# Sinusoids

Lecture #3

Chapter 2

# Complex Exponential Signals

- Note that by using Euler's formula, we can rewrite the complex exponential signal in rectangular form as:

$$z(t) = Ae^{j(\omega_o t+\theta)}$$

$$= A\cos(\omega_o t + \theta) + jA\sin(\omega_o t + \theta)$$

- Therefore real part is the cosine signal and imaginary part is a sine signal both of radial frequency $\omega_o$ and phase angle of $\theta$

# Complex Exponential Function as a function of time

- Let's look at this $z(t) = 1e^{j2\pi(1)t} = e^{j2\pi t} = \cos 2\pi t + j \sin 2\pi t$

*t=8/8 seconds*

*arg(z(t))=2π x8/8 = 2π ; z(t)= 1+ j0*

*t=2/8 seconds*

*arg(z(t))=2π x2/8= π/2; z(t)= 0 + j1*

*t=3/8 seconds*

*arg(z(t))=2πx3/8 = 3π/4;*

*z(t)= -0.707+ j0.707*

*t=1/8 seconds*

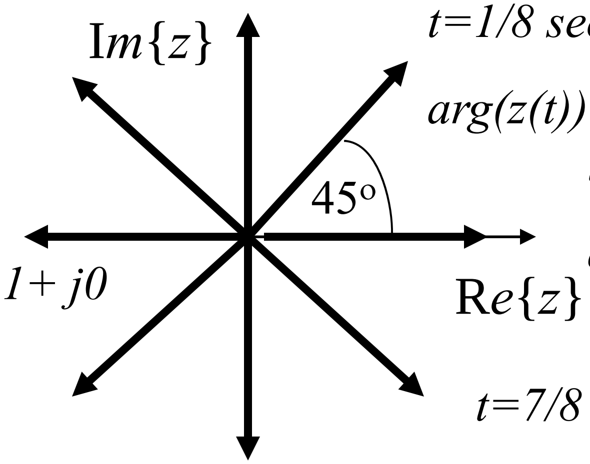
*arg(z(t))=2πx1/8=π/4; z(t)=0.707+j 0.707*

*t=0 seconds*

*arg(z(t))=2π x0=0; z(t)=1+ j0*

*t=4/8 seconds*

*arg(z(t))=2π x4/8 = π; z(t)= -1+ j0*




*t=5/8 seconds*

*arg(z(t))=2πx5/8 = 5π/4;*

*z(t)= -0.707 - j0.707*

*t=7/8 seconds*

*arg(z(t))=2π x7/8= 7π/4;*

*z(t) = 0 .707- j0.707*

*t=6/8 seconds*

*arg(z(t))=2π x6/8 = 3π/2; z(t) = 0 - j*

# Phasor Representation of a Complex Exponential Signal

- Using the multiplication rule, we can rewrite the complex exponential signal as

$$z(t) = Ae^{j(\omega_o t+\theta)} = Ae^{j\omega_o t}e^{j\theta} = Ae^{j\theta}e^{j\omega_o t} = \mathbf{X}e^{j\omega_o t}$$

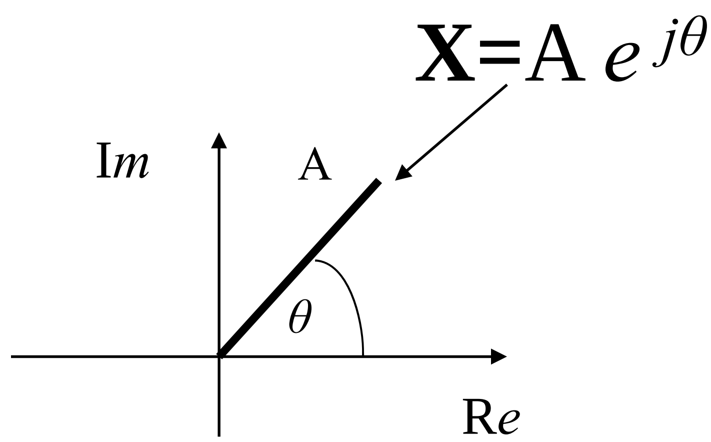
where **X** is a complex number equal to

$$\mathbf{X} = Ae^{j\theta}$$

- **X** is complex amplitude of the complex exponential signal and is also called a phasor

# Graphing a phasor

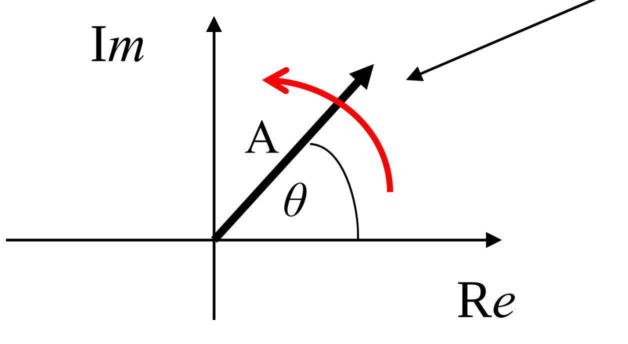
- $\mathbf{X}=A\,e^{j\theta}$ can be graphed in the complex plane with magnitude A and angle $\theta$:



# Graphing a Complex Signal in terms of its phasors

- Since a complex signal, $z(t)$, is a phasor multiplying a complex exponential signal $e^{j\omega_o t}$ , then a complex signal can be viewed as a phasor rotating in time:

$$z(t) = Ae^{j(\omega_o t + \theta)} = \mathbf{X}e^{j\omega_o t}$$



# Complex Exponential Signals

- Note that by using Euler's formula, we can rewrite the complex exponential signal in rectangular form as:

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- Therefore real part is the cosine signal and imaginary part is a sine signal both of radial frequency $\omega_o$ and phase angle of $\theta$

# Rotating Phasor

- Let’s look at this $z(t) = Ae^{j(2\pi t + \frac{\pi}{4})} = Ae^{j\frac{\pi}{4}} e^{j2\pi t} = \mathbf{X} e^{j2\pi t}$

$$\mathbf{X} = Ae^{j\frac{\pi}{4}}$$

*t=0 seconds*

*arg(z(t))=2π x0+π/4 = π/4*

*t=1/8 seconds*

*arg(z(t))=2πx1/8+π/4 = π/2*

*t=1/4 seconds*

*arg(z(t))=2π x1/4+π/4 = 3π/4*

*t=3/8 seconds*

*arg(z(t))=2π x3/8+π/4 = 4π/4 = π*

*t=1/2 seconds*

*arg(z(t))=2π x1/2+π/4 = 5π/4*

*t=5/8 seconds*

*arg(z(t))=2π x5/8+π/4 = 6π/4 = 3π/2*

*t=3/4 seconds*

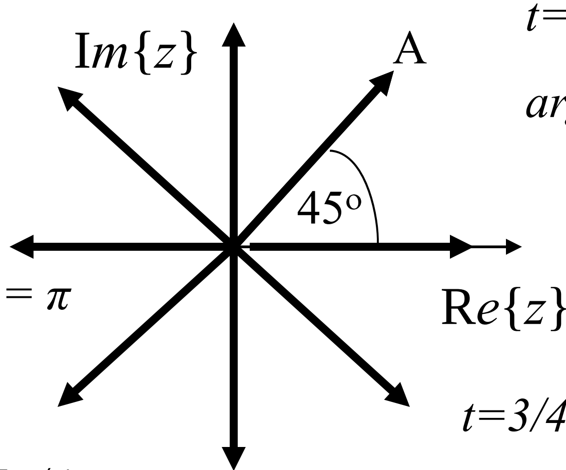
*arg(z(t))=2π x3/4+π/4 = 7π/4*

*t=7/8 seconds*

*arg(z(t))=2π x7/8+π/4 = 8π/4 = 2π*

*t=1 seconds*

*arg(z(t))=2π x1+π/4 = 9π/4 = π/4*



# Rotating Phasor

- Let's look at this $z(t) = Ae^{j(2\pi t + \frac{\pi}{4})} = Ae^{j\frac{\pi}{4}}e^{j2\pi t} = \mathbf{X}e^{j2\pi t}$

