Spectrum Representation

Lecture #5

Chapter 3

Mathematical Forms of a Decomposed Signal

Single Sided Spectrum

$$x(t) = A_o + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \theta_k) = X_o + \Re e\{\sum_{k=1}^{N} X_k e^{j2\pi f_k t}\}$$

Where X_o = DC component and $X_k = A_k e^{j\theta_k}$ is the phasor for frequency k

Two Sided Spectrum

$$x(t) = A_o + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \theta_k) = X_o + \sum_{k=1}^{N} \{\frac{A_k}{2} e^{j\theta_k} e^{j2\pi f_k t} + \frac{A_k}{2} e^{-j\theta_k} e^{-j2\pi f_k t}\}$$

$$= X_o + \sum_{k=1}^{N} \{\frac{X_k}{2} e^{j2\pi f_k t} + \frac{X_k{}^*}{2} e^{-j2\pi f_k t}\} = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

Using the subsitution $(k = -k)$ for the second summation to combine into a single summation

where $a_o = A_o$ for $k = 0; a_k = \frac{X_k}{2} = \frac{1}{2} A_k e^{j\theta_k}$ for $k \neq 0$

Periodic Waveforms

- A periodic waveform must satisfy this condition:

$$x(t+T_o) = x(t)$$

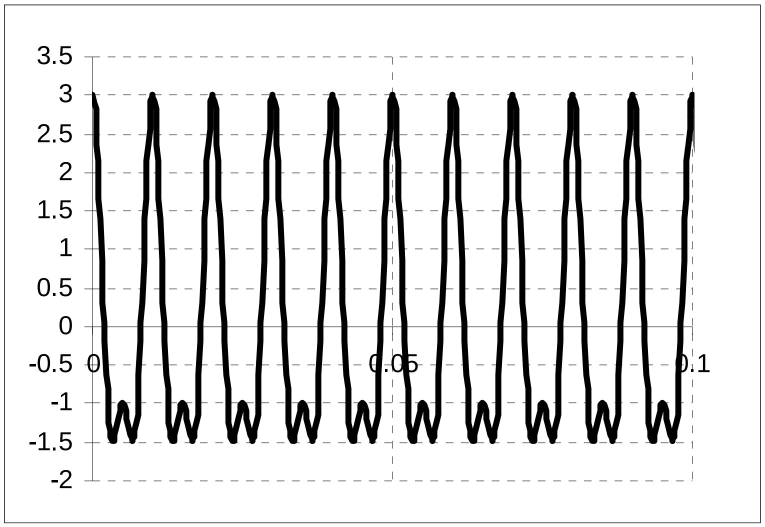
where T_o is the period of $x(t)$.

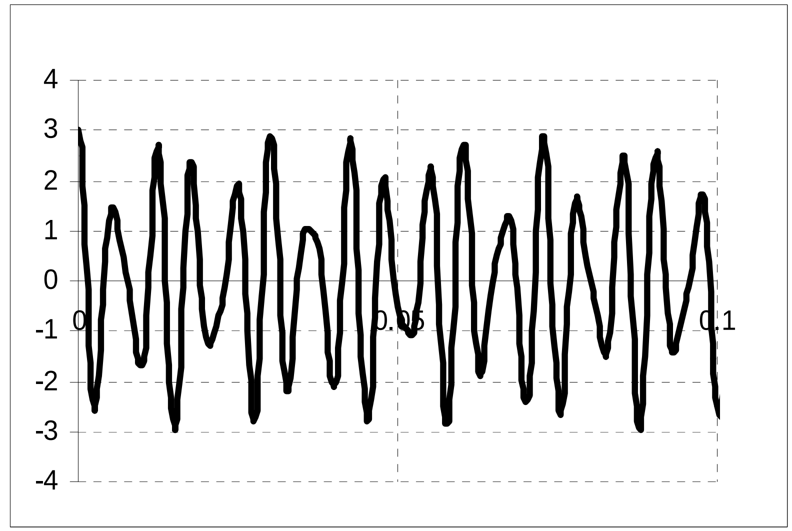
  - It can be shown that periodical waveforms are made up of sinusoidal functions which are harmonically related frequencies.
  - Harmonic frequencies are frequencies which are multiples of each other.
- If a periodic signal, $x(t)$, has a period of T_o then it has a fundamental frequency $f_o = 1/\,T_o$.
- Furthermore, we say that $x(t)$ has frequencies which are harmonics of f_o.
  - That is, the *k*th harmonic of $x(t)$ is $k\,f_o$

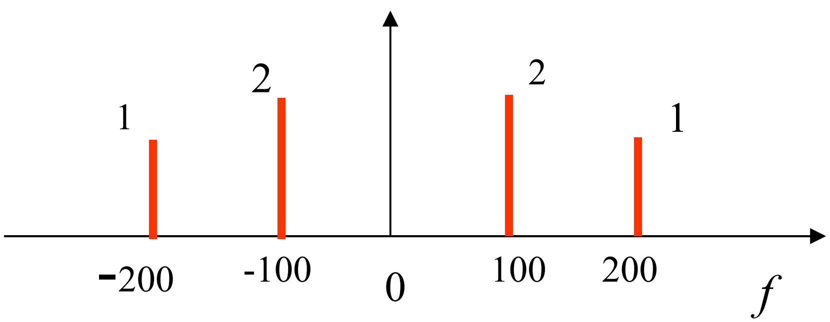
Periodic Vs Nonperiodic Signals

- We say that $x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$ is nonperiodic since there is no assumption about the individual frequencies.
- We say that $x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi k f_o t}$ is periodic since there is a fundamental period associated with the signal
- The fundamental frequency f_o is the greatest common divisor of f_k

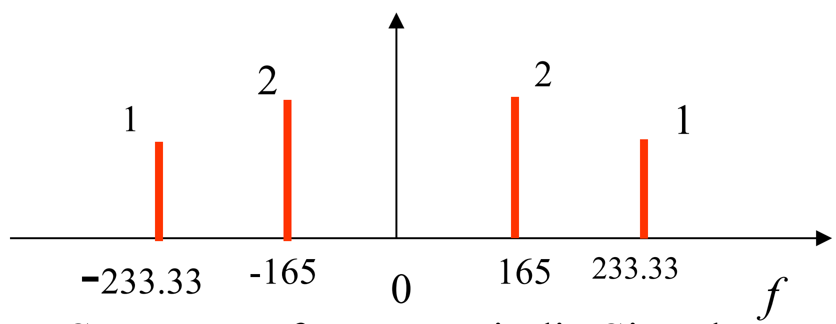
Periodic Vs Nonperiodic Signals







Spectrum of a Periodic Signal
Fundamental Frequency = 100Hz



Spectrum of a Nonperiodic Signal
No Fundamental Frequency

Fourier Analysis

- We have shown that a signal can be formulated in terms of a sum of sinusoids which defines its spectrum

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

- What we like to find are the a_k's so we can completely determine an signal's spectrum
- If a signal is periodic, (i.e., the f_k's are multiples of f_o) then the determination of these the a_k's are easily achieved using Fourier Analysis.

