Computing

Lecture #13

Chapter 13

What Is this Course All About ?

- To Gain an Appreciation of the Various Types of Signals and Systems
- To Analyze The Various Types of Systems
- To Learn the Skills and Tools needed to Perform These Analyses.
- To Understand How Computers Process Signals and Systems

What did we learn so far

- Learned about Signals and Systems
  - Continuous-time vs Discrete-time
  - Sinusoids
  - Complex Exponentials
  - Periodic Signals
- How to analyze them
  - Sampling
  - Time Domain
  - Frequency Domain
- How to Process Them
  - Filters

What do we still have left to learn

- How does a Computer handle signals?
  - Sinusoids
  - Bio Med Signals
- The Computer can’t handle Continuous-time signals
- The Computer must first sample the signal

Some more Background

- We saw that the Fourier Series can be used to handle any periodic signal since it can be decomposed into frequency components.
- But most signals are not periodic
  - ECG, EEG, EMG, etc.
  - Voice Signals
  - Video Signals

Fourier Transform

- We can handle non-periodic signals in a similar fashion as periodic signals
- That is, we can decompose them into frequency components
- However, the way we get there is different than the way we use for periodic signals
- Periodic signal frequency decomposition use the Fourier Series which generates a frequency spectrum
- Non-periodic signal frequency decomposition use the Fourier Transform which generates a frequency DENSITY spectrum.

Fourier Analysis and Fourier Transform

- Recall this is Fourier Series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_o t} = A_o + \sum_{k=1}^{\infty} A_k \cos(2\pi f_o kt + \phi_k); \text{ where } A_o = a_o; a_k = \frac{1}{2} A_k e^{j\phi_k}; a_{-k} = a_k^*; f_o = \frac{1}{T_o}$$

$$a_k = \frac{1}{T_o} \int_0^{T_o} x(t) e^{-j(\frac{2\pi}{T_o})kt} dt$$

- Here's what the Fourier Transform looks like for continuous signals, CTFT:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Fourier Transforms vs. Fourier Series

- Note that $X(j\omega)$ is a Spectral Density function; that is, if $x(t)$ is voltage, then $X(j\omega)$ is volts/rad.
  - Note in Fourier series analysis, a_k would also be volts is $x(t)$ is voltage.
- Note that $X(j\omega)$ is a continuous function of w and the limits of integration are over all values of t.
  - Note in Fourier series analysis, a_k is a discrete function of kf_o (and are f_o Hz apart) and the limits of summation are over one period of $x(t)$
- A proof of how $X(j\omega)$ is formulated is beyond our scope but, briefly, $X(j\omega)$ can be obtained by starting with the Fourier Series of $x(t)$ (as if it were periodic) and letting f_o go to zero (i.e., T_o goes to infinity which make the second repetition of $x(t)$ move to infinity and makes $x(t)$ non-periodic.
  - This would make the spectral components move closer to each other (infinitely closer – make $2\pi kf_o$ a continuous variable ω)
  - This will also make a_k approach zero but the ratio of a_k/f_o, which is a spectral density, remains finite.

Discrete-time Fourier Transform

- If this is the continuous-time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

- Then replacing t with nT_s and the integral with a summation, then the Discrete-Time Fourier Transform, DTFT, can be shown to be :

$$X(j\omega) = \sum_{n=-\infty}^{\infty} x(nT_s)e^{-j\omega nT_s}$$

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n}$$

Recalling that $\hat{\omega} = \omega T_s$

Discrete-time Fourier Transform

- Note that this looks very similar to the Frequency Response of a system

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n}$$

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$

- As a matter of fact, the Fourier Transform of the Impulse response is the Frequency Response

$$X(e^{j\hat{\omega}}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\hat{\omega}k} = \sum_{k=0}^{M} h[k]e^{-j\hat{\omega}k} = H(e^{j\hat{\omega}})$$

Discrete Fourier Transform

• The DTFT yield a spectrum which is a continuous function of $\hat{\omega}$

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n}$$

• How do we get around this? Sample the spectrum.

When we sampled in the time domain, we replaced t by nT_s where T_s is the distance (in time) between samples. Therefore to sample in the frequency domain we replace $\omega = 2\pi f$ by $2\pi k f_\Delta$ where f_Δ is the distance (in frequency) between spectrum samples.

Note that since $\hat{\omega} = \omega T_s = \frac{\omega}{f_s}; \hat{\omega} = \frac{\omega}{f_s} \Rightarrow \frac{2\pi k f_\Delta}{f_s}$

$$X(e^{j\hat{\omega} = \frac{2\pi k f_\Delta}{f_s}}) = X[k] = \sum_{n=-\infty}^{\infty} x[n]e^{-j\frac{2\pi k f_\Delta}{f_s}n}$$

•Let us assume that there are only L samples for time domain and N samples for the spectrum.

$$X[k] = \sum_{n=0}^{L-1} x[n]e^{-j\frac{2\pi k f_\Delta}{f_s}n}$$

• Since f_s is the maxium frequency in the spectrum, then $f_\Delta = \frac{f_s}{N}$. This is just the resolution of the displaced spectrum.

$$X[k] = \sum_{n=0}^{L-1} x[n]e^{-j\frac{2\pi k f_\Delta}{f_s}n} = \sum_{n=0}^{L-1} x[n]e^{-j\frac{2\pi k f_s}{f_s N}n} = \sum_{n=0}^{L-1} x[n]e^{-j\frac{2\pi}{N}kn}$$

