Spectrum Representation

Lecture #4

Chapter 3

What Is this Course All About ?

- To Gain an Appreciation of the Various Types of Signals and Systems
- To Analyze The Various Types of Systems
- To Learn the Skills and Tools needed to Perform These Analyses.
- To Understand How Computers Process Signals and Systems

What is a Spectrum?

- A signal is a function of time which can be represented by a series of sinusoidal functions or sinusoidal components.
- These sinusoidal components have different frequencies, different amplitudes, and different phases.
- Therefore, the plots of frequency versus amplitude and phase for the sinusoidal components which comprise the signal are called the Frequency Spectrum or Spectrum of the signal.

Decomposition of a signal

- We can express the following representation of a function:

$$x(t) = A_o + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \theta_k)$$

$$= X_o + \Re e\{\sum_{k=1}^{N} X_k e^{j2\pi f_k t}\}$$

Where $X_o = A_o$ is a real constant (DC component) and

$X_k = A_k e^{j\theta_k}$ is the phasor for frequency f_k

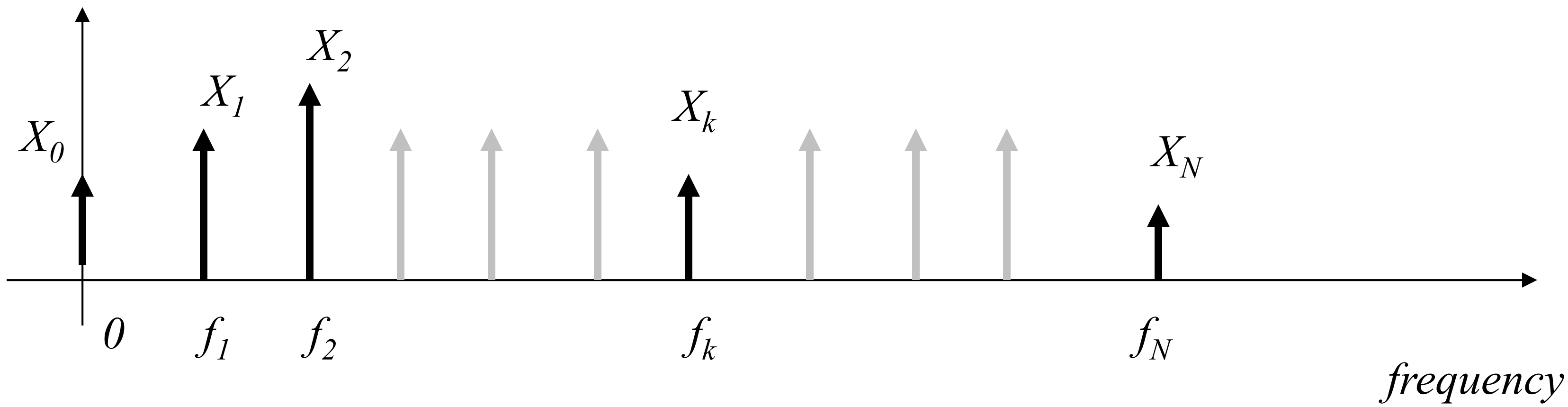
- Here we see that there are $N+1$ frequency components for $x(t)$, $0 \le k \le N$ and with each frequency there is a phasor.

Decomposition of a signal

- For example, the kth frequency has a phasor X_k with amplitude, A_k, and phase, θ_k.

$$x(t) = A_o + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \theta_k) = X_o + \Re e\{\sum_{k=1}^{N} X_k e^{j2\pi f_k t}\}$$

Where X_o = DC component and $X_k = A_k e^{j\theta_k}$ is the phasor for frequency f_k



- This is really only half of the spectrum.

