(t) = \begin{cases} 2 & 0 \le t < T_o / 2 \\ 0 & T_o / 2 \le t < T_o \end{cases}$$



Application of Ideal Filters

- Since $x(t)$ is a period, let's calculate the Fourier series for to decompose the input into its frequency components.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

$$a_k = \frac{1}{T_o}\int_{-T_o/2}^{T_o/2} x(t)e^{-jk\omega_o t}dt = \frac{1}{T_o}\int_{0}^{T_o/2} 2e^{-jk\omega_o t}dt$$

$$= \frac{2}{T_o(-jk\omega_o)}e^{-jk\omega_o t}\Big|_0^{T_o/2} = \frac{1}{-jk\pi}[e^{-jk\frac{2\pi}{T_o}\frac{T_o}{2}} - e^{-jk\frac{2\pi}{T_o}0}]$$

$$= \frac{1}{jk\pi}[1-e^{-jk\pi}]$$

Recall that $e^{-jk\pi} = \cos k\pi - j\sin k\pi = 1$ for even values of k

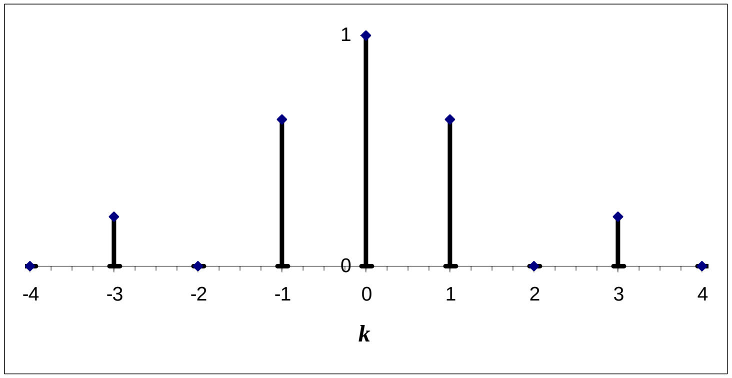
$= -1$ for odd values of k

or $e^{-jk\pi} = (-1)^k$

$$a_k = \frac{1}{jk\pi}[1-e^{-jk\pi}] = \frac{1}{jk\pi}[1-(-1)^k] = \frac{2}{jk\pi} = \frac{2}{k\pi}e^{-j\pi/2} \quad \text{for odd values of } k$$

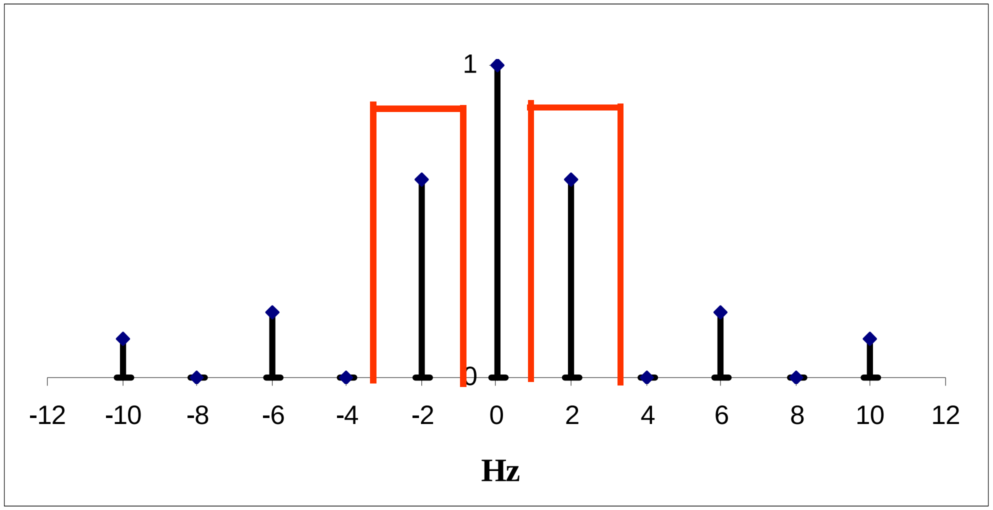
$$= 0 \quad \text{for even values of } k$$

$$a_0 = \frac{1}{T_o}\int_{-T_o/2}^{T_o/2} x(t)\,dt = \frac{1}{T_o}\int_{0}^{T_o/2} 2\,dt = 1$$



Application of Ideal Filters

- Now let's apply an ideal band pass filter with low frequency cutoff of 1,250 Hz and high frequency cutoff of 2,750 Hz which has a bandwidth of 1500 Hz and is centered around 2000 Hz which is the fundamental frequency of this square wave.



Application of Ideal Filters

- If the filter is LTI, then the output signal is also periodic with same fundamental frequency. Therefore, $y(t)$ can be written as a Fourier Series.