• This is called the Discrete Fourier Transform

Discrete Fourier Transform

- Since the computer can only process discrete functions of finite time, we have to define a new Fourier Transform called the Discrete Fourier Transform, DFT.
  - Do not confuse this with the Discrete-time Fourier Transform, DTFT.
- It is defined as

$$X(k) = \sum_{n=0}^{L-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

where there are the L samples of $x[n]$,

we evaluate the Spectrum over N frequencies, i.e., $0 \leq k \leq N-1$,

and each frequency is f_Δ apart and chose $f_\Delta = \frac{f_s}{N}$

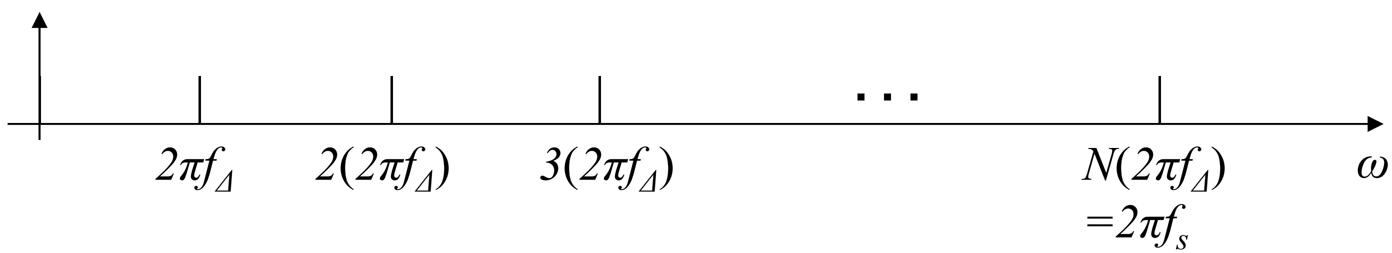
since f_s is the maximum frequency of the spectrum.

Therefore, $f_\Delta = \frac{f_s}{N} = \frac{1}{NT_s}$. We call this the resolution of the spectrum.

Discrete Fourier Transform

Let's start with the DTFT: $X(e^{j\hat{\omega}}) = \sum_{-\infty}^{\infty} x[n]e^{-j\hat{\omega}n}; \hat{\omega} = \omega T_s$

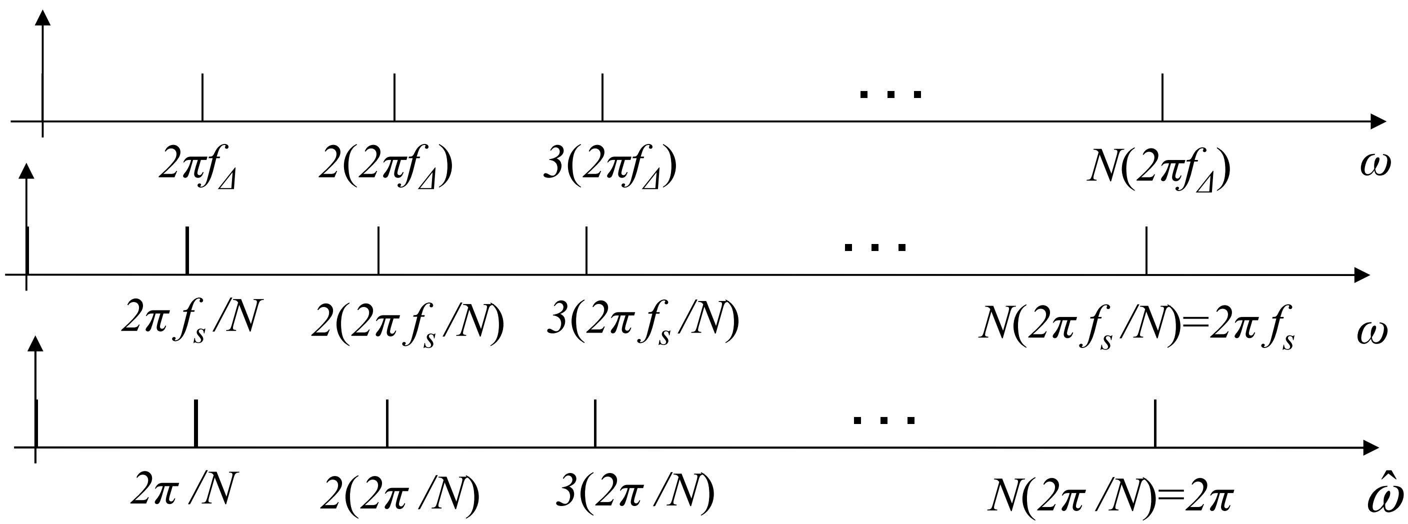
Let's divide the spectrum is into N frequencies equally spaced f_Δ Hz apart (i.e., we are sampling the spectrum).



Therefore, let's define the kth sample in the frequency domain as $\omega_k = 2\pi f_k = 2\pi k f_\Delta$ where k goes from 1 to N.

When $k = N$, the highest frequency in the spectrum is $\omega_N = \dfrac{2\pi N}{T_o} = 2\pi N f_\Delta = 2\pi f_s$.

Discrete Fourier Transform



If f_s meets the Nyquist rate, then the one-sided spectrum of $x[n] = X(e^{j\hat{\omega}})$ must end at or below $\frac{f_s}{2}$.

Therefore, $\hat{\omega}_k = \omega_k T_s = 2\pi k f_\Delta T_s = \frac{2\pi k f_s}{N} T_s = \frac{2\pi k}{N}$.

Let's substitute $\hat{\omega}_k$ for $\hat{\omega}$ in the DTFT: $X(e^{j\hat{\omega}_k}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\hat{\omega}_k n} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\frac{2\pi k}{N} n}$

This sum will only be a function of k. In addition, let's assume that there are L samples of $x[n]$.