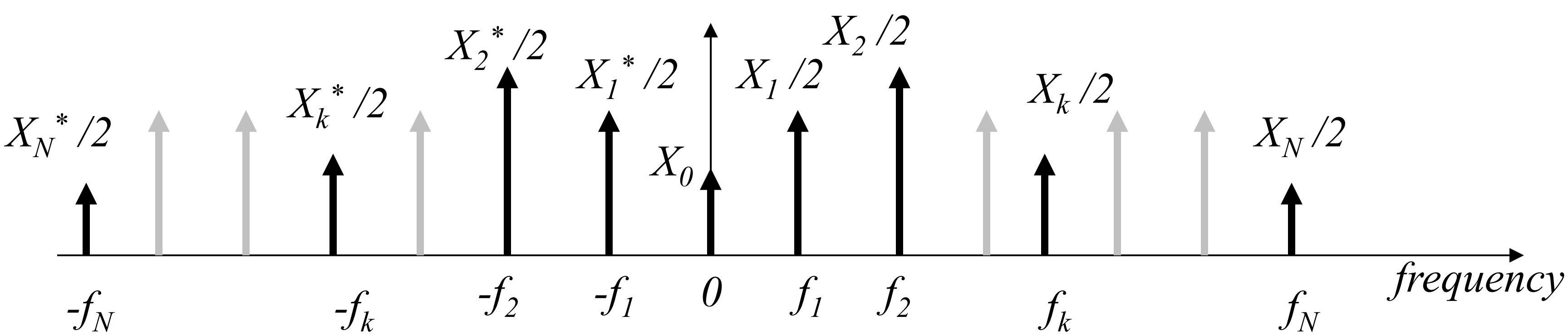
Alternative form of the Spectrum

- Using Euler's formula, let rewrite $x(t)$:

$$x(t) = A_o + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \theta_k) = X_o + \sum_{k=1}^{N} \{\frac{A_k}{2} e^{j\theta_k} e^{j2\pi f_k t} + \frac{A_k}{2} e^{-j\theta_k} e^{-j2\pi f_k t}\}$$

$$= X_o + \sum_{k=1}^{N} \{\frac{X_k}{2} e^{j2\pi f_k t} + \frac{X_k{}^*}{2} e^{-j2\pi f_k t}\}$$

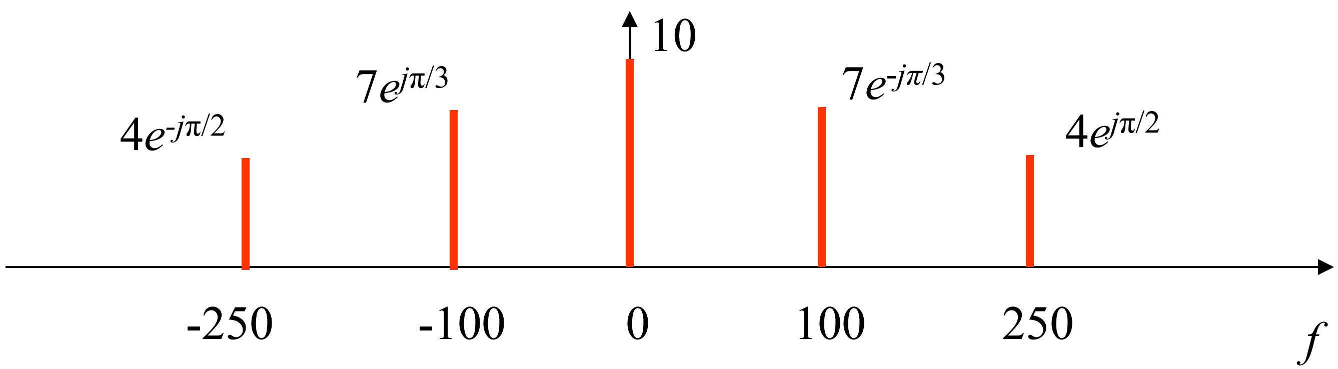
- Using this approach, we see that there are $2N+1$ frequency components
- Or we can say that for each k where $1 \le k \le N$, there is a positive frequency f_k with phasor $X_k/2$ and a negative frequency $-f_k$ with phasor $X_k^*/2$.
- Therefore, we say that the spectrum is two-sided.