$$\mathbf{X} = Ae^{j\frac{\pi}{4}}$$

*t=1 seconds*

*arg(z(t))=2π x1+π/4 = 9π/4 = π/4*

*t=0 seconds*

*arg(z(t))=2π x0+π/4 = π/4*

*t=1/8 seconds*

*arg(z(t))=2π x1/8+π/4 = π/2*

*t=1/4 seconds*

*arg(z(t))=2π x1/4+π/4 = 3π/4*

*t=3/8 seconds*

*arg(z(t))=2π x3/8+π/4 = 4π/4 = π*

*t=1/2 seconds*

*arg(z(t))=2π x1/2+π/4 = 5π/4*

*t=5/8 seconds*

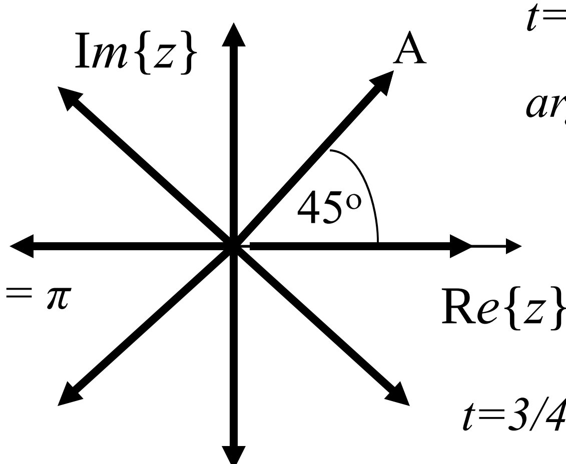
*arg(z(t))=2π x5/8+π/4 = 6π/4 = 3π/2*

*t=3/4 seconds*

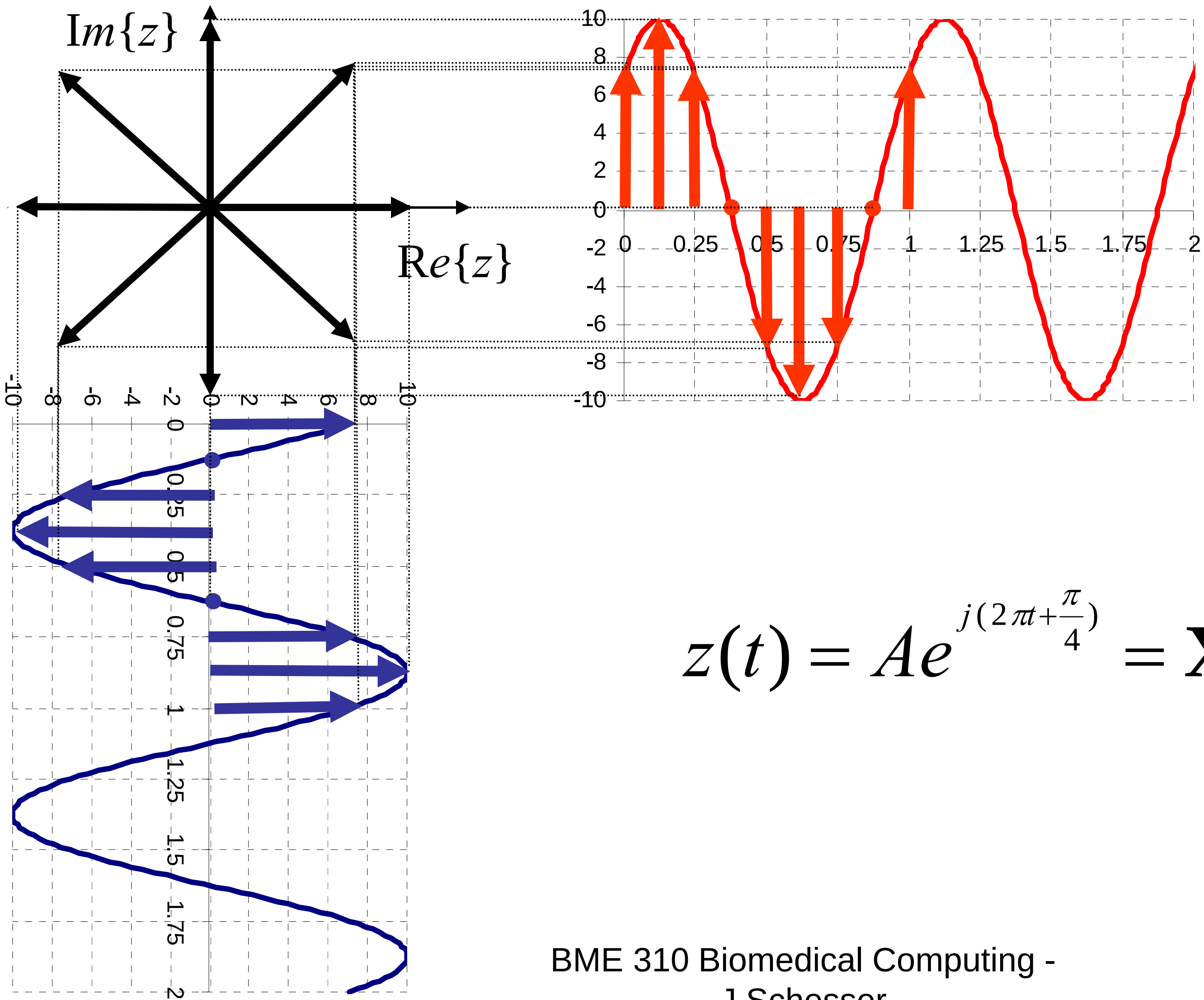
*arg(z(t))=2π x3/4+π/4 = 7π/4*

*t=7/8 seconds*

*arg(z(t))=2π x7/8+π/4 = 8π/4 = 2π*

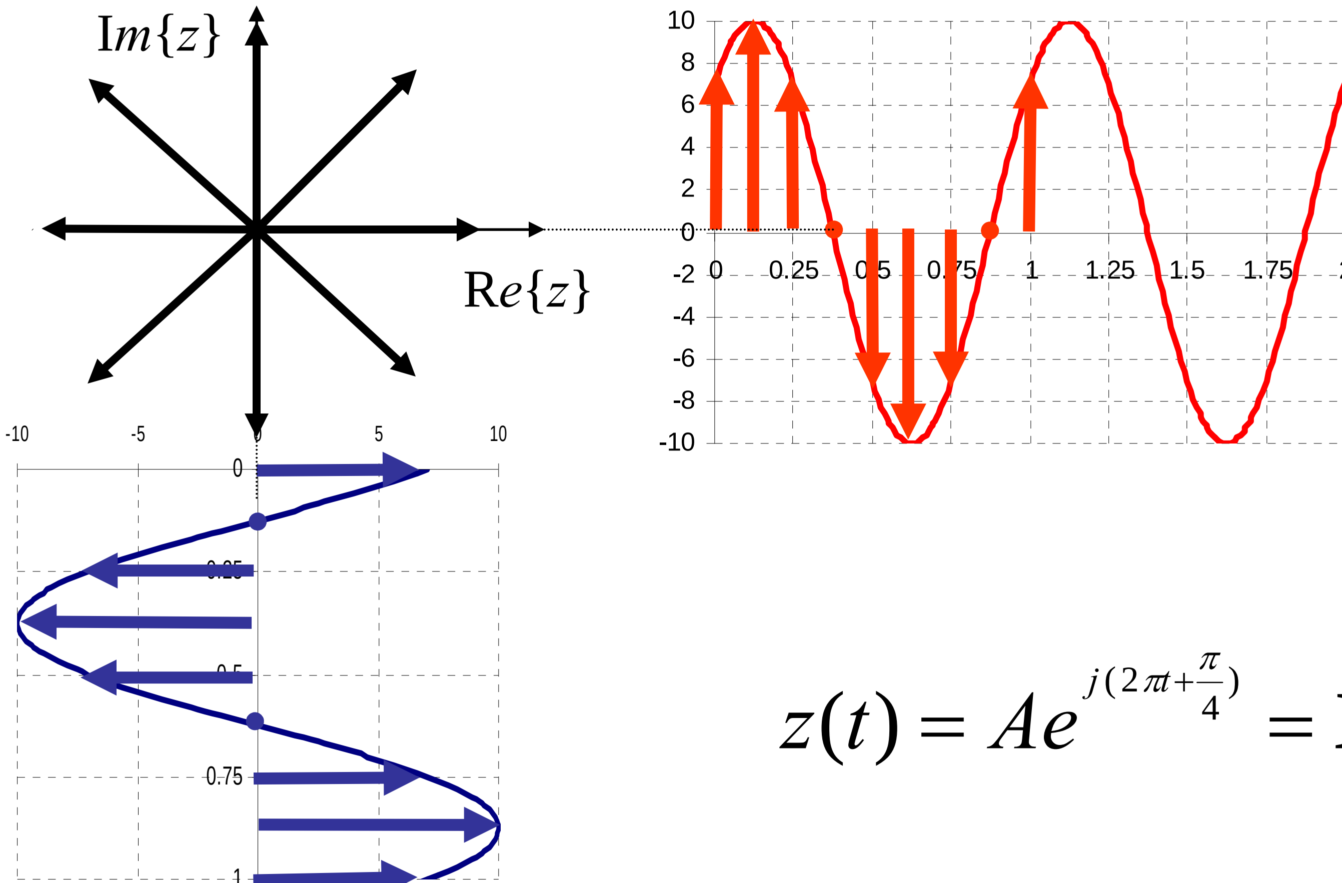


# Rotating Phasor




$$z(t) = Ae^{j(2\pi t + \frac{\pi}{4})} = \mathbf{X}e^{j2\pi t}$$

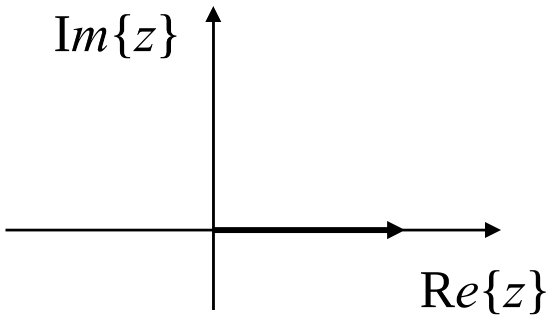
