FIR Filters

Lecture #8

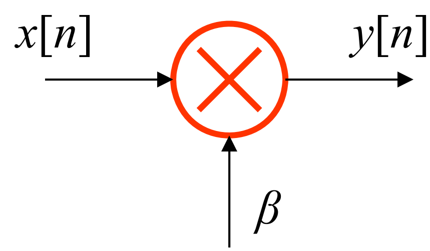
Chapter 5

Implementation of FIR

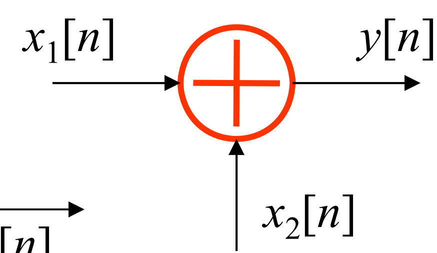
- Representing an FIR Filters using a Block Diagrams

$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k]$$

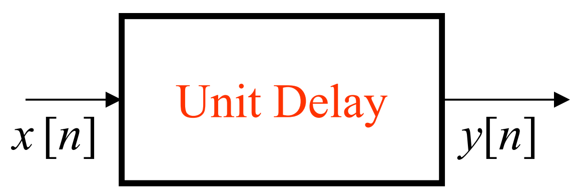
- Building Blocks
  - Multiplier: $y[n]=\beta\, x[n]$