An Example

$$x(t) = 10 + 14\cos(200\pi t - \pi/3)$$
$$\qquad + 8\cos(500\pi t + \pi/2)$$

$$= 10 + 7e^{-j\pi/3}e^{j200\pi t} + 7e^{j\pi/3}e^{-j200\pi t}$$
$$\qquad + 4e^{j\pi/2}e^{j500\pi t} + 4e^{-j\pi/2}e^{-j500\pi t}$$



Notation Change

- For the 3rd terms in this sum $x(t) = X_o + \sum_{k=1}^{N} \{ \frac{X_k}{2} e^{j2\pi f_k t} + \frac{X_k *}{2} e^{-j2\pi f_k t} \}$
- Let's define:

$$A_{-k} = A_k$$
$$\theta_{-k} = -\theta_k$$
$$f_{-k} = -f_k$$

Then,

$$X_k * = (A_k e^{j\theta_k})* = A_k e^{-j\theta_k} = A_{-k} e^{j\theta_{-k}} = X_{-k}$$

- Furthermore, define

$$a_k = \begin{cases} A_o & \text{for } k = 0 \\ \frac{1}{2} A_k e^{j\theta_k} & \text{for } k \neq 0 \end{cases}$$

Using the new notation another (simpler) form appears

$$x(t) = A_o + \sum_{k=1}^{N} \{\frac{X_k}{2} e^{j2\pi f_k t} + \frac{X_k^*}{2} e^{-j2\pi f_k t}\} = A_o + \sum_{k=1}^{N} \frac{X_k}{2} e^{j2\pi f_k t} + \sum_{k=1}^{N} \frac{X_k^*}{2} e^{-j2\pi f_k t}$$

$$= A_o + \sum_{k=1}^{N} \frac{A_k}{2} e^{j\theta_k} e^{j2\pi f_k t} + \sum_{k=1}^{N} (\frac{A_k}{2} e^{j\theta_k})^* e^{-j2\pi f_k t} = A_o + \sum_{k=1}^{N} \frac{A_k}{2} e^{j\theta_k} e^{j2\pi f_k t} + \sum_{k=1}^{N} \frac{A_k}{2} e^{-j\theta_k} e^{-j2\pi f_k t}$$

Using the new notation, we now have

$$= A_o + \sum_{k=1}^{N} \frac{A_k}{2} e^{j\theta_k} e^{j2\pi f_k t} + \sum_{k=1}^{N} \frac{A_{-k}}{2} e^{j\theta_{-k}} e^{j2\pi f_{-k} t}$$

In the second sum, we replace k with $-k$

$$= A_o + \sum_{k=1}^{N} \frac{A_k}{2} e^{j\theta_k} e^{j2\pi f_k t} + \sum_{k=-1}^{-N} \frac{A_k}{2} e^{j\theta_k} e^{j2\pi f_k t}$$

$$= a_o + \sum_{k=1}^{N} a_k e^{j2\pi f_k t} + \sum_{k=-1}^{-N} a_k e^{j2\pi f_k t} = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

The Spectrum is now associated with the $2 \times \mathrm{N} + 1$ a_k's

Multiplication of Sinusoids

- A beat note (or frequency) is the result of multiplying sinusoids
- We see this in many applications:
  - Music and broadcasting,
- What is the spectrum of the product of sinusoids?