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi k f_o t}$$

Another Mathematical Form of a Decomposed Signal

Two Sided Spectrum with $N \rightarrow \infty$

$$x(t) = A_o + \sum_{k=1}^{\infty} A_k \cos(2\pi f_k t + \theta_k) = X_o + \sum_{k=1}^{\infty} \{\frac{A_k}{2} e^{j\theta_k} e^{j2\pi f_k t} + \frac{A_k}{2} e^{-j\theta_k} e^{-j2\pi f_k t}\}$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{j2\pi f_k t}$$

where $f_k \rightarrow kf_o;\ a_k = \ A_o$ for $k = 0; a_k = \ \dfrac{X_k}{2} = \dfrac{1}{2} A_k e^{j\theta_k}$ for $k \neq 0$

$$= a_o + \sum_{k=1}^{\infty} a_k e^{j2\pi k f_o t} + a_k * e^{-j2\pi k f_o t} = a_o + \sum_{k=1}^{\infty} 2\Re e\{a_k e^{j2\pi k f_o t}\}$$

since $s + s* = 2\Re e\{s\}$

$$= a_o + \sum_{k=1}^{\infty} 2\Re e\{\left|a_k\right| e^{j\theta_k} e^{j2\pi k f_o t}\} = a_o + \sum_{k=1}^{\infty} 2\left|a_k\right| \cos(2\pi k f_o t + \theta_k)$$

Summary of Mathematical Forms of a Decomposed Signal

Single Sided Spectrum with $N \to \infty$

$$x(t) = A_o + \sum_{k=1}^{\infty} A_k \cos(2\pi f_k t + \theta_k) = X_o + \Re e\{\sum_{k=1}^{\infty} X_k e^{j2\pi f_k t}\}$$

Where X_o = DC component and $X_k = A_k e^{j\theta_k}$ is the phasor for frequency k

Two Sided Spectrum with $N \to \infty$

$$x(t) = A_o + \sum_{k=1}^{\infty} A_k \cos(2\pi f_k t + \theta_k) = X_o + \sum_{k=1}^{\infty}\{\frac{A_k}{2} e^{j\theta_k} e^{j2\pi f_k t} + \frac{A_k}{2} e^{-j\theta_k} e^{-j2\pi f_k t}\}$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{j2\pi f_k t} \text{ where } a_k = A_o \text{ for } k = 0; a_k = \frac{X_k}{2} = \frac{1}{2} A_k e^{j\theta_k} \text{ for } k \neq 0$$

$$= a_o + \sum_{k=1}^{\infty} 2\Re e\{a_k e^{j2\pi k f_o t}\}$$

$$= a_o + \sum_{k=1}^{\infty} 2|a_k| \cos(2\pi k f_o t + \theta_k)$$

Fourier Analysis and Synthesis

- Given a Fourier Series (note the limits go to -∞ to ∞)

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_o t} = A_o + \sum_{k=1}^{\infty} A_k \cos(2\pi f_o kt + \theta_k)$$

$$\text{Where } A_o = a_o ; a_k = \frac{1}{2} A_k e^{j\theta_k} ; a_{-k} = a_k^* ; f_o = \frac{1}{T_o}$$

- Fourier Analysis is the technique to determine the a_k's from $x(t)$.
- Fourier Synthesis is the (opposite) technique to determine $x(t)$ from the a_k's.

Fourier Analysis

- The way to perform Fourier Analysis is by using the Fourier Series Integral:

$$a_k = \frac{1}{T_o} \int_0^{T_o} x(t) e^{-j(\frac{2\pi}{T_o})kt} dt$$

- This states to obtain the a_k's
  1. Take the periodic function $x(t)$ and multiply it by $e^{-j2\pi/T_o kt}$
  2. integrate the product over 1 period, T_o, of $x(t)$
  3. Divide the result by T_o

Derivation of the Fourier Integral

- Some background: The Orthogonality Property
- This is a property of certain functions
- Orthogonality means perpendicular
- So if two functions are orthogonal they are perpendicular as defined by the following:

$$\int_0^{T_o} v_k(t) v_l^*(t)\, dt = \begin{cases} 0 & \text{if } k \neq l \\ T_o & \text{if } k = l \end{cases}$$

Sinusoids and Orthogonality

- Note that sinusoidal and complex exponential functions are orthogonal.
- First recall:

$$1)\ \int_0^{T_o} e^{j\frac{2\pi}{T_o}kt}\,dt = \left.\frac{e^{j\frac{2\pi}{T_o}kt}}{j\frac{2\pi}{T_o}k}\right|_0^{T_o}$$

$$= \frac{e^{j\frac{2\pi}{T_o}kT_o}-1}{j\frac{2\pi}{T_o}k} = \frac{e^{j2\pi k}-1}{j\frac{2\pi}{T_o}k} = 0;\ \text{for } k \neq 0$$

$$\text{since } e^{j2\pi k} = \cos 2\pi k + j\sin 2\pi k = 1 + j0 = 1$$

Sinusoids and Orthogonality

- Note that sinusoidal and complex exponential functions are orthogonal.
- First recall:

$$1)\ \int_0^{T_o} e^{j\frac{2\pi}{T_o}kt}\,dt = \left.\frac{e^{j\frac{2\pi}{T_o}kt}}{j\frac{2\pi}{T_o}k}\right|_o^{T_o}$$

$$= \frac{e^{j\frac{2\pi}{T_o}kT_o} - 1}{j\frac{2\pi}{T_o}k} = \frac{e^{j2\pi k} - 1}{j\frac{2\pi}{T_o}k} = 0;\ \text{for } k \neq 0$$

$$\text{since } e^{j2\pi k} = \cos 2\pi k + j\sin 2\pi k = 1 + j0 = 1$$

Sinusoids and Orthogonality

- Note that sinusoidal and complex exponential functions are orthogonal.
- For $k = 0$:

$$\int_0^{T_o} e^{j\frac{2\pi}{T_o}kt}\,dt = \int_0^{T_o} e^{j\frac{2\pi}{T_o}0t}\,dt = \int_0^{T_o} 1dt = t\,\Big|_0^{T_o}$$

$$= T_o - 0 = T_o$$

Sinusoids and Orthogonality

- Another way

$$2)\ \int_0^{T_o} e^{j\frac{2\pi}{T_o}kt}dt = \int_0^{T_o}\cos\frac{2\pi}{T_o}ktdt + j\int_0^{T_o}\sin\frac{2\pi}{T_o}ktdt$$

$$= \left.\frac{\sin\frac{2\pi}{T_o}kt}{\frac{2\pi}{T_o}k}\right|_0^{T_o} - j\left.\frac{\cos\frac{2\pi}{T_o}kt}{\frac{2\pi}{T_o}k}\right|_0^{T_o}$$

$$= \frac{\sin\frac{2\pi}{T_o}kT_o - \sin\frac{2\pi}{T_o}k0}{\frac{2\pi}{T_o}k} - j\frac{\cos\frac{2\pi}{T_o}kT_o - \cos\frac{2\pi}{T_o}k0}{\frac{2\pi}{T_o}k}$$

$$= \frac{\sin 2\pi k - \sin 0}{\frac{2\pi}{T_o}k} - j\frac{\cos 2\pi k - \cos 0}{\frac{2\pi}{T_o}k}$$

$$= \frac{0-0}{\frac{2\pi}{T_o}k} - j\frac{1-1}{\frac{2\pi}{T_o}k} = 0 \text{ for } k \neq 0$$

for $k = 0$

$$\int_0^{T_o} e^{j\frac{2\pi}{T_o}kt}dt = \int_0^{T_o}\cos\frac{2\pi}{T_o}ktdt + j\int_0^{T_o}\sin\frac{2\pi}{T_o}ktdt = \int_0^{T_o}\cos\frac{2\pi}{T_o}0tdt + j\int_0^{T_o}\sin\frac{2\pi}{T_o}0tdt = \int_0^{T_o}1dt + j\int_0^{T_o}0dt = \int_0^{T_o}1dt = t\,|_0^{T_o} = T_o$$

Sinusoids and Orthogonality

- Now let's check sinusoids and complex exponential function for orthogonality:

$$\int_0^{T_o} v_k(t)v_l^*(t)dt = \int_0^{T_o} e^{j\frac{2\pi}{T_o}kt} e^{-j\frac{2\pi}{T_o}lt} dt$$

$$= \int_0^{T_o} e^{j\frac{2\pi}{T_o}(k-l)t} dt$$

$$= 0$$

$$\text{if } k \neq l$$

$$\int_0^{T_o} v_k(t)v_l^*(t)dt = \int_0^{T_o} e^{j\frac{2\pi}{T_o}kt} e^{-j\frac{2\pi}{T_o}lt} dt$$

$$= \int_0^{T_o} e^{j\frac{2\pi}{T_o}(k-l)t} dt = \int_0^{T_o} e^{j\frac{2\pi}{T_o}(l-l)t} dt$$

$$= \int_0^{T_o} e^{j0} dt = \int_0^{T_o} (\cos 0 + j\sin 0)dt$$

$$= \int_0^{T_o} (1 + j0)dt = \int_0^{T_o} dt$$

$$= T_o$$

$$\text{if } k = l$$

Proof of Fourier Series Integral

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T_o}kt}$$

$$x(t)e^{-j\frac{2\pi}{T_o}lt} = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T_o}kt} \times e^{-j\frac{2\pi}{T_o}lt} = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T_o}(k-l)t}$$

$$\int_0^{T_o} x(t)e^{-j\frac{2\pi}{T_o}lt}dt = \int_0^{T_o}\{\sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T_o}(k-l)t}\}dt$$

$$= \sum_{k=-\infty}^{\infty} a_k \int_0^{T_o} e^{j\frac{2\pi}{T_o}(k-l)t}dt$$

$$= \cdots + a_{l-1}\int_0^{T_o} e^{j\frac{2\pi}{T_o}(l-1-l)t}dt + a_l\int_0^{T_o} e^{j\frac{2\pi}{T_o}(l-l)t}dt + a_{l+1}\int_0^{T_o} e^{j\frac{2\pi}{T_o}(l+1-l)t}dt + \cdots$$

$$= \cdots + a_{l-1}0 + a_l T_o + a_{l+1}0 + \cdots$$

$$= a_l T_o \quad \Rightarrow \quad a_k = \frac{1}{T_o}\int_0^{T_o} x(t)e^{-j(\frac{2\pi}{T_o})kt}dt$$

Fourier Series

- Fourier Analysis:

$$a_k = \frac{1}{T_o}\int_0^{T_o} x(t)e^{-j(\frac{2\pi}{T_o})kt}dt$$

- Fourier Synthesis:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T_o}kt}$$

Example

- Find the spectrum of: $\sin^3(3\pi t)$

$$a_k = \frac{1}{T_o}\int_0^{T_o} \sin^3(3\pi t) e^{-j\frac{2\pi}{T_o}kt}\,dt$$

$$= \frac{1}{T_o}\int_0^{T_o} \left(\frac{e^{j3\pi t} - e^{-j3\pi t}}{2j}\right)^3 e^{-j\frac{2\pi}{T_o}kt}\,dt$$

$$= \frac{1}{T_o}\int_0^{T_o} \left(\frac{e^{j9\pi t} - 3e^{j6\pi t}e^{-j3\pi t} + 3e^{j3\pi t}e^{-j6\pi t} - e^{-j9\pi t}}{-8j}\right) e^{-j\frac{2\pi}{T_o}kt}\,dt$$

$$= \frac{1}{T_o}\int_0^{T_o} \left(\frac{e^{j9\pi t} - 3e^{j3\pi t} + 3e^{-j3\pi t} - e^{-j9\pi t}}{-8j}\right) e^{-j\frac{2\pi}{T_o}kt}\,dt$$

$$\omega_o = 3\pi; f_o = \frac{3\pi}{2\pi} = \frac{3}{2}; T_o = \frac{2}{3}$$

Example

- Find the spectrum of: $\sin^3(3\pi t)$

$$a_k = \frac{3}{2}\int_0^{2/3}\left(\frac{e^{j9\pi t} - 3e^{j3\pi t} + 3e^{-j3\pi t} - e^{-j9\pi t}}{-8j}\right)e^{-j\frac{2\pi}{2/3}kt}\,dt$$