# Rotating Phasor

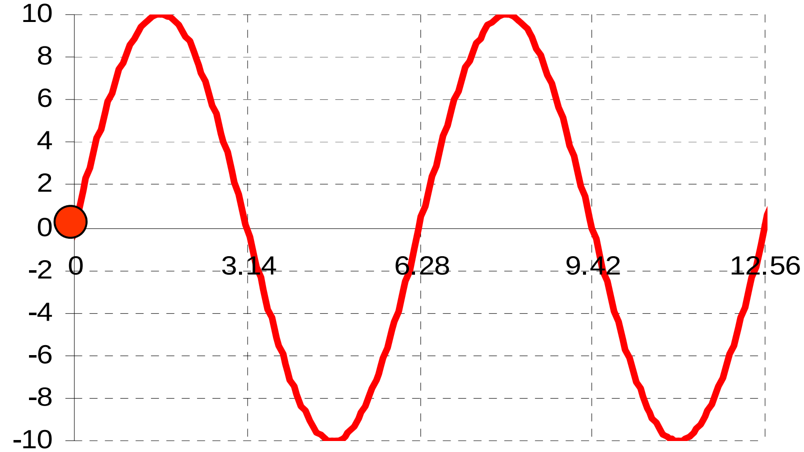


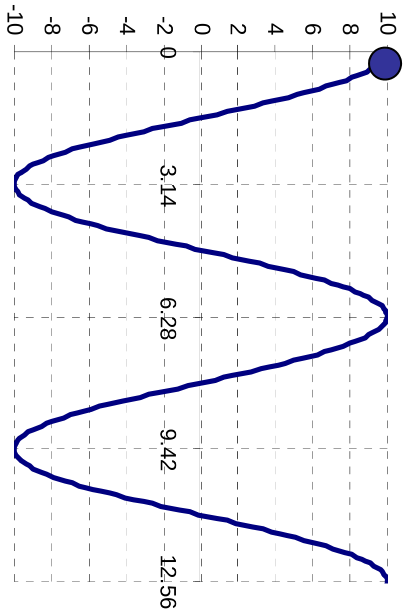

$$z(t) = Ae^{j(2\pi t + \frac{\pi}{4})} = \mathbf{X}e^{j2\pi t}$$

# Rotating Phasor

$$z(t) = Ae^{j(2\pi t)} = \mathbf{X}e^{j\omega_o t}$$



# Inverse Euler Formulas

- The inverse Euler formulas how the cosine and sine functions consist of complex exponentials

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

# Sinusoidal Signals

$$A\cos(\omega_o t+\theta) = A\left(\frac{e^{j(\omega_o t+\theta)}+e^{-j(\omega_o t+\theta)}}{2}\right)$$

$$= A\left(\frac{e^{j\omega_o t}e^{j\theta}+e^{-j\omega_o t}e^{-j\theta}}{2}\right)$$

$$A\cos(\omega_o t+\theta) = \frac{\mathbf{X}e^{j\omega_o t}+\mathbf{X}^{*}e^{-j\omega_o t}}{2}$$

$$= \frac{1}{2}z(t)+\frac{1}{2}z^{*}(t)$$

$$= \Re e\{z(t)\}$$