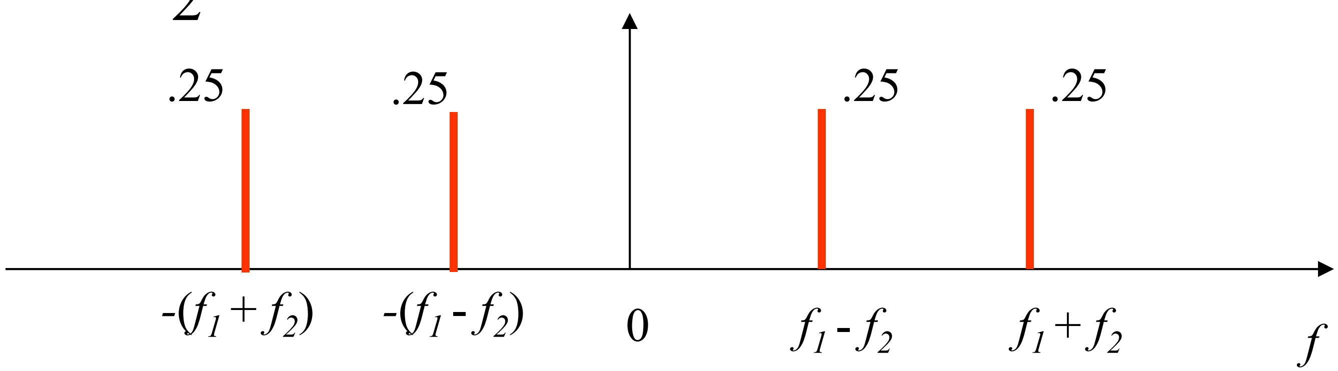
Product of 2 Sinusoids

$$x(t) = \cos(2\pi f_1 t)\ \cos(2\pi f_2 t);\ \text{let's assume } f_1 > f_2$$

$$= \left(\frac{e^{j2\pi f_1 t} + e^{-j2\pi f_1 t}}{2}\right)\left(\frac{e^{j2\pi f_2 t} + e^{-j2\pi f_2 t}}{2}\right)$$

$$= \frac{1}{2}\left(\frac{e^{j2\pi(f_1+f_2)t} + e^{-j2\pi(f_1+f_2)t} + e^{j2\pi(f_1-f_2)t} + e^{-j2\pi(f_1-f_2)t}}{2}\right)$$

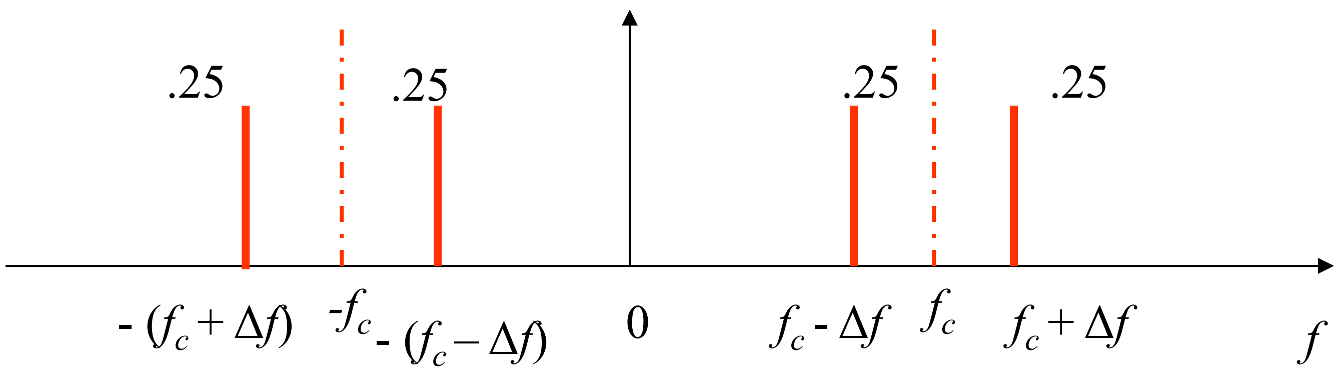
$$= \frac{1}{2}(\cos 2\pi(f_1+f_2)t + \cos 2\pi(f_1-f_2)t)$$