$$= \frac{3}{2}\int_0^{2/3}\left(\frac{e^{j9\pi t} - 3e^{j3\pi t} + 3e^{-j3\pi t} - e^{-j9\pi t}}{-8j}\right)e^{-j3\pi kt}\,dt$$

$$= \frac{3}{2}\int_0^{2/3}\left(\frac{e^{j(9-3k)\pi t} - 3e^{j(3-3k)\pi t} + 3e^{-j(3+3k)\pi t} - e^{-j(9+3k)\pi t}}{-8j}\right)dt$$

$$= -\frac{3}{j16}\int_0^{2/3} e^{j(3-k)3\pi t}\,dt + \frac{9}{j16}\int_0^{2/3} e^{j(1-k)3\pi t}\,dt - \frac{9}{j16}\int_0^{2/3} e^{-j(1+k)3\pi t}\,dt + \frac{3}{j16}\int_0^{2/3} e^{-j(3+k)3\pi t}\,dt$$

Example

- Find the spectrum of: $\sin^3(3\pi t)$

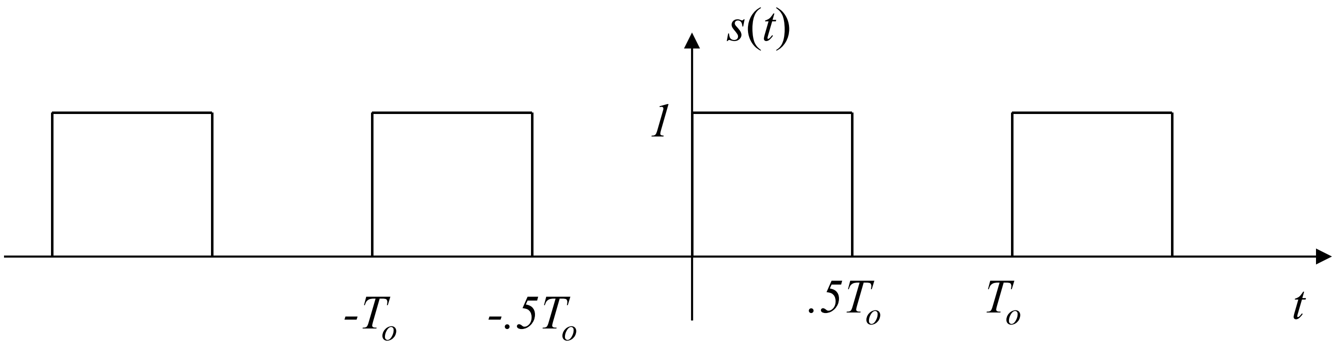
$$a_k = -\frac{3}{j16}\int_0^{2/3} e^{j(3-k)3\pi t}dt = -\frac{3}{j16}\int_0^{2/3} dt = -\frac{3}{j16}\times\frac{2}{3} = -\frac{1}{j8} = \frac{j}{8};\ \text{for } k=3; \Rightarrow a_3 = \frac{j}{8} = \frac{1}{8}e^{j\frac{\pi}{2}}$$

$$= \frac{9}{j16}\int_0^{2/3} e^{j(1-k)3\pi t}dt = \frac{9}{j16}\int_0^{2/3} dt = \frac{9}{j16}\times\frac{2}{3} = \frac{3}{j8} = -\frac{j3}{8};\ \text{for } k=1; \Rightarrow a_1 = -\frac{j3}{8} = \frac{3}{8}e^{-j\frac{\pi}{2}}$$

$$= -\frac{9}{j16}\int_0^{2/3} e^{j(1+k)3\pi t}dt = -\frac{9}{j16}\int_0^{2/3} dt = \frac{9}{j16}\times\frac{2}{3} = \frac{3}{j8} = \frac{j3}{8};\ \text{for } k=-1; \Rightarrow a_{-1} = \frac{j3}{8} = \frac{3}{8}e^{j\frac{\pi}{2}}$$

$$= \frac{3}{j16}\int_0^{2/3} e^{j(3+k)3\pi t}dt = \frac{3}{j16}\int_0^{2/3} dt = \frac{3}{j16}\times\frac{2}{3} = \frac{1}{j8} = -\frac{j}{8};\ \text{for } k=-3; \Rightarrow a_{-3} = -\frac{j}{8} = \frac{1}{8}e^{-j\frac{\pi}{2}}$$

Fourier Series of a Square Wave



$$s(t) = \begin{cases} 1 & \text{for } 0 \le t < \frac{1}{2}T_o \\ 0 & \text{for } \frac{1}{2}T_o \le t \le T_o \end{cases}$$

Fourier Series of a Square Wave Analysis

$$a_k = \frac{1}{T_o}\int_0^{T_o} s(t) e^{-j(\frac{2\pi}{T_o})kt} dt$$

$$= \frac{1}{T_o}\int_0^{T_o/2} 1 e^{-j(\frac{2\pi}{T_o})kt} dt + \frac{1}{T_o}\int_{T_o/2}^{T_o} 0 e^{-j(\frac{2\pi}{T_o})kt} dt$$

$$= \frac{1}{T_o}\int_0^{T_o/2} 1 e^{-j(\frac{2\pi}{T_o})kt} dt = \frac{1}{T_o}\frac{e^{-j(\frac{2\pi}{T_o})kt}}{-j(\frac{2\pi}{T_o})k}\Big|_0^{T_o/2}$$

$$= \frac{1}{T_o}\frac{e^{-j(\frac{2\pi}{T_o})k\frac{T_o}{2}} - e^{-j(\frac{2\pi}{T_o})k0}}{-j(\frac{2\pi}{T_o})k} = \frac{1-e^{-j\pi k}}{j2\pi k}$$