- Note that * means complex conjugate and $z(t)$ and $z^*(t)$ are called conjugate pairs
- This means that a cosine function consists of two complex exponential functions: one with positive frequency and one with negative frequency
- The amplitudes are complex conjugates

# Complex Conjugate

- A conjugate of a complex number has the same real part but negative imaginary part of the complex number

$$x = a + jb$$

$$x^* = a - jb$$

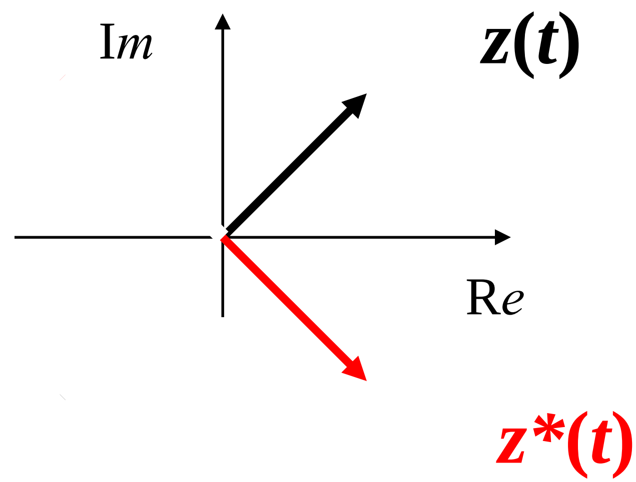
- Note the following important properties

$$x + x^* = 2\,\mathrm{R}e\{x\} = a + jb + (a - jb) = 2a$$

$$x - x^* = j2\,\mathrm{I}m\{x\} = a + jb - (a - jb) = j\,2b$$

# Rotating Conjugate Pairs

Note that the imaginary part is zero since the imaginary parts of each cancel each other.