Then, we have the Discrete Fourier Transform, DFT as $X[k] = X(e^{j\hat{\omega}_k}) = \sum_{n=0}^{L-1} x[n] e^{-j\frac{2\pi k}{N} n}$

Computer Processing

- Computers use the DFT to determine the spectrum of a signal $x(t)$.
- There are different computer algorithms for processing the DFT
  - The most widely used algorithm is called the Fast Fourier Transform: FFT
- Note that the DFT is just like a discrete Fourier Series in the Frequency Domain

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T_o}kt} \quad \Leftrightarrow \quad X[k] = \sum_{n=0}^{L-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

How to Evaluate the DFT Method 1: Expand n first, then k.

- What is the DFT for the $x[n]=\{1, 1, 0, 0\}$ assuming $N=4$

$$X[k]=\sum_{n=0}^{3} x[n]e^{-j\frac{2\pi}{4}kn}=\sum_{n=0}^{3} x[n]e^{-j\frac{\pi}{2}kn}$$

$$=x[0]e^{-j\frac{\pi}{2}k0}+x[1]e^{-j\frac{\pi}{2}k1}+x[2]e^{-j\frac{\pi}{2}k2}+x[3]e^{-j\frac{\pi}{2}k3}$$

$$=1+1e^{-j\frac{\pi}{2}k}+0e^{-j\frac{\pi}{2}k2}+0e^{-j\frac{\pi}{2}k3}$$

$$=1+1e^{-j\frac{\pi}{2}k}$$

$$X[0]=1+1e^{-j\frac{\pi}{2}0}=2$$

$$X[1]=1+1e^{-j\frac{\pi}{2}1}=1-j=\sqrt{2}e^{-j\frac{\pi}{4}}$$

$$X[2]=1+1e^{-j\frac{\pi}{2}2}=1+1e^{-j\pi}=1-1=0$$

$$X[3]=1+1e^{-j\frac{\pi}{2}3}=1+j=\sqrt{2}e^{j\frac{\pi}{4}}$$

$$X[k]=\{2,\sqrt{2}e^{-j\frac{\pi}{4}},0,\sqrt{2}e^{j\frac{\pi}{4}}\}$$

Method 2: Expand k first, then n.

- What is the DFT for the $x[n]=\{1, 1, 0, 0\}$ assuming N=4

$$X[k]=\sum_{n=0}^{3}x[n]e^{-j\frac{2\pi}{4}kn}$$

$$X[0]=\sum_{n=0}^{3}x[n]e^{-j\frac{2\pi}{4}0n}$$
$$=x[0]e^{-j0}+x[1]e^{-j0}+x[2]e^{-j0}+x[3]e^{-j0}$$
$$=1+1+0+0=2$$

$$X[1]=\sum_{n=0}^{3}x[n]e^{-j\frac{2\pi}{4}1n}$$
$$=x[0]e^{-j0}+x[1]e^{-j\frac{\pi}{2}1}+x[2]e^{-j\pi}+x[3]e^{-j\frac{3\pi}{2}}$$
$$=1+(-j)+0+0=\sqrt{2}e^{-j\frac{\pi}{4}}$$

$$X[2]=\sum_{n=0}^{3}x[n]e^{-j\frac{2\pi}{4}2n}$$
$$=x[0]e^{-j\pi 0}+x[1]e^{-j\pi}+x[2]e^{-j\pi 2}+x[3]e^{-j3\pi}$$
$$=1+(-1)+0+0=0$$

$$X[3]=\sum_{n=0}^{3}x[n]e^{-j\frac{2\pi}{4}3n}$$
$$=x[0]e^{-j\pi 0}+x[1]e^{-j\frac{3\pi}{2}}+x[2]e^{-j3\pi}+x[3]e^{-j\frac{9\pi}{2}}$$
$$=1+j+0+0=\sqrt{2}e^{j\frac{\pi}{4}}$$

$$X[k]=\{2,\sqrt{2}e^{-j\frac{\pi}{4}},0,\sqrt{2}e^{j\frac{\pi}{4}}\}$$

Another Example: Method 1

- What is the DFT for the $x[n]$={1, 1, 1, 0, 0, 0} assuming N=6

$$X[k]=\sum_{n=0}^{5}x[n]e^{-j\frac{2\pi}{6}kn}=\sum_{n=0}^{5}x[n]e^{-j\frac{\pi}{3}kn}$$

$$=x[0]e^{-j\frac{\pi}{3}k0}+x[1]e^{-j\frac{\pi}{3}k1}+x[2]e^{-j\frac{\pi}{3}k2}+x[3]e^{-j\frac{\pi}{3}k3}+x[4]e^{-j\frac{\pi}{3}k4}+x[5]e^{-j\frac{\pi}{3}k5}$$

$$=1+e^{-j\frac{\pi}{3}k}+e^{-j\frac{\pi}{3}k2}+0e^{-j\frac{\pi}{3}k3}+0e^{-j\frac{\pi}{3}k4}+0e^{-j\frac{\pi}{3}k5}$$

$$=1+e^{-j\frac{\pi}{3}k}+e^{-j\frac{\pi}{3}k2}$$

$$X[0]=1+e^{-j\frac{\pi}{3}0}+e^{-j\frac{\pi}{3}02}=3$$

$$X[1]=1+e^{-j\frac{\pi}{3}}+e^{-j\frac{\pi}{3}2}=1+.5-j0.86-.5-j0.86=1-j2(0.86)=2e^{-j\frac{\pi}{3}}$$

$$X[2]=1+e^{-j\frac{\pi}{3}2}+e^{-j\frac{\pi}{3}4}=1-.5-j0.86+-.5+j0.86=0$$

$$X[3]=1+e^{-j\frac{\pi}{3}3}+e^{-j\frac{\pi}{3}6}=1-1+1=1$$

$$X[4]=1+e^{-j\frac{\pi}{3}4}+e^{-j\frac{\pi}{3}8}=0$$

$$X[5]=1+e^{-j\frac{\pi}{3}5}+e^{-j\frac{\pi}{3}10}=1+.5+j0.86-.5+j0.86=2e^{j\frac{\pi}{3}}$$