$$= \frac{1-(-1)^k}{j2\pi k}; k \neq 0$$

$$e^{-j\pi k} = \cos(-\pi k) + j\sin(-\pi k) = (-1)^k + j0 = (-1)^k$$

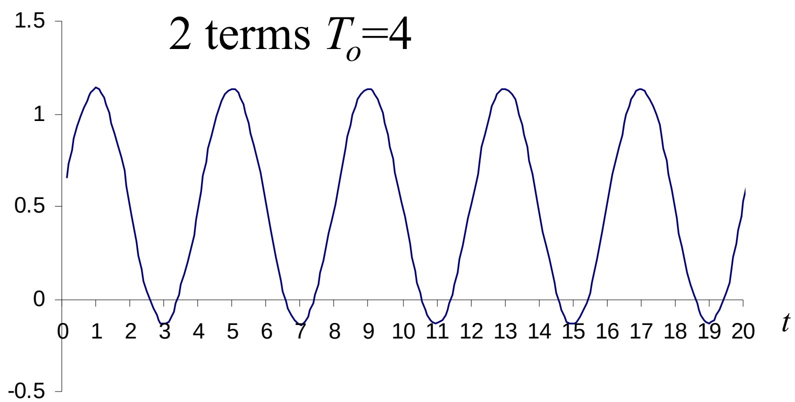
$$a_k = \frac{1-(-1)^k}{j2\pi k}; k \neq 0$$

$$a_k = \frac{1-(-1)}{j2\pi k} = \frac{1}{j\pi k} = \frac{1}{\pi k} e^{-j\frac{\pi}{2}}; k \ \text{ odd}$$

$$= \frac{1-(1)}{j2\pi k} = 0; k \ \text{ even}$$

$$a_0 = \frac{1}{T_o}\int_0^{T_o} s(t)dt = \frac{1}{T_o}\int_0^{T_o/2} 1dt = \frac{1}{2}$$

Fourier Series of a Square Wave Synthesis

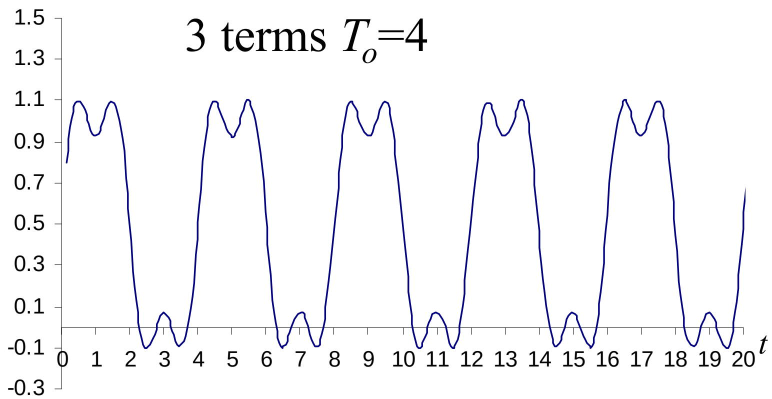


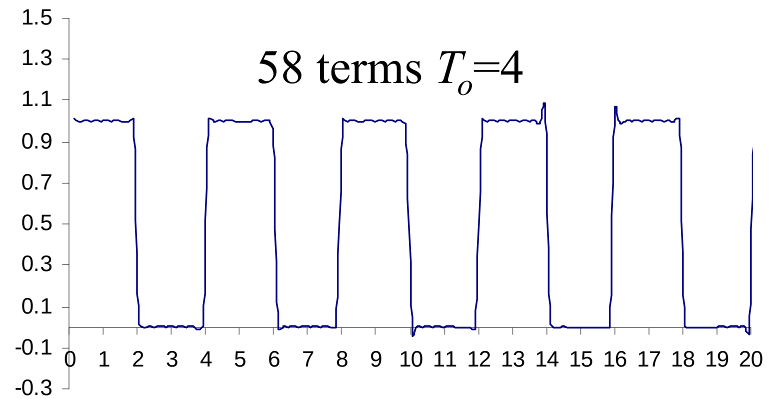
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T_o}kt} = a_0 + \sum_{k=1}^{\infty} 2\Re e\{a_k e^{j\frac{2\pi}{T_o}kt}\}$$

$$= \frac{1}{2} + 2\sum_{k=odd}^{\infty} \Re e\{\frac{e^{-j\frac{\pi}{2}}}{\pi k} e^{j\frac{2\pi}{T_o}kt}\}$$

$$= \frac{1}{2} + 2\sum_{k=odd}^{\infty} \Re e\{\frac{1}{\pi k} e^{j(\frac{2\pi}{T_o}kt - \frac{\pi}{2})}\}$$

$$= \frac{1}{2} + 2\sum_{k=odd}^{\infty} \frac{1}{\pi k}\cos(\frac{2\pi}{T_o}kt - \frac{\pi}{2})$$





Spectrum of a Square Wave

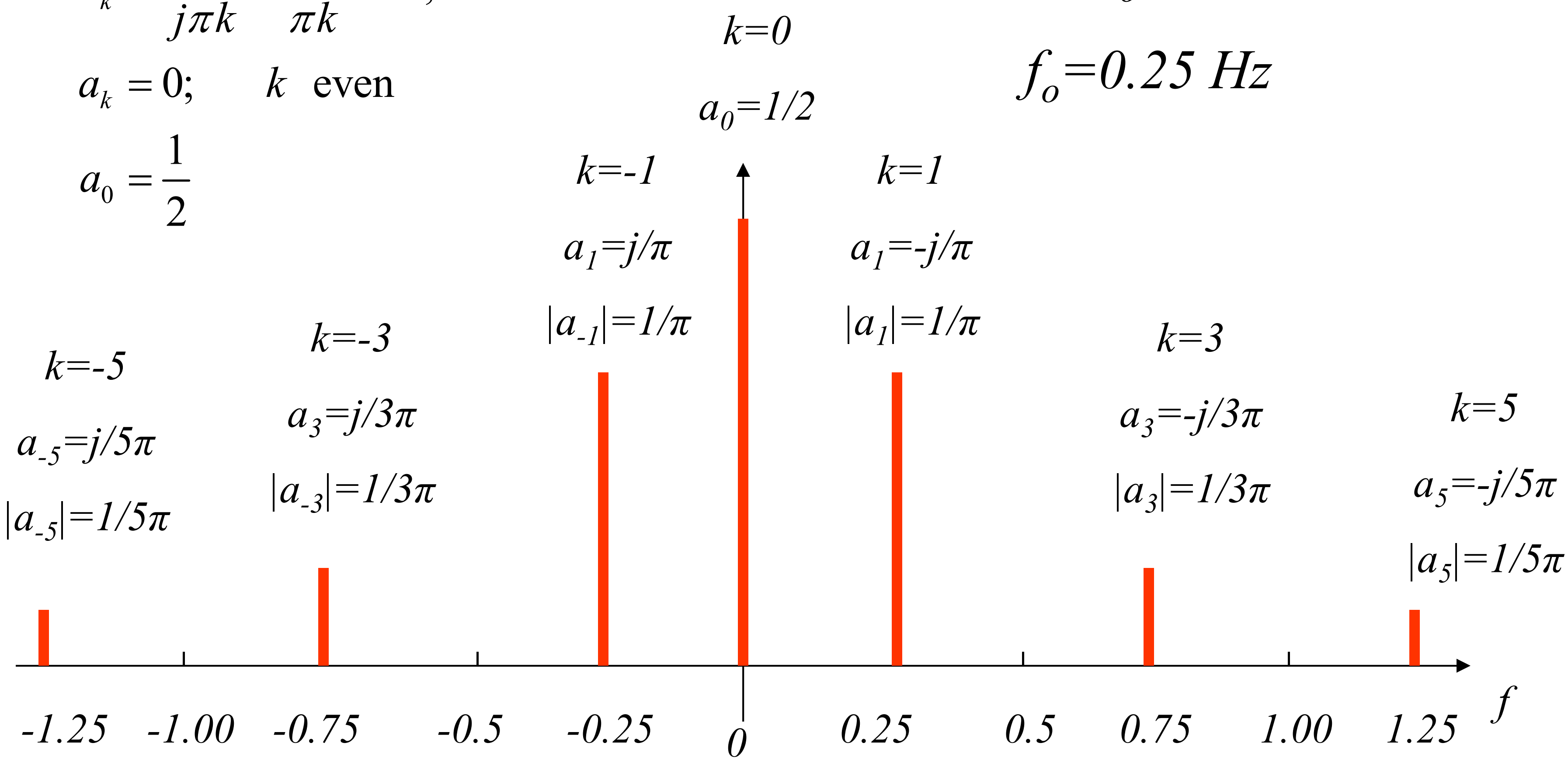
$$a_k = \frac{1}{j\pi k} = \frac{1}{\pi k} e^{-j\frac{\pi}{2}}; \quad k \ \text{odd}$$