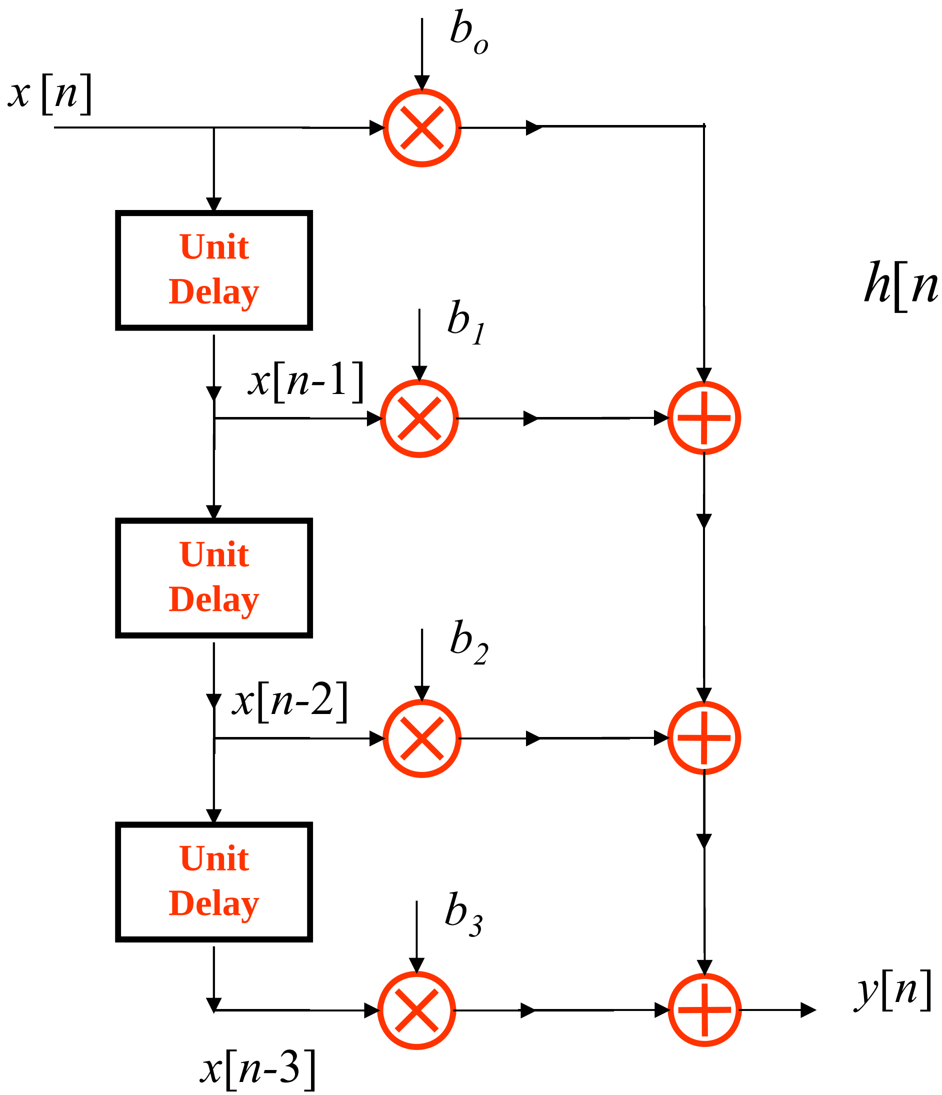
  - Adder: $y[n]= x_1[n] + x_2[n]$



  - Unit Delay



Direct Form FIR



$$h[n] = \sum_{k=0}^{3} b_k \delta[n-k]$$

$$= (((b_0 x[n] + b_1 x[n-1])$$

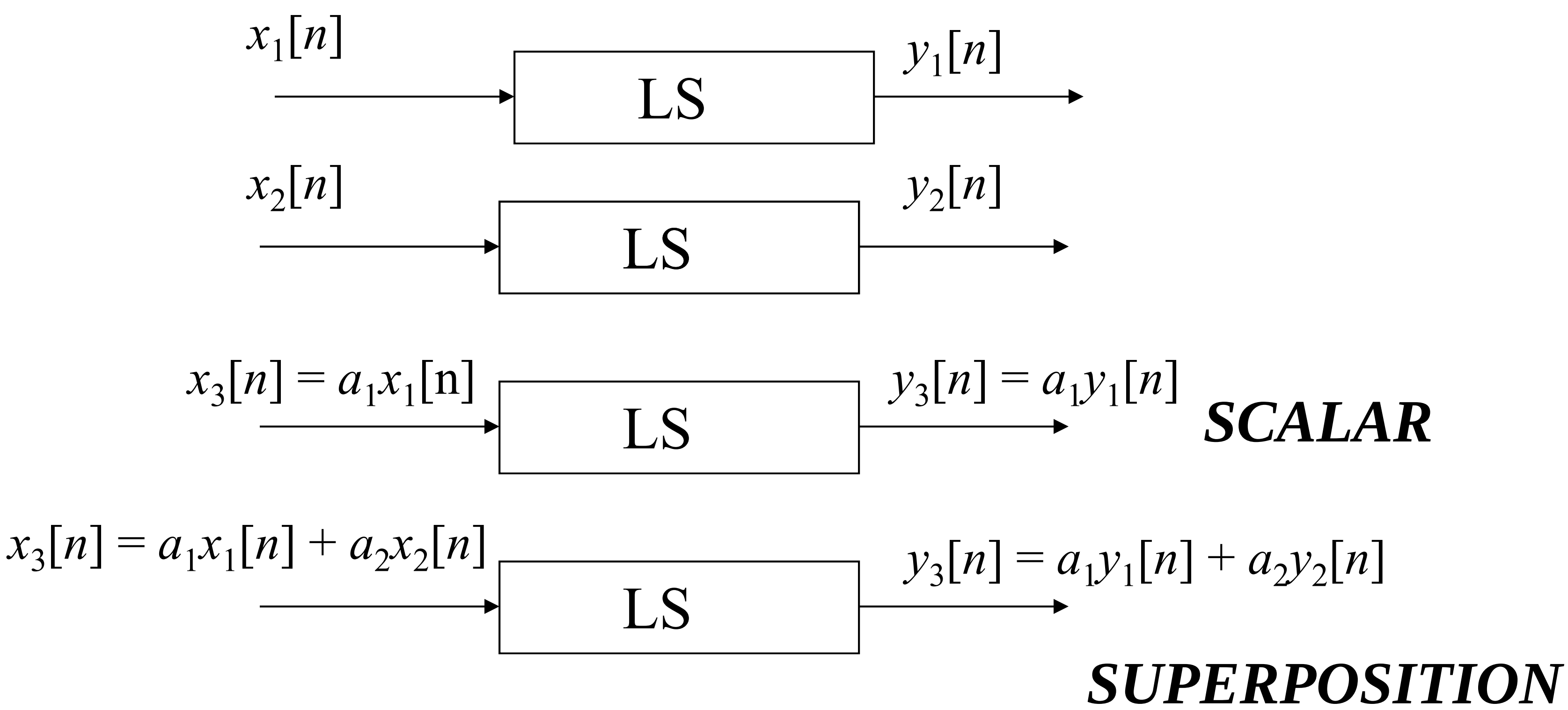
$$+ b_2 x[n-2])$$

$$+ b_3 x[n-3])$$

Some Basic Properties of Linear Systems

- If a system is Linear, or better yet Linear and Time Invariant (LTI), it is easier to analyze and understand than systems that are non-linear and/or vary with time.
- All LTI systems must be
  - Linear and support superposition
  - Causal
  - Time Invariant

Linear Systems



Shorthand

$$x_k[n] \longrightarrow y_k[n]$$

$$\Sigma_k \, a_k x_k[n] \longrightarrow \Sigma_k \, a_k y_k[n]$$

Time Invariance

$$x_k[n] \longmapsto y_k[n]$$

Delay $x[n]$ by n_0 yields same response only later

$$x_k[n - n_0] \longmapsto y_k[n - n_0]$$

Testing whether a system is LTI

- A system must meet all three criteria to be LTI
- General test procedure:
  1. Take input and apply the criteria and send it through the system
  2. Compare this result with output if the system is LTI
  3. If they are the same then system meets that requirement.

Example

- Is the squarer system, $y[n]=(x[n])^2$, LTI?