Product of 2 Sinusoids

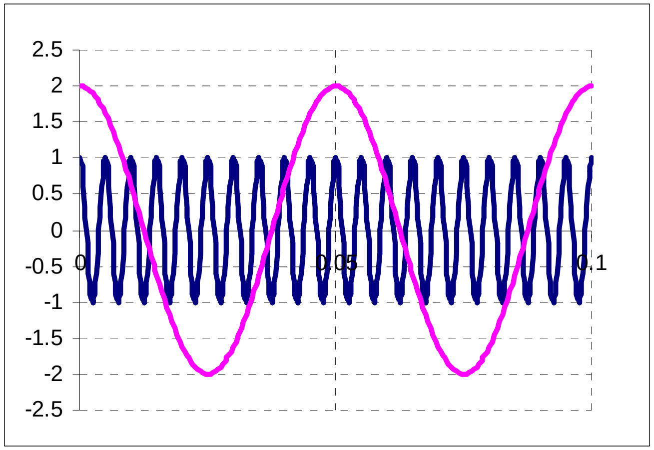
- The product of 2 sinusoids produce a signal which is the sum of 2 sinusoids whose frequencies are the sum and difference of the frequencies of the original sinusoids.
- In fact, if we define $f_1 = f_c$ as a center frequency and $f_c >> f_2$, such that $f_2 = \Delta f$ (a small value compared to f_C), then the product of the two sinusoids will yet frequencies $\pm \Delta f$ around f_c

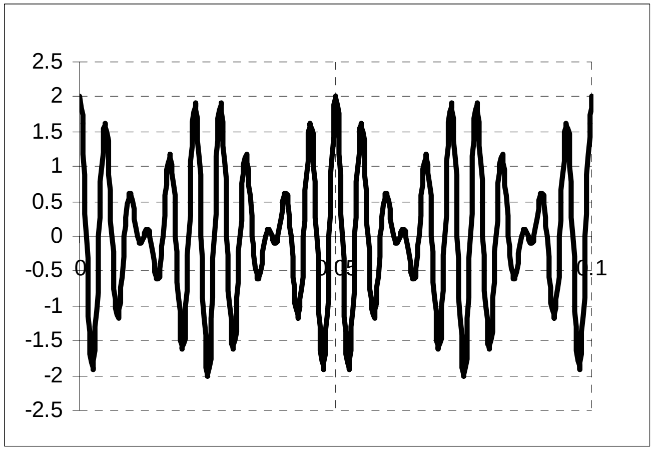
$$x(t) = \cos(2\pi f_c t)\ \cos(2\pi \Delta f t) = \frac{1}{2}(\cos 2\pi (f_c + \Delta f)t + \cos 2\pi (f_c - \Delta f)t)$$



Examples

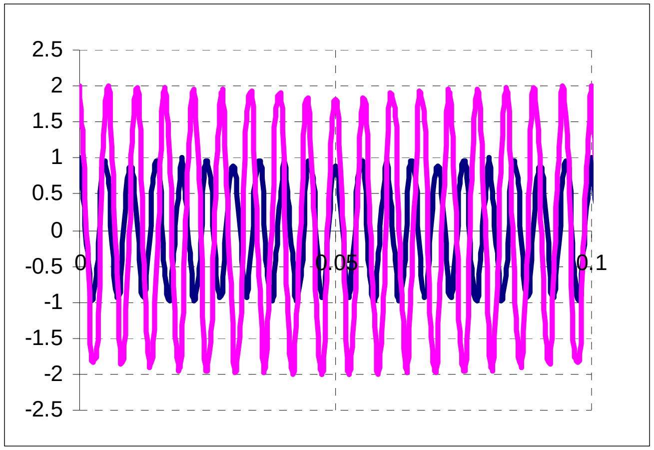
- Two signals 20 Hz and 200 Hz
- Resultant product signal at 220 Hz and 180 Hz
- An envelop effect results (Beat Signal)

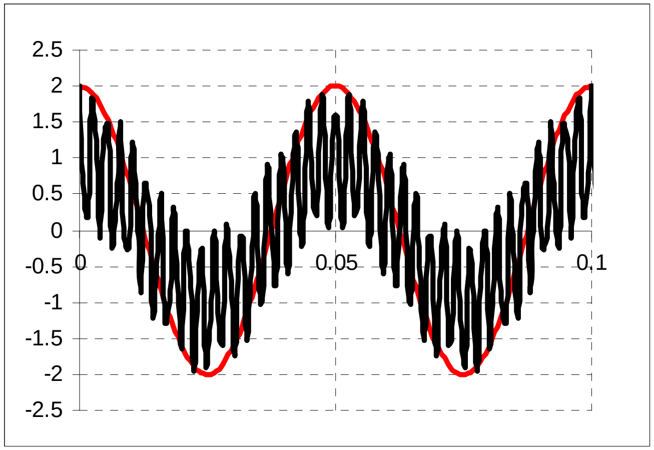




Examples

- Two signals 180 Hz and 200 Hz
- Resultant product signal at 20 Hz and 380 Hz
- Notice the 20 Hz component (A single 20 Hz signal is added to emphasis the 20 Hz component)