Method 2

- What is the DFT for the $x[n]$={1, 1, 1, 0, 0, 0} assuming N=6

$$X[k]=\sum_{n=0}^{5}x[n]e^{-j\frac{2\pi}{6}kn}=\sum_{n=0}^{2}e^{-j\frac{\pi}{3}kn}=1+e^{-j\frac{\pi}{3}k1}+e^{-j\frac{\pi}{3}k2}=1+e^{-j\frac{\pi}{3}k}+e^{-j\frac{2\pi}{3}k}$$

$$X[0]=1+e^{-j\frac{\pi}{3}0}+e^{-j\frac{2\pi}{3}0}=3$$

$$X[1]=1+e^{-j\frac{\pi}{3}1}+e^{-j\frac{2\pi}{3}1}=1+e^{-j\frac{\pi}{3}}+e^{-j\frac{2\pi}{3}}=2e^{-j\frac{\pi}{3}}$$

$$X[2]=1+e^{-j\frac{\pi}{3}2}+e^{-j\frac{2\pi}{3}2}=1+e^{-j\frac{2\pi}{3}}+e^{-j\frac{4\pi}{3}}=0$$

$$X[3]=1+e^{-j\frac{\pi}{3}3}+e^{-j\frac{2\pi}{3}3}=1+e^{-j\pi}+e^{-j2\pi}=1$$

$$X[4]=1+e^{-j\frac{\pi}{3}4}+e^{-j\frac{2\pi}{3}4}=1+e^{-j\frac{4\pi}{3}}+e^{-j\frac{8\pi}{3}}=0$$

$$X[5]=1+e^{-j\frac{\pi}{3}5}+e^{-j\frac{2\pi}{3}5}=1+e^{-j\frac{5\pi}{3}}+e^{-j\frac{10\pi}{3}}=2e^{j\frac{\pi}{3}}$$

What is the right hand side of the FFT.vi spectrum?

- Note that N and L are usually taken to be the same in order to easily calculate $X[k]$.
- Note that due to this fact $X[N - k]=X[-k]=X^*[k]$

$$X[k]=\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$$

$$X[N-k]=\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}(N-k)n} = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}Nn}e^{j\frac{2\pi}{N}kn}$$

$$=\sum_{n=0}^{N-1} x[n]e^{-j2\pi n}e^{j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} x[n]e^{j\frac{2\pi}{N}kn}$$

$$= X[-k] = X^*[k]$$

- Therefore, $X[-k]$ will show up as $X[N-k]$

Take a look at our Example

- What is the DFT for the $x[n]=\{1, 1, 0, 0\}$ assuming $N=4$

$$X[k]=\sum_{n=0}^{3} x[n]e^{-j\frac{2\pi}{4}kn}$$

$$X[-1]=\sum_{n=0}^{3} x[n]e^{j\frac{2\pi}{4}1n}$$

$$= x[0]e^{-j0} + x[1]e^{j\frac{\pi}{2}1} + x[2]e^{j\pi} + x[3]e^{j\frac{3\pi}{2}}$$

$$= 1 + (j) + 0 + 0 = \sqrt{2}e^{j\frac{\pi}{4}}$$

$$= X[4-1] = X[3]$$

Another Example

Here we have a signal whose digitized frequency is $\hat{\omega}_o$

$$x_1[n] = e^{j(\hat{\omega}_o n+\phi)} \quad \text{for } n = 0,\ 1,\ 2,\ \ldots,\ N-1$$

We now want to obtain the spectrum of our signal using the DFT. If the DFT is done correctly, we would expect a the spectrum to show a single component at $\hat{\omega}_o$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} e^{j(\hat{\omega}_o n+\phi)} e^{-j(2\pi/N)kn}$$

$$= e^{j\phi} \sum_{n=0}^{N-1} e^{-j[(2\pi k/N)-\hat{\omega}_o]n}$$

Another Example Continued

Using the partial sum of a geometric series: $\sum_{k=0}^{L-1} \alpha^k = \frac{1-\alpha^L}{1-\alpha}$

$$X[k] = e^{j\phi}\left(\frac{1-e^{-j[(2\pi k/N)-\hat{\omega}_o]N}}{1-e^{-j[(2\pi k/N)-\hat{\omega}_o]}}\right) = e^{j\phi}\left(\frac{e^{-j[(2\pi k/N)-\hat{\omega}_o]N/2}(e^{+j[(2\pi k/N)-\hat{\omega}_o]N/2} - e^{-j[(2\pi k/N)-\hat{\omega}_o]N/2})}{e^{-j[(2\pi k/N)-\hat{\omega}_o]/2}(e^{j[(2\pi k/N)-\hat{\omega}_o]/2} - e^{-j[(2\pi k/N)-\hat{\omega}_o]/2})}\right)$$

$$= e^{j\phi}\left(\frac{e^{-j[(2\pi k/N)-\hat{\omega}_o](N-1)/2}\sin([(2\pi k/N)-\hat{\omega}_o]N/2)}{\sin([(2\pi k/N)-\hat{\omega}_o]/2)}\right)$$

$$= e^{j\phi}e^{-j[(2\pi k/N)-\hat{\omega}_o](N-1)/2}\frac{\sin([(2\pi k/N)-\hat{\omega}_o]N/2)}{\sin([(2\pi k/N)-\hat{\omega}_o]/2)} = e^{j\phi}e^{-j[(2\pi k/N)-\hat{\omega}_o](N-1)/2}D_N(e^{+j[(2\pi k/N)-\hat{\omega}_o]N})$$

where

$$D_N(e^{+j[(2\pi k/N)-\hat{\omega}_o]N}) = \frac{\sin([(2\pi k/N)-\hat{\omega}_o]N/2)}{\sin([(2\pi k/N)-\hat{\omega}_o]/2)},\ \text{the Dirichlet function}$$

Another Example Continued

$$X[k] = e^{j\phi} e^{-j[(2\pi k/N)-\hat{\omega}_o](N-1)/2} \frac{\sin([(2\pi k/N)-\hat{\omega}_o]N/2)}{\sin([(2\pi k/N)-\hat{\omega}_o]/2)}$$

$$\hat{\omega}_o = 2\pi k_o/N \Rightarrow \omega_0 = \hat{\omega}_o f_s = \frac{2\pi k_o}{N} f_s \Rightarrow f_0 = \frac{k_o}{N} f_s$$

Case 1: k_o is NOT an integer;

that is, $\hat{\omega}_o$ is NOT a multiple of $2\pi/N$ and therefore f_0 is NOT a multiple of f_s,

which is the resolution of the sampled spectrum obtained using DFT.