- First test Time Invariant:

$$x_k[n - n_0] \longmapsto y_k[n - n_0]$$

$$x_k[n\ ] \longmapsto y_k[n\ ] = (x[n])^2$$

$$x_k[n - n_0] \longmapsto (x[n - n_0])^2 = y_k[n - n_0\ ]$$

OK

Example Continued

- Next test Linear:

$$x_3[n] = a_1x_1[n] + a_2x_2[n] \longmapsto y_3[n] = a_1y_1[n] + a_2y_2[n]$$

$$x_k[n\ ] \longmapsto y_k[n\ ] = (x[n])^2$$

$$a_1x_1[n] + a_2x_2[n] \longmapsto (a_1x_1[n] + a_2x_2[n])^{\,2}$$

$$= (a_1x_1[n])^{\,2} + (a_2x_2[n])^{\,2} + 2\ a_1x_1[n]\ a_2x_2[n]$$

$$\neq a_1y_1[n] + a_2y_2[n] = a_1(x_1[n])^2 + a_2(x_2[n])^2$$

NOT OK

FIR

- Is the FIR, $y[n] = \sum_{k=0}^{M} b_k x[n-k]$, LTI?
- First test Time Invariant: OK

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

$$x[n-n_o] \mapsto \sum_{k=0}^{M} b_k x[(n-n_o)-k]$$

$$= y[n-n_o] = \sum_{k=0}^{M} b_k x[n-n_o-k]$$

FIR Continued

- Next test Linear:

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

$$a_1 x_1[n] + a_2 x_2[n] \mapsto \sum_{k=0}^{M} b_k (a_1 x_1[n-k] + a_2 x_2[n-k])$$

$$= \sum_{k=0}^{M} b_k (a_1 x_1[n-k]) + \sum_{k=0}^{M} b_k (a_2 x_2[n-k])$$

$$= a_1 \sum_{k=0}^{M} b_k x_1[n-k] + a_2 \sum_{k=0}^{M} b_k x_2[n-k]$$

$$= a_1 y_1[n] + a_2 y_2[n]$$

OK ANY FIR is LTI!!!!