Amplitude Modulation

- Amplitude Modulation or AM is the technique used to broadcast AM radio.
- The “information” (e.g., the newscaster’s voice) is modulates a carrier signal.
- Usually, the carrier signal is at a frequency (the carrier frequency) which is much higher than the “information” to be broadcast (the AM band is 660 kHz through 1600 kHz while the information is a voice signal which is between 150 Hz and 4 kHz)
- The form of AM is:

$$x(t)=v(t)\cos(2\pi f_c t)$$

Spectrum of an AM Signal

- Let $v(t) = A + B\cos(2\pi f_i t)$
- Then the AM signal becomes:

$$x(t) = v(t)\cos(2\pi f_c t) = (A + B\cos(2\pi f_i t))\cos(2\pi f_c t)$$

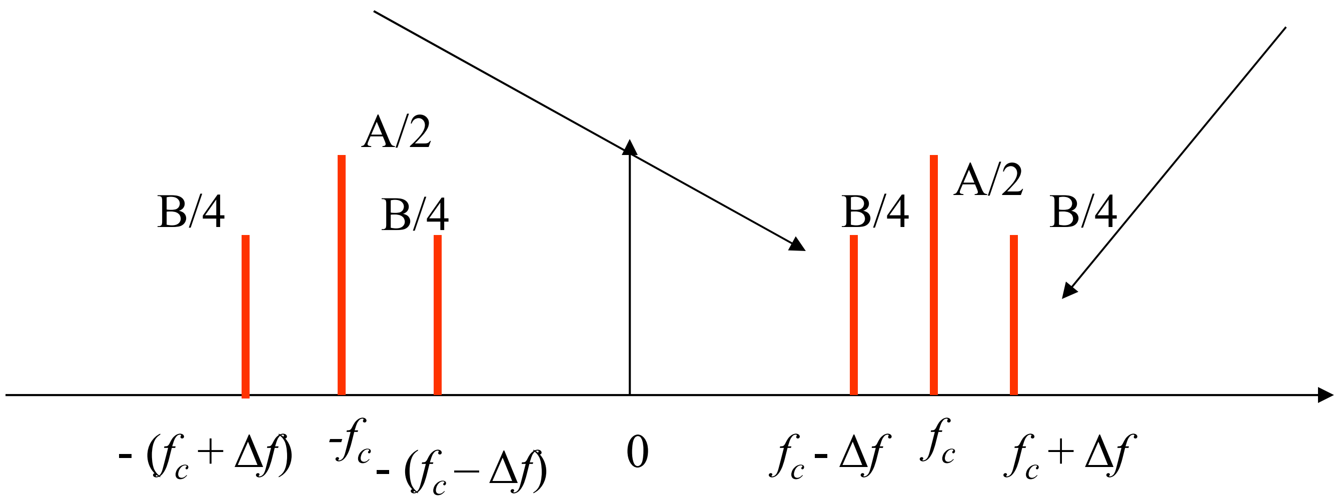
$$= A\cos(2\pi f_c t) + B\cos(2\pi f_i t)\cos(2\pi f_c t)$$