$\therefore k - k_o \neq 0$ for any value of k

$$= e^{j\phi} e^{-j[(2\pi k/N)-2\pi k_o/N](N-1)/2} \frac{\sin([(2\pi k/N)-2\pi k_o/N]N/2)}{\sin([(2\pi k/N)-2\pi k_o/N]/2)}$$

$$= e^{j\phi} e^{-j[2\pi/N(k-k_o)](N-1)/2} \frac{\sin([2\pi/N(k-k_o)]N/2)}{\sin([2\pi/N(k-k_o)]/2)}$$

$$= e^{j\phi} e^{-j[2\pi/N(k-k_o)](N-1)/2} \frac{\sin[\pi(k-k_o)]}{\sin[\pi/N(k-k_o)]}; \text{ for all } k$$

Two Cases for our Example

say let $k_o = 2.5$ and $N = 40$; $\hat{\omega}_o = \dfrac{2\pi(2.5)}{40} = 2.5\dfrac{\pi}{20} = \dfrac{\pi}{8} = 0.125\pi$

If we do this, the figure below shows that do not have a single component at $\hat{\omega}_o$

$k_o = 2.5; N = 40$

$$|X[k]| = \left|\frac{\sin([\pi(k-2.5)])}{\sin([\pi/40(k-2.5)])}\right|$$

$$|X[0]| = \left|\frac{\sin([\pi(0-2.5)])}{\sin([\pi/40(0-2.5)])}\right| = 5.1$$

$$|X[1]| = \left|\frac{\sin([\pi(1-2.5)])}{\sin([\pi/40(1-2.5)])}\right| = \left|\frac{\sin([\pi(-1.5)])}{\sin([\pi/40(-1.5)])}\right| = 8.5$$

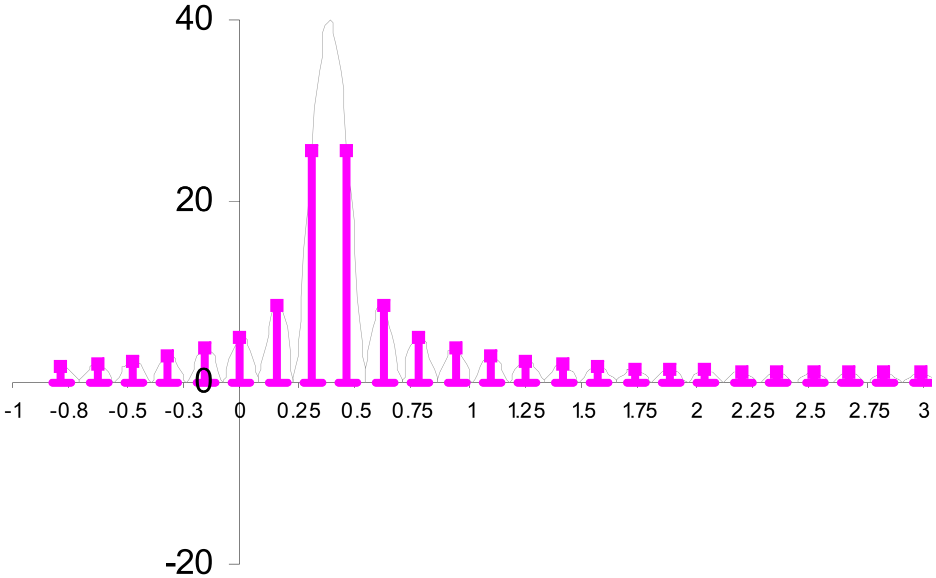
$$|X[2]| = \left|\frac{\sin([\pi(2-2.5)])}{\sin([\pi/40(2-2.5)])}\right| = \left|\frac{\sin([\pi(-0.5)])}{\sin([\pi/40(-0.5)])}\right| = 25.5$$

$$|X[3]| = \left|\frac{\sin([\pi(3-2.5)])}{\sin([\pi/40(3-2.5)])}\right| = \left\|\frac{\sin([\pi(0.5)])}{\sin([\pi/40(0.5)])}\right\| = 25.5$$

$$|X[4]| = \left|\frac{\sin([\pi(4-2.5)])}{\sin([\pi/40(4-2.5)])}\right| = \left|\frac{\sin([\pi(1.5)])}{\sin([\pi/40(1.5)])}\right| = 8.5$$

$\vdots$

etc.