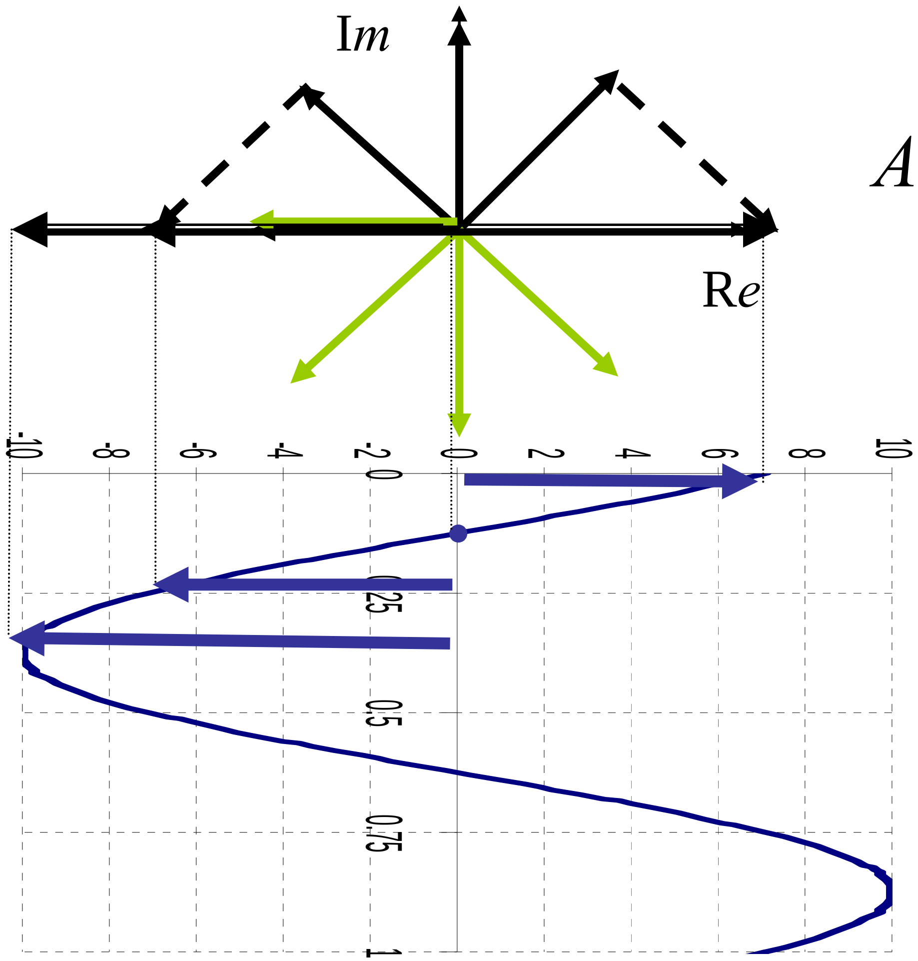

$$A\cos(\omega_o t+\theta)=\frac{1}{2}z(t)+\frac{1}{2}z^*(t)$$

$$=\frac{1}{2}[z(t)+z^*(t)]=\frac{1}{2}[\Re e\{z(t)\}+j\Im m\{z(t)\}+\Re e\{z*(t)\}+j\Im m\{z*(t)\}]$$

$$=\frac{1}{2}[2\Re e\{z(t)\}+j[\Im m\{z(t)\}+\Im m\{z*(t)\}]]=\frac{1}{2}[2\Re e\{z(t)\}+j[\Im m\{z(t)\}-\Im m\{z(t)\}]]$$

$$=\Re e\{z(t)\}$$

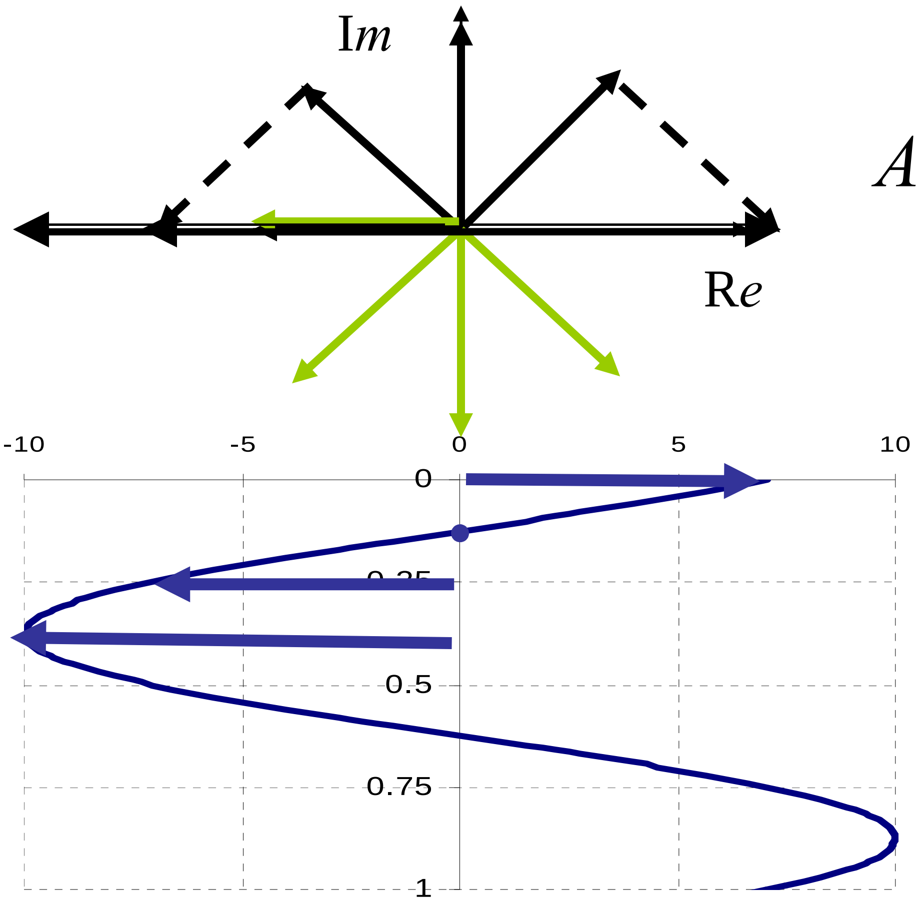
# Rotating Conjugate Pairs




$$A\cos(\omega_o t + \theta) = \frac{1}{2}z(t) + \frac{1}{2}z^*(t)$$

$$= \Re e\{z(t)\}$$

# Rotating Conjugate Pairs




$$A\cos(\omega_o t+\theta)=\frac{1}{2}z(t)+\frac{1}{2}z^*(t)$$
$$=\Re e\{z(t)\}$$