$$= A\cos(2\pi f_c t) + B/2(\cos(2\pi[f_c + f_i]t) + \cos(2\pi[f_c - f_i]t)$$

Spectrum of an AM Signal

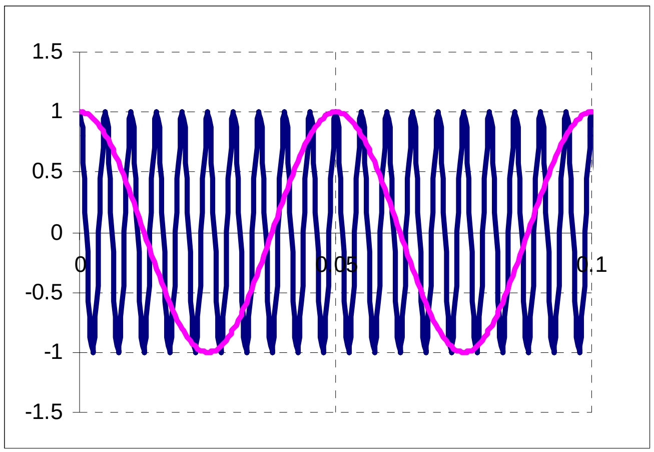
$x(t) = A\cos(2\pi f_c t) + B/2(\cos(2\pi[f_c + f_i]t) + \cos(2\pi[f_c - f_i]t)$

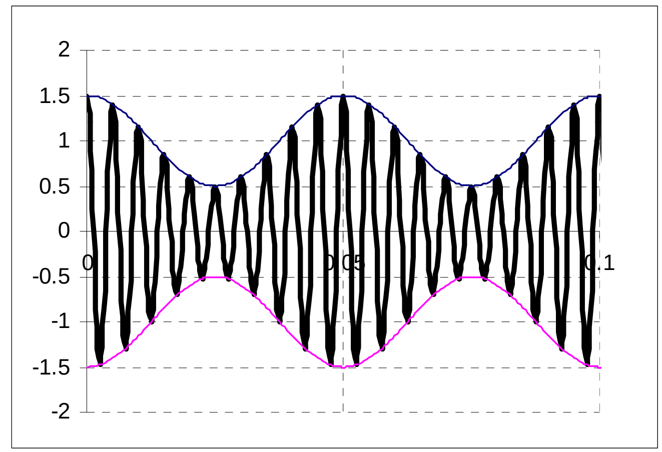
- The spectrum has 3 components: at f_c, $f_c + f_i$ and $f_c - f_i$ where the latter 2 are called the sidebands of the AM signal.
- $f_c - f_i$ is the lower sideband and $f_c + f_i$ is the upper sideband

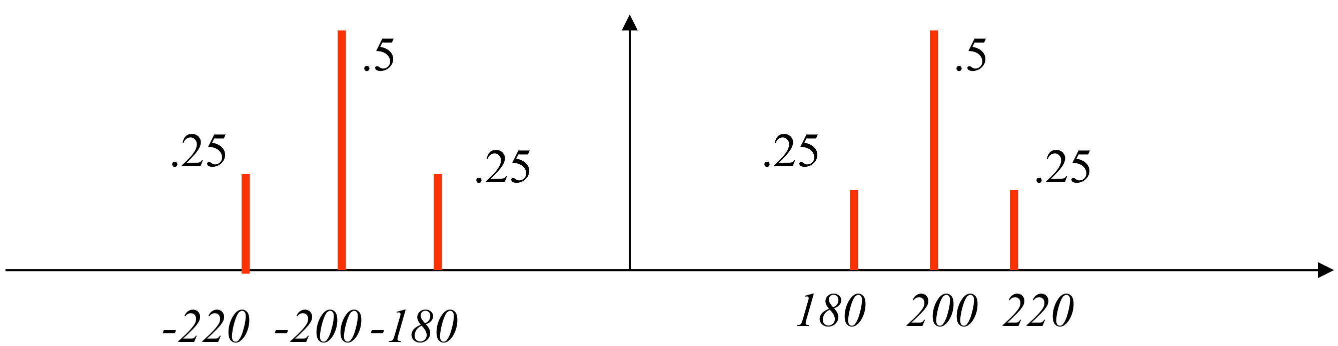


The waveform of an AM Signal

$$x(t)= (1+0.5 \cos(2\pi 20t)) \cos(2\pi 200t)$$







Homework

- Exercises:
  - 3.1 – 3.3
- Problems:

$$-3.1, -3.2$$

$$-3.3 \text{ Instead use } x(t) = \sin^3(3\pi t)$$