Another Example Continued

$$X[k]=e^{j\phi}e^{-j[(2\pi k/N)-\hat{\omega}_o](N-1)/2}\frac{\sin([(2\pi k/N)-\hat{\omega}_o]N/2)}{\sin([(2\pi k/N)-\hat{\omega}_o]/2)}$$

Case 2: $k_o$is an integer; that is, $\hat{\omega}_o$ is an integer multiple of $2\pi/N$ and f_0 is a multiple of f_s.
the resolution of the sampled spectrum obtained from the DFT.
$\therefore$ $k-k_o$ will be a non-zero integer = l except when $k=k_o$

$$=e^{j\phi}e^{-j[(2\pi k/N)-2\pi k_o/N](N-1)/2}\frac{\sin([(2\pi k/N)-2\pi k_o/N]N/2)}{\sin([(2\pi k/N)-2\pi k_o/N]/2)}$$

$$=e^{j\phi}e^{-j[2\pi/N(k-k_o)](N-1)/2}\frac{\sin([2\pi/N(k-k_o)]N/2)}{\sin([2\pi/N(k-k_o)]/20}$$

$$=e^{j\phi}e^{-j[2\pi/N(k-k_o)](N-1)/2}\frac{\sin[\pi(k-k_o)]}{\sin[\pi/N(k-k_o)]}=e^{j\phi}e^{-j[2\pi l/N](N-1)/2}\frac{\sin[\pi l]}{\sin[\pi l/N]}$$

$=0 \quad k\neq k_o$ since the $\sin(\pi l)=0$

$=Ne^{j\phi} \quad k=k_o$ $(l=0)$ since $X[k_o]=e^{j\phi}1\frac{0}{0}$;

using L'Hopital's rule $\lim_{l\to 0}\frac{\sin[\pi l]}{\sin[\pi l/N]}=\lim_{l\to 0}\frac{\pi\cos[\pi l]}{\frac{\pi}{N}\cos[\pi l/N]}=N$

Two Cases for our Example

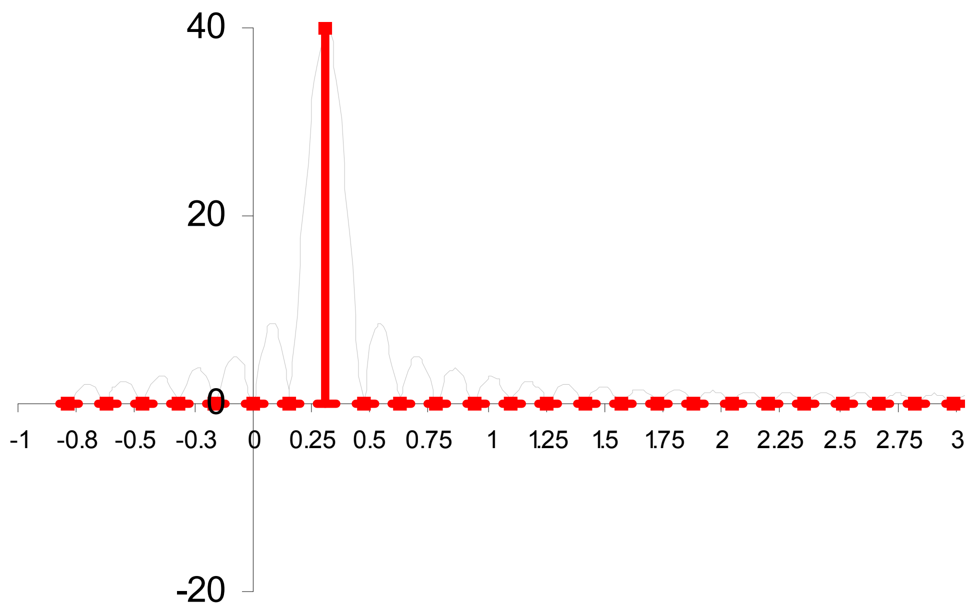
Case 2: when the fundamental frequency of our signal is a multiple of $\frac{f_s}{N}$,

which is the resolution of the sampled spectrum obtained from the DFT.

that is say $k_o = 2$ and $N = 40$; $\hat{\omega}_o = \frac{2\pi(2)}{40} = 2\frac{\pi}{20} = 0.1\pi$

If we do this, the figure below shows that we have a single component at $\hat{\omega}_o$ or at $k = k_o = 2$.

In order words: $X[k] = Ne^{j\phi}\delta(k-2)$ which corresponds to $2 \times 2\pi / 40 = 0.1\pi = \hat{\omega}_o$