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_o t}$$

By superposition of these complex exponential signals at $k\omega_o$

$$b_k = H(jk\omega_o) a_k$$

But since $H(j\omega)$ is only defined for $1250 \le \omega \le 2750$, then only the terms for which $|k| = 1$ will be left upon this multiplication

$$b_k = H(jk\omega_o) a_k = \frac{2}{\pi k} e^{-j\pi/2} \quad \text{for } |k| = 1$$

$$\begin{aligned}
y(t) &= b_1 e^{j2\pi(2000)t} + b_{-1} e^{-j2\pi(2000)t} \\
&= \frac{2}{\pi(1)} e^{-j\pi/2} \; e^{j2\pi(2000)t} + \frac{2}{\pi(-1)} e^{-j\pi/2} e^{-j2\pi(2000)t}) \\
&= \frac{2}{\pi}(\; e^{j(2\pi(2000)t-\pi/2)} - e^{-j(2\pi(2000)t+\pi/2)}) \\
&= \frac{2}{\pi}(\; e^{j(2\pi(2000)t-\pi/2)} + e^{j\pi} e^{-j(2\pi(2000)t+\pi/2)}) \\
&= \frac{2}{\pi}(\; e^{j(2\pi(2000)t-\pi/2)} + e^{-j(2\pi(2000)t-\pi/2)}) \\
&= \frac{4}{\pi} \cos(2\pi(2000)t - \pi/2)
\end{aligned}$$

Time Domain or Frequency Domain

- We have seen that a LTI can be represented by its impulse response in the time domain and by its frequency response in the frequency domain.
- In general when working with sinusoids (or complex exponentials) either single or summed signals, it is easier to work in the Frequency Domain.
- If the signal consists of impulses, step functions, or other non-sinusoidal signals (e.g., signals which are progressive integrations of the impulse function), convolution of the impulse response (Time Domain) is usually easiest.

An Example

- An LTI system has an impulse response of

$$h(t) = \delta(t) - 200\pi e^{-200\pi t}u(t)$$

- The following signal is applied:

$$x(t)=10+20\delta(t - 0.1)+40\cos(200\pi\, t+0.3\pi) \text{ for all } t$$

- The input has 3 parts: a constant, an impulse and a cosine wave. We will take each part separately and use the easiest method to find the solution.

An Example

- Let's first find the frequency response of the system from the impulse response:

$$H(j\omega)=\int_{-\infty}^{\infty}[\delta(t)-200\pi e^{-200\pi t}u(t)]e^{-j\omega t}dt$$

$$=\int_{-\infty}^{\infty}\delta(t)e^{-j\omega t}dt-200\pi\int_{-\infty}^{\infty}e^{-200\pi t}u(t)e^{-j\omega t}dt$$

$$=1-200\pi\int_{0}^{\infty}e^{-(200\pi+j\omega)t}dt=1+\frac{200\pi}{200\pi+j\omega}e^{-(200\pi+j\omega)t}\Big|_0^{\infty}$$

$$=1-\frac{200\pi}{200\pi+j\omega}=\frac{j\omega}{200\pi+j\omega}$$

An Example

- Now let's take the first (constant, $\omega = 0$) part and the third (cosine) part and evaluate the solution using the frequency response:

$$\text{The first part of the input}: 10$$

$$H(j\omega) = \frac{j\omega}{200\pi + j\omega}$$

$$10 \mapsto H(j0)10 = \frac{j0}{200\pi + j0}10 = 0$$

$$\text{The third part of the input}: 40\cos(200\pi t + 0.3\pi)$$

$$H(j\omega) = \frac{j\omega}{200\pi + j\omega}$$

$$40\cos(200\pi t + 0.3\pi) \mapsto 40\left|H(j200\pi)\right|\cos[200\pi t + 0.3\pi + \angle H(j200\pi)]$$

$$H(j200\pi) = \frac{j200\pi}{200\pi + j200\pi} = \frac{j}{1+j} = \frac{1\angle\frac{\pi}{2}}{\sqrt{2}\angle\frac{\pi}{4}} = \frac{1}{\sqrt{2}}\angle\frac{\pi}{4}$$

$$40\cos(200\pi t + 0.3\pi) \mapsto 40\frac{1}{\sqrt{2}}\cos[200\pi t + 0.3\pi + .25\pi] = \frac{40}{\sqrt{2}}\cos[200\pi t + 0.55\pi]$$

An Example

- Now for the second part of the input (the impulse function), we will apply the impulse response:

$$\text{The second part of the input}: 20\delta(t-0.1)$$

$$20\delta(t-0.1) \mapsto 20h(t-0.1) = 20[\delta(t-0.1) - 200\pi e^{-200\pi(t-0.1)}u(t-0.1)]$$

$$20\delta(t-0.1) \mapsto 20\delta(t-0.1) - 4000\pi e^{-200\pi(t-0.1)}u(t-0.1)$$

- The Complete solution by superposition is:

$$\begin{aligned} y(t) &= 0 \\ &+ 20\delta(t-0.1) - 4000\pi e^{-200\pi(t-0.1)}u(t-0.1) \\ &+ \frac{40}{\sqrt{2}}\cos(200\pi t + 0.55\pi) \end{aligned}$$

Homework

- Exercises:
  - 10.4-10.7
- Problems:
  - 10.5, 10.6,
  - 10.7 Use Matlab to plot $x(t)$; show your code