Convolution and FIRs

- The general expression for FIR filters can be derived in terms of the unit impulse response
- This expression uses the mathematical operation called convolution.
- Convolution is defined as:

$$g[n]=\Sigma\, y[m]\, x[n-m]$$

- We will see that the more powerful expression to relate the output of an FIR filter to any input using the unit impulse response is:

$$y[n]=\Sigma\, x[m]\, h[n-m]$$

Convolution and the FIR

- Recall

$$x[n] = \sum_{l} x[l]\delta[n-l] = \cdots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \cdots$$

- But the definition of the unit impulse response:

$$\delta[n] \mapsto h[n]$$

$$\delta[n-1] \mapsto h[n-1]$$

$$\delta[n-2] \mapsto h[n-2]$$

$$\delta[n-l] \mapsto h[n-l] \text{ for any } l$$

Convolution and the FIR

- Using scalar property

$$x[0]\delta[n] \mapsto x[0]h[n]$$

$$x[1]\delta[n-1] \mapsto x[1]h[n-1]$$

$$x[2]\delta[n-2] \mapsto x[2]h[n-2]$$

$$x[l]\delta[n-l] \mapsto x[l]h[n-l] \text{ for any } l$$

Convolution and the FIR

- And now superposition

$$
\begin{array}{cc}
\vdots & \\
+x[-1]\delta[n+1] & +x[-1]h[n+1] \\
+x[0]\delta[n] & +x[0]h[n] \\
+x[1]\delta[n-1] & +x[1]h[n-1] \\
\vdots & \\
= & = \\
\end{array}
$$

$$x[n] \mapsto \sum_l x[l]h[n-l]$$

$$x[n] \mapsto y[n]$$

Convolution and the FIR

$$y[n] = \sum_l x[l]h[n-l]$$

- This states all that is needed to determine the output of an FIR is
  - The unit-impulse response
  - The desired input
- Convolving the input and the unit-impulse response yields the output response of this input.
- Only need to solve the difference equation ONCE for the unit impulse sequence and you'll know the output for any other input sequence.

Properties of Convolution

- First some shorthand:

$$\sum_l x_1[l]x_2[n-l] \Rightarrow x_1[n] \otimes x_2[n]$$

- Commutative:

$$x_1[n] \otimes x_2[n] = x_2[n] \otimes x_1[n]$$

- Associative:

$$x_1[n] \otimes [x_2[n] \otimes x_3[n]] = [x_1[n] \otimes x_2[n]] \otimes x_3[n]$$

- Distributive:

$$x_1[n] \otimes [x_2[n] + x_3[n]] = [x_1[n] \otimes x_2[n]] + [x_1[n] \otimes x_3[n]]$$

Convolution and the FIR

$$y[n] = \sum_{l=-\infty}^{\infty} x[l]h[n-l]$$

$$h[n] = \sum_{k=0}^{M} h[k]\delta[n-k] = \sum_{k=0}^{M} b_k \delta[n-k]$$

$$h[n-l] = \sum_{k=0}^{M} b_k \delta[(n-l)-k]$$

$$y[n] = \sum_{l=-\infty}^{\infty} x[l]h[n-l] = \sum_{l=-\infty}^{\infty} x[l] \sum_{k=0}^{M} b_k \delta[(n-l)-k] = \sum_{l=-\infty}^{\infty} \sum_{k=0}^{M} x[l] b_k \delta[(n-l)-k]$$

$$= \sum_{k=0}^{M} \sum_{l=-\infty}^{\infty} x[l] b_k \delta[(n-l)-k]$$

Since

$\delta[(n-l)-k] = 1$ when $(n-l)-k = 0$ or $l = n-k$; and $\delta[(n-l)-k] = 0$ elsewhere

Then, each of the $M+1$ summations reduces to a single term.

For example, lets look at the 3 consective points $n-k-1, n-k, n-k+1$

$$\sum_{l=-\infty}^{\infty} x[l] b_k \delta[(n-l)-k] = \cdots + x[n-k-1] b_k \delta[(n-(n-k-1))-k]$$

$$+ x[n-k] b_k \delta[(n-(n-k))-k]$$

$$+ x[n-k+1] b_k \delta[(n-(n-k+1))-k] + \cdots$$

$$= \cdots + x[n-k-1] b_k \delta[1] + x[n-k] b_k \delta[0] + x[n-k+1] b_k \delta[-1] + \cdots$$

$$= x[n-k] b_k \delta[0] = x[n-k] b_k = b_k x[n-k]$$

$$y[n] = \sum_{k=0}^{M} \sum_{l=-\infty}^{\infty} x[l] b_k \delta[(n-l)-k] = \sum_{k=0}^{M} b_k x[n-k]$$

This becomes the difference equation.