Time Domain & Frequency Domain

- Time Domain

Input
Time signal
CT: $x(t)$
DT: $x[n]$

LTI System

CT: Differential Eq
DT: Difference Eq

CT: $a\ddot{y}(t) + b\dot{y}(t) + cy(t) = d\dot{x}(t) + ex(t)$

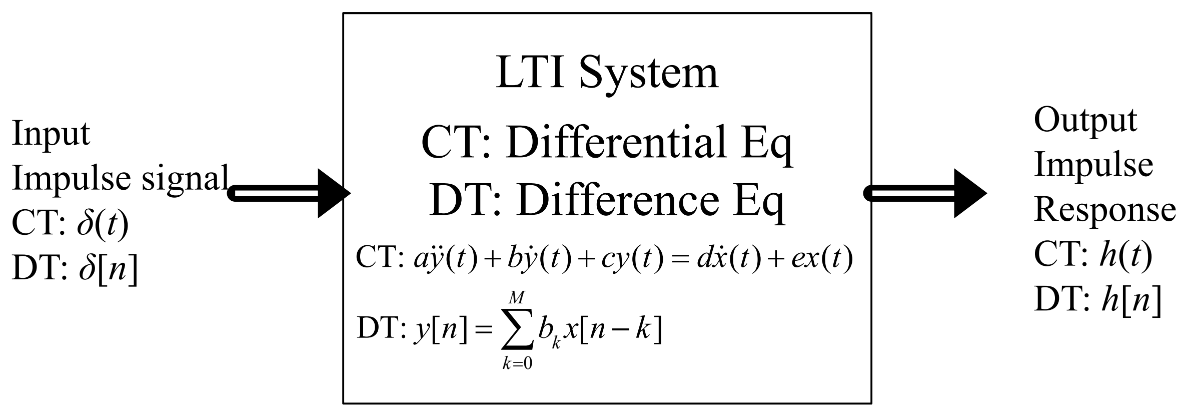
DT: $y[n] = \sum_{k=0}^{M} b_k x[n-k]$

Output
Time Signal
CT: $y(t)$
DT: $y[n]$

Time Domain & Frequency Domain

Time Domain

- Impulse Response



Time Domain & Frequency Domain

- Time Domain

Input
Time signal
CT: $x(t)$
DT: $x[n]$

→

LTI System

CT: Differential Eq
DT: Difference Eq

CT: $a\ddot{y}(t)+b\dot{y}(t)+cy(t)=d\dot{x}(t)+ex(t)$

DT: $y[n]=\sum_{k=0}^{M} b_k x[n-k]$

Convolution

CT: $y(t)=\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$

DT: $y[n]=\sum_{l=-\infty}^{\infty} x[l]h[n-l]$

→

Output
Time signal
CT: $y(t)$
DT: $y[n]$

Time Domain & Frequency Domain

- Frequency Domain

Input Signal Spectrum

CT: $x(t) = A_0 + \sum_{k=0}^{M} A_k \cos(2\pi f_k t + \theta_k)$

DT: $x[n] = A_0 + \sum_{k=0}^{M} A_k \cos(\omega_k T_s n + \theta_k)$

$= A_0 + \sum_{k=0}^{M} A_k \cos(\hat{\omega}_k n + \theta_k)$

LTI System Frequency Response

CT: $H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t}\, dt = M(\omega) e^{-j\psi(\omega)}$

DT: $H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega} k} = \sum_{k=0}^{M} h[k] e^{-j\hat{\omega} k}$

$= M(\hat{\omega}) e^{-j\psi(\hat{\omega})}$

Output Signal Spectrum

CT: $y(t) = A_0 M(0) e^{-j\psi(0)} + \sum_{k=0}^{M} A_k M(2\pi f_k) \cos(2\pi f_k t + \theta_k - \psi(2\pi f_k))$

DT: $y[n] = A_0 M(0) e^{-j\psi(0)} + \sum_{k=0}^{M} A_k M(\hat{\omega}_k) \cos(\hat{\omega}_k n + \theta_k - \psi(\hat{\omega}_k))$

Fourier Transform Signals and Systems

Signals

- The FT of a signal transforms it from the time domain, $x(t)$ or $x[n]$, to the frequency domain to yield its spectral representation, $X(j\omega)$ or $X[k]$.
- The inverse FT transforms the signal's spectrum, $X(j\omega)$ or $X[k]$, in the frequency domain to the time domain, $x(t)$ or $x[n]$.
- The FT can also be applied to systems.

Systems

- The FT of a system's impulse response, $h(t)$ or $h[n]$, transforms it into the frequency response, $H(j\omega)$ or $H[k]$.
- The inverse FT transforms a system's frequency response, $H(j\omega)$ or $H[k]$ to the impulse response, $h(t)$ or $h[n]$.
- Note that the impulse response is really the output signal of a system when the input signal is the impulse function, $\delta(t)$ or $\delta[n]$.

Fourier Transform
Signals and Systems
The Math is the Same

Signals

- For continuous signals, its spectral representation is:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

- For discrete signals, its spectral representation is discrete frequency function of *k* and is calculated as:

$$X[k] = \sum_{n=0}^{L-1} x[n]e^{-j\frac{2\pi k}{N}n}$$

Systems

- For systems which process continuous signals, the frequency response is calculated as:

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\,dt$$

- For systems which process discrete signals, the frequency response is a continuous frequency function of $\hat{\omega}$ and is calculated as:

$$H(j\hat{\omega}) = H(e^{j\hat{\omega}}) = \sum_{k=0}^{M-1} h[k]e^{-j\hat{\omega}k} = \sum_{k=0}^{M-1} b_k e^{-j\hat{\omega}k}$$

Fourier Transform
Signals and Systems
The Math is the Same

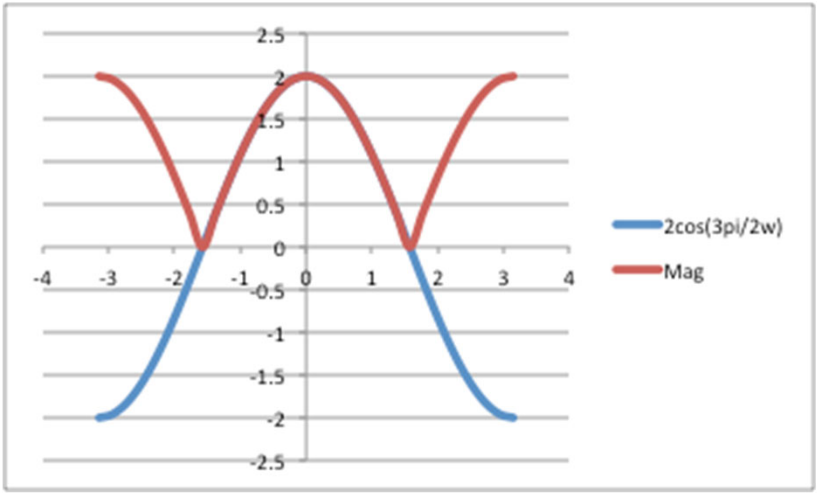
Frequency Response is a continuous function