# Sinusoid Signal Addition

- Adding several sinusoid signals with the **<u>same frequency</u>** but with different amplitudes and phase angles to be in a form of a single sinusoidal signal

$$\sum_{1}^{N} A_k \cos(\omega_o t + \theta_k) = B \cos(\omega_o t + \phi)$$

- Proof of this uses identity 5:

$$A \cos(\omega_o t + \theta) = A \cos\theta \cos\omega_o t - A \sin\theta \sin\omega_o t$$

# Proof of Sinusoid Signal Addition Algorithm

- The following shows how to calculate the sum of 2 phasors; it can be easily extended to more than 2 phasors:

$$A_1 \cos(\omega_o t + \theta_1) \quad + A_2 \cos(\omega_o t + \theta_2) \quad = B\cos(\omega_o t + \phi)$$

$$A_1 \cos\omega_o t \cos\theta_1 - A_1 \sin\omega_o t \sin\theta_1 + A_2 \cos\omega_o t \cos\theta_2 - A_2 \sin\omega_o t \sin\theta_2 = B\cos\omega_o t \cos\phi - B\sin\omega_o t \sin\phi$$

$$\underbrace{(A_1 \cos\theta_1 + A_2 \cos\theta_2)}\cos\omega_o t - \underbrace{(A_1 \sin\theta_1 + A_2 \sin\theta_2)}\sin\omega_o t = \underbrace{B\cos\phi}\cos\omega_o t - \underbrace{B\sin\phi}\sin\omega_o t$$

Matching terms, we have:

$$B\cos\phi = A_1 \cos\theta_1 + A_2 \cos\theta_2$$

$$B\sin\phi = A_1 \sin\theta_1 + A_2 \sin\theta_2$$

# Proof of Sinusoid Signal Addition Algorithm

$$B\cos\phi = A_1\cos\theta_1 + A_2\cos\theta_2$$

$$B\sin\phi = A_1\sin\theta_1 + A_2\sin\theta_2$$

Since

$$\sqrt{(B\cos\phi)^2 + (B\sin\phi)^2} = B$$

So

$$B = \sqrt{(A_1\cos\theta_1 + A_2\cos\theta_2)^2 + (A_1\sin\theta_1 + A_2\sin\theta_2)^2}$$

And

$$\frac{B\sin\phi}{B\cos\phi} = \tan\phi$$

$$\phi = \arctan\left(\frac{B\sin\phi}{B\cos\phi}\right)$$

$$= \arctan\left(\frac{A_1\sin\theta_1 + A_2\sin\theta_2}{A_1\cos\theta_1 + A_2\cos\theta_2}\right)$$

# Example

$$A\cos(20\pi t+\phi)=1.7\cos(20\pi t+70\pi/180)+1.9\cos(20\pi t+200\pi/180)$$

$$A=\sqrt{(1.7\cos(70\pi/180)+1.9\cos(200\pi/180))^2+(1.7\sin(70\pi/180)+1.9\sin(200\pi/180))^2}$$

$$=1.532$$

$$\phi=\arctan(\frac{1.7\sin(70\pi/180)+1.9\sin(200\pi/180)}{1.7\cos(70\pi/180)+1.9\cos(200\pi/180)}=2.47rad\Rightarrow 141.79^\circ$$

$$=1.532\cos(20\pi t+2.47)$$

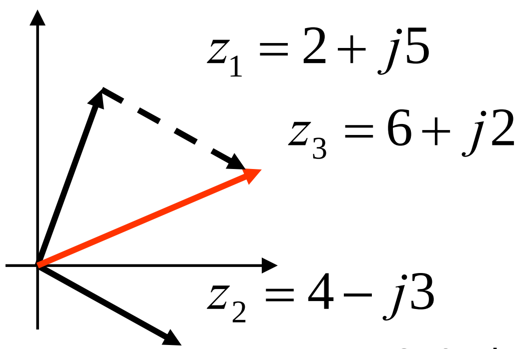
# RECALL
# Addition of Complex Numbers Rectangular form

- The real part of the sum is the sum of the real parts and imaginary part of the sum is the sum of the imaginary parts.