Convolution
Mathematical Approach

$$y[n]=\sum_{-\infty}^{\infty}x[l]h[n-l]$$

Given:

$$x[n]=1 \quad n=0,1,$$
$$=2 \quad n=2,3$$
$$h[n]=(\delta[n]+\delta[n-1]+\delta[n-2])$$

$$h[n-l]=(\delta[n-l]+\delta[n-1-l]+\delta[n-2-l])$$

$$y[n]=\sum_{-\infty}^{\infty}x[l]h[n-l]=\sum_{-\infty}^{\infty}x[l](\delta[n-l]+\delta[n-1-l]+\delta[n-2-l])$$

$$=\sum_{-\infty}^{\infty}x[l]\delta[n-l]+\sum_{-\infty}^{\infty}x[l]\delta[n-1-l]+\sum_{-\infty}^{\infty}x[l]\delta[n-2-l])$$

Since $x[l]$ is nonzero for when l is $0,1,2,3$ we have

$$=x(0)\delta[n-0]+x(1)\delta[n-1]+x(2)\delta[n-2]+x(3)\delta[n-3]$$
$$+x(0)\delta[n-1-0]+x(1)\delta[n-1-1]+x(2)\delta[n-1-2]+x(3)\delta[n-1-3]$$
$$+x(0)\delta[n-2-0]+x(1)\delta[n-2-1]+x(2)\delta[n-2-2]+x(3)\delta[n-2-3]$$
$$=(\delta[n]+\delta[n-1]+2\delta[n-2]+2\delta[n-3])$$
$$+(\delta[n-1]+\delta[n-2]+2\delta[n-3]+2\delta[n-4])$$
$$+(\delta[n-2]+\delta[n-3]+2\delta[n-4]+2\delta[n-5])$$
$$=(\delta[n]+2\delta[n-1]+4\delta[n-2]+5\delta[n-3]+4\delta[n-4]+2\delta[n-5])$$

Convolution

Tabular Approach

$$y[n] = \sum_{-\infty}^{\infty} x[l]h[n-l]$$

$$x[l] = 1 \quad l = 0,1,$$
$$= 2 \quad l = 2,3$$
$$h[l] = (\delta[l] + \delta[l-1] + \delta[l-2])$$
$$h[n-l] = (\delta[n-l] + \delta[n-1-l] + \delta[n-2-l])$$

- Note that the first row contains the time, *l*.
- The second row contains the values of h[l]
- The third row contains the $x[l]$ values $x[l]$ = 1 for l=0,1 & $x[l]$ = 2 for l=2,3
- The subsequent rows (columns 3 to 14) are the values of the convoluted (flipped l => $-l$ and shifted by n) version of $h[l]$, $h[n-l]$,where the values of n are in the first column and the values of l are in the first row.
- The values of $y[n]$ are found in column 2.

		l	-4	-3	-2	-1	0	1	2	3	4	5	6	7
		h(l)	0	0	0	0	1	1	1	0	0	0	0	0
n	y[n]	x[l]	0	0	0	0	1	1	2	2	0	0	0	0
-2	0	h(n-l]	1	1	1	0	0	0	0	0	0	0	0	0
-1	0	h(n-l]	0	1	1	1	0	0	0	0	0	0	0	0
0	1	h(n-l]	0	0	1	1	1	0	0	0	0	0	0	0
1	2	h(n-l]	0	0	0	1	1	1	0	0	0	0	0	0
2	4	h(n-l]	0	0	0	0	1	1	1	0	0	0	0	0
3	5	h(n-l]	0	0	0	0	0	1	1	1	0	0	0	0
4	4	h(n-l]	0	0	0	0	0	0	1	1	1	0	0	0
5	2	h(n-l]	0	0	0	0	0	0	0	1	1	1	0	0
6	0	h(n-l]	0	0	0	0	0	0	0	0	1	1	1	0