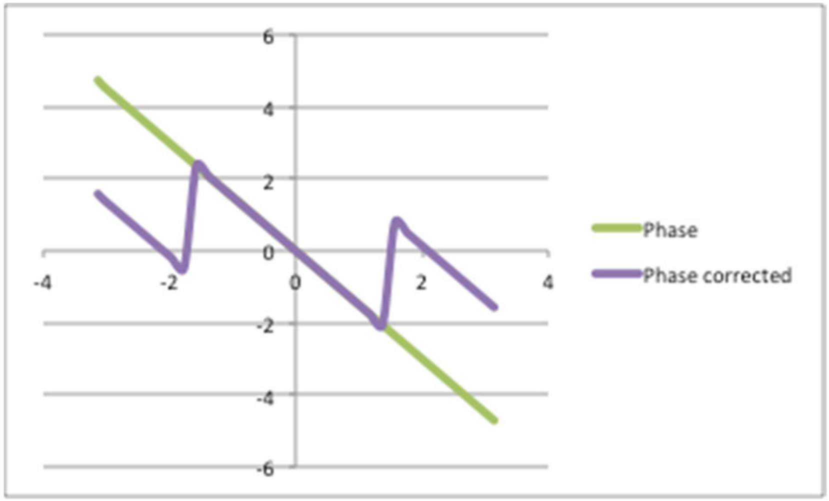
$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M-1} b_k e^{-j\hat{\omega}k}$$

$$b_k = \{1, 0, 0, 1\}$$

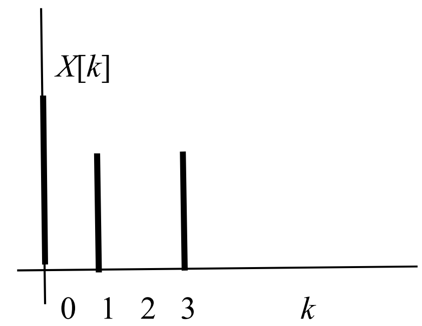
$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{3} b_k e^{-j\hat{\omega}k} = 1 + 0 + 0 + e^{-j\hat{\omega}3} = 1 + e^{-j\hat{\omega}3}$$

$$= e^{-j\frac{3\hat{\omega}}{2}}(e^{-j\frac{3\hat{\omega}}{2}} + e^{j\frac{3\hat{\omega}}{2}}) = 2\cos(\frac{3\hat{\omega}}{2})e^{-j\frac{3\hat{\omega}}{2}}$$





Fourier Transform
Signals and Systems
The Math is the Same



Signal Spectral Density is a discrete function

$$X[k] = \sum_{n=0}^{L-1} x[n] e^{-j\frac{2\pi k}{N}n}$$

$$x[n] = \{1, 0, 0, 1\}$$

$$X[k] = \sum_{n=0}^{3} x[n] e^{-j\frac{2\pi k}{4}n} = 1 + 0 + 0 + e^{-j\frac{\pi k}{2}3} = 1 + e^{-j\frac{3\pi k}{2}}$$

$$X[k] = 1 + e^{-j\frac{3\pi k}{2}}$$

$$X[0] = 1 + e^{-j\frac{3\pi}{2}0} = 2$$

$$X[1] = 1 + e^{-j\frac{3\pi}{2}1} = 1 + j = \sqrt{2}e^{j\frac{\pi}{4}}$$

$$X[2] = 1 + e^{-j\frac{3\pi}{2}2} = 1 - 1 = 0$$

$$X[3] = 1 + e^{-j\frac{3\pi}{2}3} = 1 + e^{-j\frac{\pi}{2}} = 1 - j = \sqrt{2}e^{-j\frac{\pi}{4}}$$

OR

$$X[k] = \sum_{n=0}^{3} x[n] e^{-j\frac{2\pi k}{4}n} = 1 + 0 + 0 + e^{-j\frac{\pi k}{2}3} = 1 + e^{-j\frac{3\pi k}{2}}$$

$$= e^{-j\frac{3\pi k}{4}} (e^{j\frac{3\pi k}{4}} + e^{-j\frac{3\pi k}{4}}) = 2\cos(\frac{3\pi k}{4}) e^{-j\frac{3\pi k}{4}}$$

$$X[k] = 2\cos(\frac{3\pi k}{4}) e^{-j\frac{3\pi k}{4}}$$

$$X[0] = 2\cos(\frac{3\pi 0}{4}) e^{-j\frac{3\pi 0}{4}} = 2$$

$$X[1] = 2\cos(\frac{3\pi}{4}) e^{-j\frac{3\pi}{4}} = -\sqrt{2}e^{-j\frac{3\pi}{4}} = \sqrt{2}e^{-j\frac{3\pi}{4}} e^{j\pi} = \sqrt{2}e^{j\frac{\pi}{4}}$$

$$X[2] = 2\cos(\frac{3\pi}{2}) e^{-j\frac{3\pi}{2}} = 0$$

$$X[3] = 2\cos(\frac{9\pi}{4}) e^{-j\frac{9\pi}{4}} = 2\cos(\frac{\pi}{4}) e^{-j\frac{\pi}{4}} = \sqrt{2}e^{-j\frac{\pi}{4}}$$

Time Domain to Frequency Domain Transformations

Time Domain		Transformation Type	Frequency Domain	
Time Signal Type	Computer processing & storage		Frequency Spectrum Type	Computer processing & storage
Periodic & Continuous $x(t)$	No	Fourier Series $a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j\frac{2\pi k}{T}t}dt$	Discrete Spectrum a_k	Yes
Non-Periodic & Continuous $x(t)$	No	Continuous Time Fourier Transform CTFT $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$	Continuous Spectral Density $X(j\omega)$	No
Discrete $x[n]$	Yes	Discrete Time Fourier Transform DTFT $X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n}$	Continuous Spectral Density $X(e^{j\hat{\omega}})$	No
Discrete $x[n]$	Yes	Discrete Fourier Transform DFT $X[k] = \sum_{n=-\infty}^{\infty} x[n]e^{-j\frac{2\pi}{N}nk}$	Discrete Spectrum $X[k]$	Yes

Homework

- Exercises:
  - 13.2- 13.4, 13.6 – 13.8
- Problems:
  - 13.1 Use Matlab to plot the spectrum; submit your code
  - 13.4, 13.5, 13.6