$$z_1 = x_1 + jy_1; z_2 = x_2 + jy_2$$

$$z_3 = z_1 + z_2 = x_1 + jy_1 + x_2 + jy_2$$

$$= (x_1 + x_2) + j(y_1 + y_2)$$




$$z_1 = 2 + j5; z_2 = 4 - j3$$

$$z_3 = z_1 + z_2 = 2 + j5 + 4 - j3$$

$$= (2+4) + j(5-3)$$

$$= 6 + j2$$

# Phasor Addition

$$B\cos(\omega_o t + \phi) = \sum_1^N A_k \cos(\omega_o t + \theta_k)$$

$$\Re e\{Be^{j(\omega_o t + \phi)}\} = \sum_1^N \Re e\{A_k e^{j(\omega_o t + \theta_k)}\}$$

$$\Re e\{Be^{j\phi} e^{j\omega_o t}\} = \sum_1^N \Re e\{A_k e^{j\theta_k} e^{j\omega_o t}\}$$

$$\Re e\{Be^{j\phi} e^{j\omega_o t}\} = \Re e\{(\sum_1^N A_k e^{j\theta_k})e^{j\omega_o t}\}$$

Therefore,

$$Be^{j\phi} = \sum_1^N A_k e^{j\theta_k}$$

# An Easier Method for Adding Sinusoids using Phasors

1. Represent the sinusoidal signals by complex exponential signals
2. From these exponential signals, take the each phasor in polar form and convert them to the Cartesian complex number form
3. Add the complex number to obtain a single complex number
4. Convert this complex number into its polar form
5. Using the phasor, reformate the sinusoidal signal by multiplying the phasor with $e^{j\omega_o t}$ and taking its real part

# Example

$x_1(t) = 1.7\cos(20\pi t + 70\pi/180)$

Step 1) Formulate the phasor for signal 1.

$X_1 = A_1 e^{j\phi_1} = 1.7 e^{j70\pi/180}$

Step 2) Convert its phasor to rectangular form

$X_1 = 1.7\cos 70\pi/180 + j1.7\sin 70\pi/180$

$= 0.5814 + j1.597$

$x_2(t) = 1.9\cos(20\pi t + 200\pi/180)$

Step 1) Formulate the phasor for signal 2.

$X_2 = A_2 e^{j\phi_1} = 1.9 e^{j200\pi/180}$

Step 2) Convert its phasor to rectangular form

$X_2 = 1.9\cos 200\pi/180 + j1.9\sin 200\pi/180$

$= -1.785 - j0.6498$

Step 3) Add the 2 phasors in rectangular form

$X_3 = X_1 + X_2$

$= 0.5814 + j1.597 + (-1.785 - j0.6498)$

$= -1.204 + j0.948$

Step 4) Convert the resultant phasor to polar form

$X_3 = -1.204 + j0.948 = 1.532 e^{j2.475}$

Step 5) Formulate the resultant signal from the phasor

$x_3(t) = 1.532\cos(20\pi t + 2.475)$

# Solution of Differential Equations of a Tuning Fork

- Hooke's law defines the deformation force of a tuning fork where $k$ is the elastic constant of the material and $x$ is the amount of deformation along an axis:

$$F = - k x$$

- Newton's second law defines the acceleration of the deformation:

$$F = ma = m \, d^2x/dt^2$$

- Equating the two, we have the following differential equation:

$$m \, d^2x/dt^2 = - k x$$

- What function whose second derivation is proportional to itself?

# Solution continued

- The cosine function meets this criterion:

$$x(t) = \cos\omega_o t$$

$$dx/dt = -\omega_o \sin\omega_o t$$

$$d^2x/dt^2 = -\omega_o^2 \cos\omega_o t$$

- Then:

$$-m\omega_o^2 \cos\omega_o t = -k\cos\omega_o t$$

$$-m\omega_o^2 = -k$$

- Finally, we have:

$$\omega_o = \pm\sqrt{\frac{k}{m}}$$

$$x(t) = \cos(\sqrt{\frac{k}{m}}t)$$

# Tuning Fork Interpretation

- This states that the frequency of oscillation of a tuning fork is solely a function of the tuning fork's mass and its elastic constant
- Note that both of these signals are also a solution of this second order differential equation:

$$z(t) = Ae^{j(\omega_o t+\theta)} = \mathbf{X}e^{j\omega_o t}; z^*(t) = Ae^{-j(\omega_o t+\theta)} = \mathbf{X}^* e^{-j\omega_o t}$$

# Homework

- Exercises:
  - 2.5 – 2.10
- Problems:

–2.5

–2.6 Also use DeMoiver's formula to evaluate $(4 - j3)^{-20}$ and $(-4 + j3)^{-20}$

–2.7 Also $f)\ \Im m\{\frac{j}{e^{-j\frac{\pi}{2}}}\}$

–2.9, –2.10, –2.11, –2.15

–2.17 Instead use $x(t) = 4\cos(\omega t + 3\pi/2) + 3\cos(\omega t + 2\pi/3) + 2\cos(\omega t + \pi/3)$