Difference Equation

Mathematical Approach

$$y[n] = \sum_{k=0}^{2} b_k x[n-k]$$

$$x[n] = 1 \quad n = 0,1,$$

$$= 2 \quad n = 2,3$$

$$x[n] = \delta[n] + \delta[n-1] + 2\delta[n-2] + 2\delta[n-3]$$

$$x[n-k] = \delta[n-k] + \delta[n-k-1] + 2\delta[n-k-2] + 2\delta[n-k-3]$$

$$b_k = \{1,1,1\}$$

$$y[n] = \sum_{k=0}^{2} b_k x[n-k] = x[n-0] + x[n-1] + x[n-2]$$

$$= \delta[n-0] + \delta[n-0-1] + 2\delta[n-0-2] + 2\delta[n-0-3]$$

$$+ \delta[n-1] + \delta[n-1-1] + 2\delta[n-1-2] + 2\delta[n-1-3]$$

$$+ \delta[n-2] + \delta[n-2-1] + 2\delta[n-2-2] + 2\delta[n-2-3]$$

$$= \delta[n] + \delta[n-1] + 2\delta[n-2] + 2\delta[n-3]$$

$$+ \delta[n-1] + \delta[n-2] + 2\delta[n-3] + 2\delta[n-4]$$

$$+ \delta[n-2] + \delta[n-3] + 2\delta[n-4] + 2\delta[n-5]$$

$$= (\delta[n] + 2\delta[n-1] + 4\delta[n-2] + 5\delta[n-3] + 4\delta[n-4] + 2\delta[n-5])$$