$$a_k = 0; \qquad k \ \text{even}$$

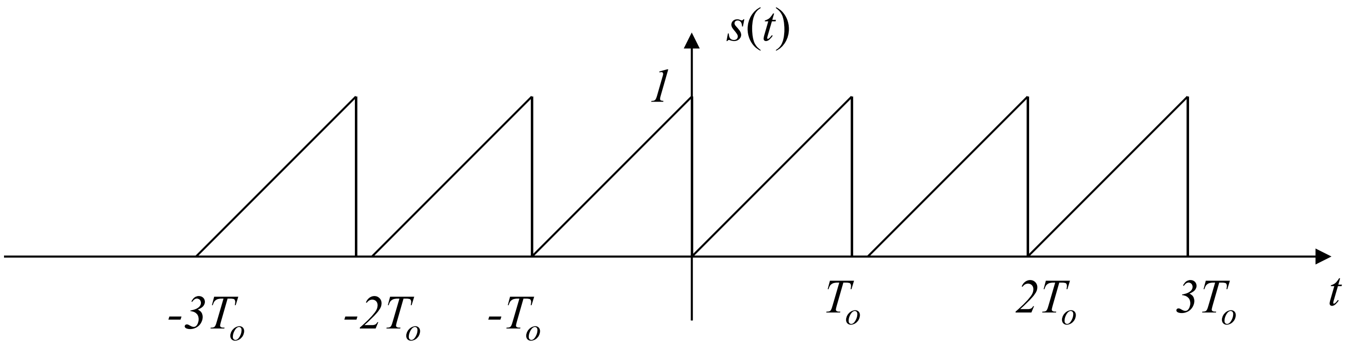
$$a_0 = \frac{1}{2}$$

$T_o = 4$

$f_o = 0.25\ Hz$



Fourier Series of a Sawtooth Wave



$$s(t) = t / T_o \qquad \text{for } 0 \le t \le T_o$$

Fourier Series of a Sawtooth Wave Analysis

$$a_k = \frac{1}{3}\int_0^3 s(t)e^{-j2\pi kt/3}dt = \frac{1}{3}\int_0^3 \frac{t}{3}e^{-j\frac{2\pi}{3}kt}dt$$

$$u = \frac{t}{3}, du = \frac{1}{3}dt$$

$$dv = e^{-j\frac{2\pi}{3}kt}dt, v = \frac{1}{-j2\pi k/3}e^{-j\frac{2\pi}{3}kt}$$

$$a_k = \frac{1}{3}\int_0^3 \frac{t}{3}e^{-j\frac{2\pi}{3}kt}dt$$

$$= \frac{1}{9}[\frac{t}{-j2\pi k/3}e^{-j\frac{2\pi}{3}kt}\Big|_0^3 - \int_0^3 (\frac{1}{-j2\pi k/3})e^{-j2\pi kt/3}dt]$$

$$= \frac{1}{9}[\frac{1}{-j2\pi k/3}\{3e^{-j2\pi k} - 0\} - (\frac{1}{-j2\pi k/3})(\frac{1}{-j2\pi k/3}\{e^{-j2\pi k} - 1\}$$

$$\text{since } e^{-j2\pi k} = \cos(-2\pi k) + j\sin(-2\pi k) = 1 + j0$$

Fourier Series of a Sawtooth Wave Analysis

$$a_k = \frac{1}{9}[\frac{1}{-j2\pi k/3}\{3\times 1\} - (\frac{1}{-j2\pi k/3})(\frac{1}{-j2\pi k/3}\{1-1\}$$

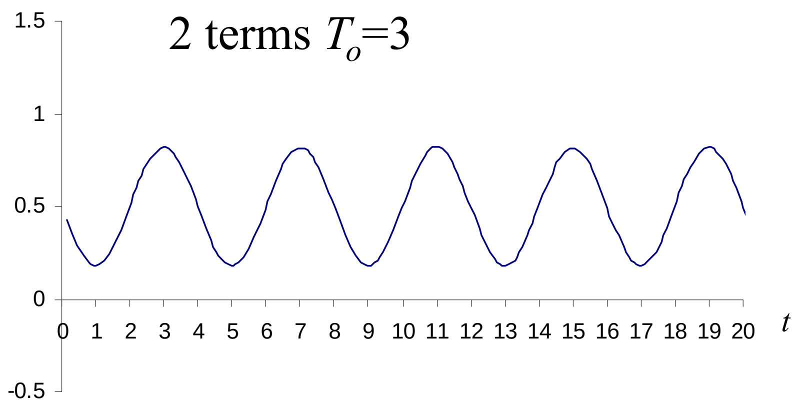
$$= \frac{1}{9}[\frac{1}{-j2\pi k/3}\{3\times 1\} - 0$$

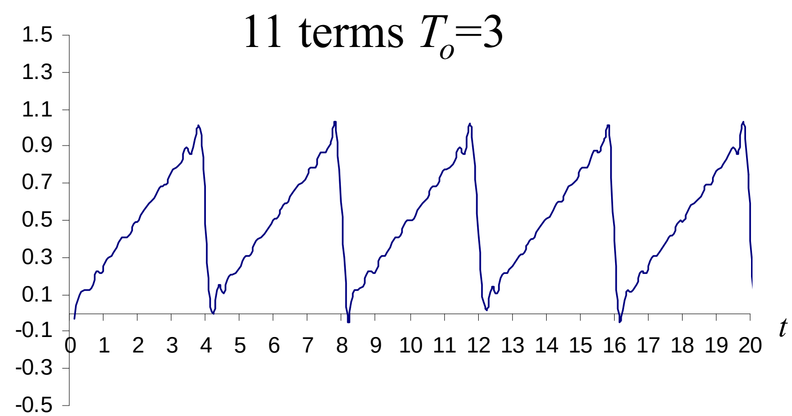
$$= \frac{1}{3}\frac{1}{(-j2\pi k/3)} = j\frac{1}{2\pi k} = \frac{1}{2\pi k}e^{j\frac{\pi}{2}}; k \neq 0$$

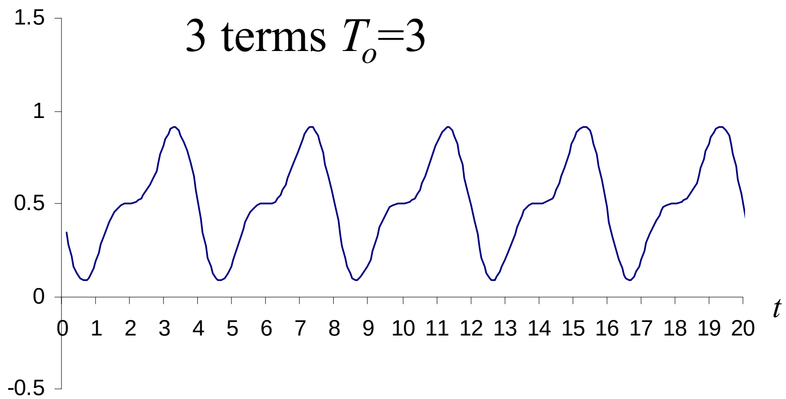
$$a_0 = \frac{1}{3}\int_0^3 f(t)dt = \frac{1}{3}\int_0^3 \frac{t}{3}dt = (\frac{1}{9})\frac{9}{2} = 0.5$$

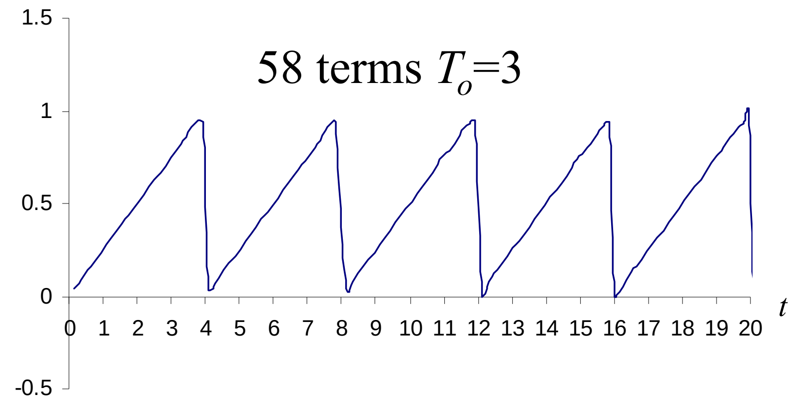
$$s(t) = \frac{1}{2} + 2\sum_1^\infty \frac{1}{2\pi k}\cos(\frac{2\pi}{3}kt + \frac{\pi}{2})$$