Difference Equation
Tabular Approach

$$y[n] = \sum_{k=0}^{2} b_k x[n-k]$$

$$x[n] = 1 \quad n = 0,1,$$

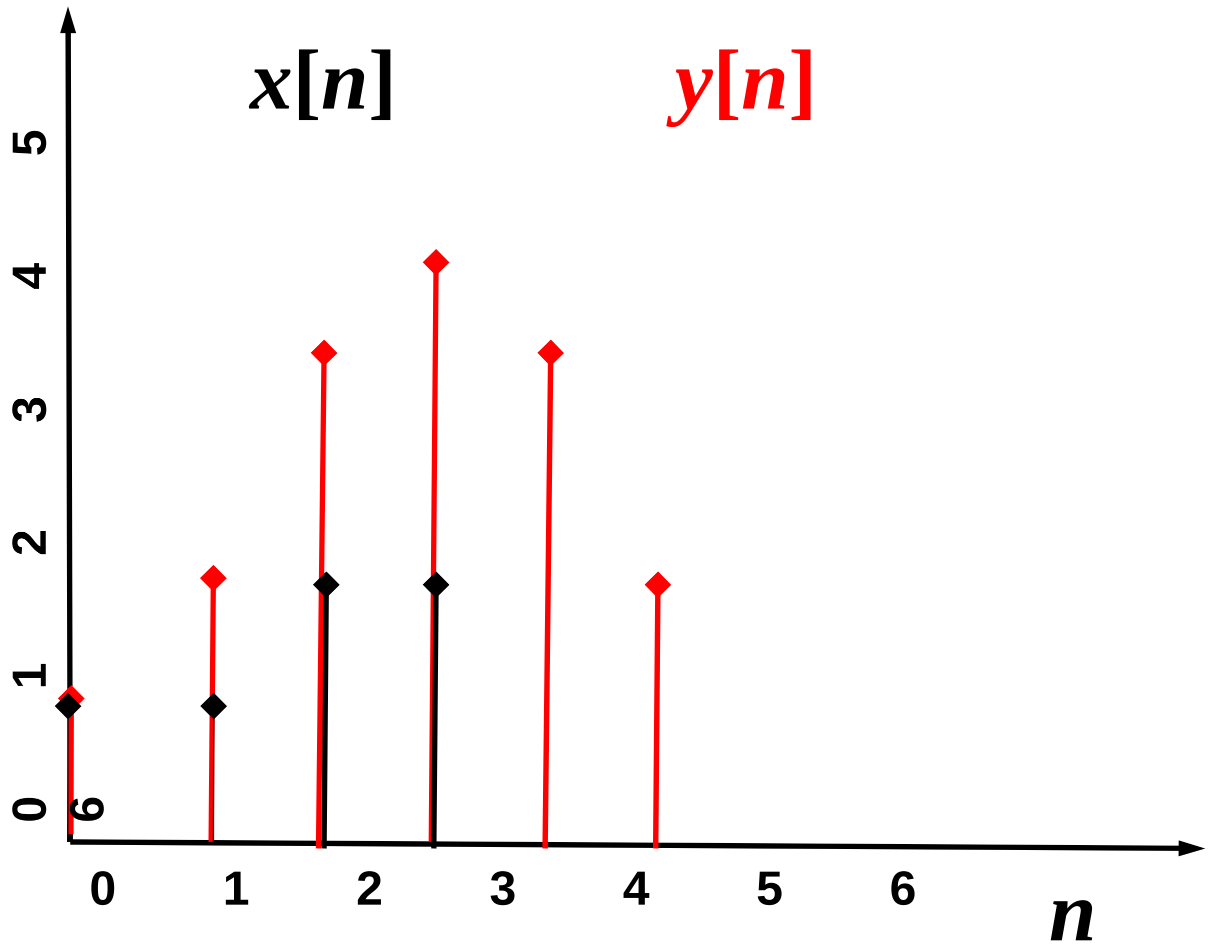
$$= 2 \quad n = 2,3$$

$$b_k = \{1,1,1\}$$

$$y[n] = \sum_{k=0}^{2} b_k x[n-k] = x[n-0] + x[n-1] + x[n-2]$$

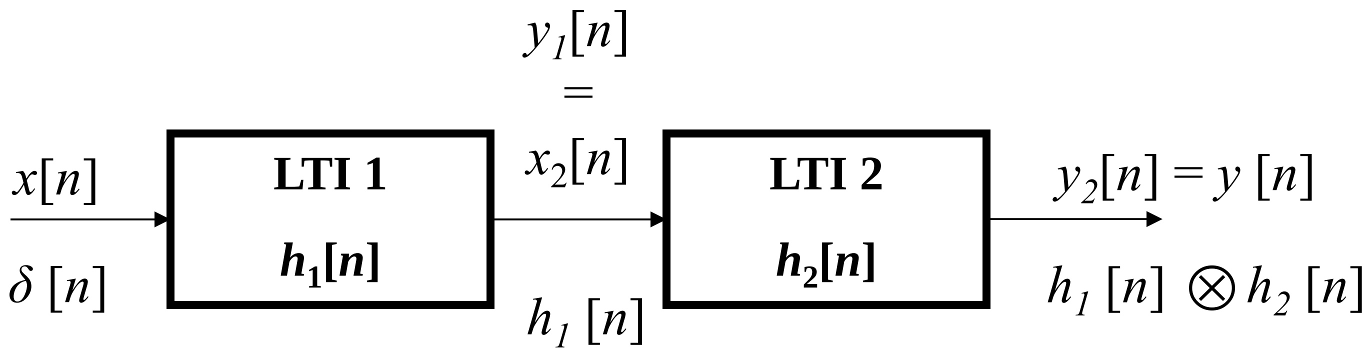
n	x[n]	y[n]
-3	0	0
-2	0	0
-1	0	0
0	1	1
1	1	2
2	2	4
3	2	5
4	0	4
5	0	2
6	0	0
7	0	0
8	0	0
9	0	0

Plot of Example



Cascaded System

- Two systems where the output of the first is the input to the second.



- This states that a cascaded system can be replaced a single system with unit impulse response, $h[n]=h_1[n]\otimes h_2[n]$

Homework

- Exercises:
  - 5.6 – 5.11
- Problems:

$$-5.7$$

$$-5.8 \text{ Instead use } \{b_k\} = \{13, 13, 13\}$$

$$-5.9, -5.11, -5.12, -5.13, -5.14$$