Fourier Series of a Sawtooth Wave Synthesis









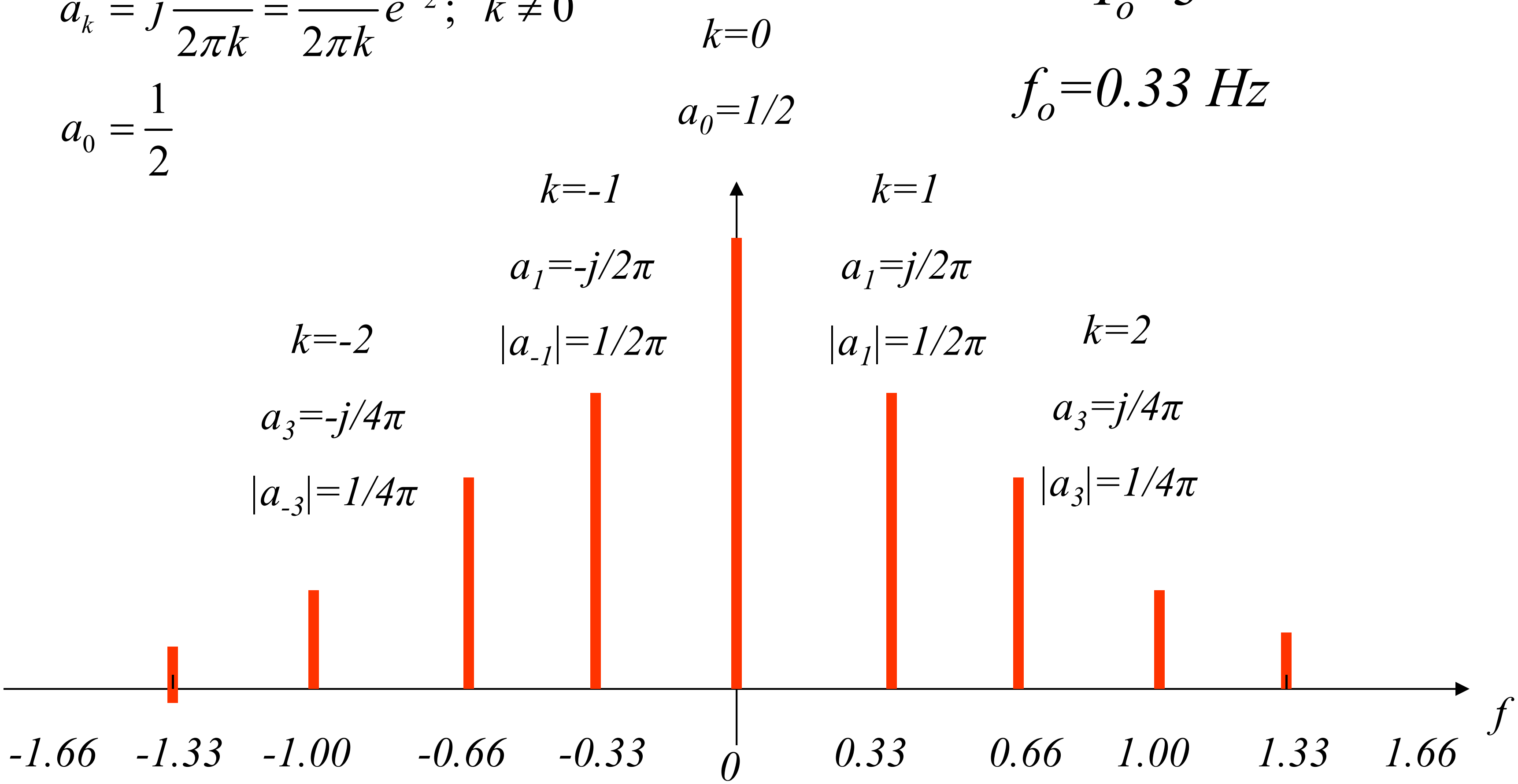
Spectrum of a Sawtooth Wave

$$a_k = j\frac{1}{2\pi k} = \frac{1}{2\pi k}e^{j\frac{\pi}{2}}; \quad k \neq 0$$

$$a_0 = \frac{1}{2}$$

$T_o=3$

$f_o=0.33\ Hz$

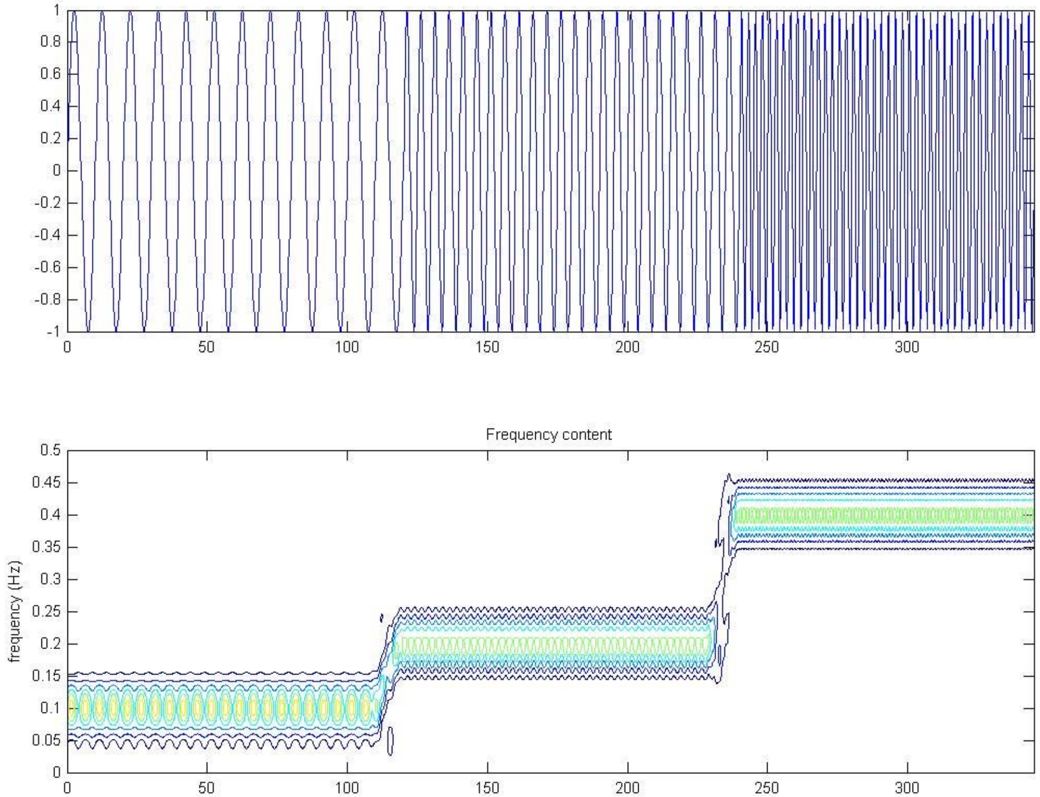


Time-Frequency Spectrum

- Up till now we have seen signals whose frequencies do not change with time.
- However, in reality, signals can produce different frequencies at different times
  - Music
  - Voice
  - Frequency Modulation
- To plot the spectrum of such signals, we can use a 3-D plot called a spectrogram.
  - Plots time on the x-axis, frequency on the y-axis and magnitude on the z-axis (out of the plane of the paper)

Example of a Spectrogram

- Three sinusoids (0.1 Hz, 0.2 Hz, 0.4Hz) in sequence



Frequency Modulation

- We have signals of the type:

$$x(t) = \Re e\{Ae^{j(\omega_o t+\theta)}\} = A\cos(\omega_o t + \theta)$$

where the angle (of the cosine) varies linearly with time and the time derivative of the angle is ω_o, constant

- However, we can general this signal such that the angle varies with time such that its derivative is not constant.

$$x(t) = \Re e\{Ae^{j\psi(t)}\} = A\cos(\psi(t))$$

$$\omega(t) = \frac{d}{dt}\psi(t) \quad \text{Instantaneous Frequency}$$

FM Radio

- Frequency Modulation is the scheme used in FM broadcast.
- For example, if $\omega(t)=\omega_c+m(t)$ where ω_c is called the carrier frequency and $m(t)$ contains the "information" (voice or music), then FM broadcast is:

$$\psi(t)=\int\{\omega_c+m(t)\}dt=\omega_c t+\int m(t)dt$$

Homework

- Exercises:
  - 3.4 – 3.8
- Problems:

–3.5 Instead use $x(t) = 10 + 20\cos(2\pi(200)t + \frac{1}{4}\pi) + 10\cos(2\pi(250)t)$

–3.8,

–3.9 Use Matlab for plotting the spectrum; submit your code

–3.12 Instead use $x(t) = \begin{cases} 0 & \text{for } 0 < t \le 2.5 \\ 2 & \text{for } 2.5 < t \le 5 \end{cases}$ Use Matlab for plotting the spectrum; submit your code

–3.13, –3.14