Frequency Response of FIR Filters

Lecture #9

Chapter 6

What Is this Course All About ?

- To Gain an Appreciation of the Various Types of Signals and Systems
- To Analyze The Various Types of Systems
- To Learn the Skills and Tools needed to Perform These Analyses.
- To Understand How Computers Process Signals and Systems

Frequency Response

- Study FIR systems, where the input is a discrete-time sinusoidal signal, in particular, a discrete-time complex exponential signal with normalized radian frequency $\hat{\omega}$
- Let's substitute this signal into the general expression of a FIR.

$$x(t) = Ae^{j\theta}e^{j\omega t}$$

$$x[n] = x(nT_s) = Ae^{j\theta}e^{j\omega nT_s}; \quad \hat{\omega} = \omega T_s$$

$$= Ae^{j\theta}e^{j\hat{\omega}n} \quad -\infty < n < \infty$$

Frequency Response Function

$$x[n] = Ae^{j\theta}e^{j\hat{\omega}n}$$

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

$$= \sum_{k=0}^{M} b_k Ae^{j\theta}e^{j\hat{\omega}(n-k)}$$

$$y[n] = (\sum_{k=0}^{M} b_k e^{-j\hat{\omega}k})Ae^{j\theta}e^{j\hat{\omega}n}$$

$$= H(\hat{\omega})Ae^{j\theta}e^{j\hat{\omega}n}$$

$$H(\hat{\omega}) = H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$

- We define this function $H(e^{j\hat{\omega}})$ as the Frequency Response Function or the Frequency Response

Frequency Response Function Another Approach using Convolution

$$x[n] = Ae^{j\theta}e^{j\hat{\omega}n} \Rightarrow x[n-l] = Ae^{j\theta}e^{j\hat{\omega}(n-l)}$$

$$h[n] = \sum_{k=0}^{M} h[k]\delta[n-k]$$

$$y[n] = \sum_{l} h[l]x[n-l] = \sum_{l} h[l]Ae^{j\theta}e^{j\hat{\omega}(n-l)} = \sum_{l} h[l]Ae^{j\theta}e^{j\hat{\omega}n}e^{-j\hat{\omega}l}$$

$$= (\sum_{l} h[l]e^{-j\hat{\omega}l})Ae^{j\theta}e^{j\hat{\omega}n} = (\sum_{l}\sum_{k=0}^{M} h[k]\delta[l-k]e^{-j\hat{\omega}l})Ae^{j\theta}e^{j\hat{\omega}n}$$

Since $\delta[l-k] = 0$ for $l \neq k$ and

$\delta[l-k] = 1$ for $l = k$, then the outer summation can be removed.

$$= (\sum_{k=0}^{M} h[k]e^{-j\hat{\omega}k})Ae^{j\theta}e^{j\hat{\omega}n}$$

Frequency Response of a FIR

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k} = \sum_{k=0}^{M} h[k] e^{-j\hat{\omega}k}$$

1. When the input is a discrete-time complex exponential signal, the output of an FIR filter is also a discrete-time complex exponential signal with a different amplitude but same frequency
2. The Frequency Response Function only applies to inputs which are discrete-time complex exponential signals
3. The Frequency Response is a complex number which can be represented in Cartesian or polar form
4. Therefore, the amplitude of the output is product of the amplitude of the input signal times the amplitude of the Frequency Response (polar form)
5. The phase of the output is the sum of the phase of the input and the phase of the Frequency Response

Polar Form of the Frequency Response

$$H(e^{j\hat{\omega}}) = \left|H(e^{j\hat{\omega}})\right| e^{j\angle H(e^{j\hat{\omega}})}$$

$$y[n] = H(e^{j\hat{\omega}}) A e^{j\theta} e^{j\hat{\omega}n}$$

$$= \left|H(e^{j\hat{\omega}})\right| e^{j\angle H(e^{j\hat{\omega}})} A e^{j\theta} e^{j\hat{\omega}n}$$

$$= (A \times \left|H(e^{j\hat{\omega}})\right|) e^{j(\angle H(e^{j\hat{\omega}})+\theta)} e^{j\hat{\omega}n}$$

Note that this:

$$y[n] = H(e^{j\hat{\omega}}) x[n]$$

is only meaningful for

$$x[n] = A e^{j\theta} e^{j\hat{\omega}n}$$

An Example

Let $\{b_k\} = \{1, 2, 1\}$

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{2} b_k e^{-j\hat{\omega}k}$$

$$= 1 + 2e^{-j\hat{\omega}} + 1e^{-j\hat{\omega}2}$$

$$= e^{-j\hat{\omega}}(e^{j\hat{\omega}} + 2 + e^{-j\hat{\omega}})$$

$$= e^{-j\hat{\omega}}(2 + 2\cos\hat{\omega})$$

$$\left|H(e^{j\hat{\omega}})\right| = 2 + 2\cos\hat{\omega}$$

$$\angle H(e^{j\hat{\omega}}) = -\hat{\omega}$$

Example continued

For this system let $x[n] = 2e^{j\frac{\pi}{4}} e^{j\frac{\pi}{3}n}$

since $\hat{\omega} = \frac{\pi}{3}$,

$$H(e^{j\frac{\pi}{3}}) = e^{-j\frac{\pi}{3}}(2 + 2\cos\frac{\pi}{3})$$

$$= 3e^{-j\frac{\pi}{3}}$$

$$y[n] = H(e^{j\frac{\pi}{3}})x[n]$$

$$= H(e^{j\frac{\pi}{3}})2e^{j\frac{\pi}{4}} e^{j\frac{\pi}{3}n} = 3e^{-j\frac{\pi}{3}} 2e^{j\frac{\pi}{4}} e^{j\frac{\pi}{3}n}$$

$$= 6e^{-j(\frac{\pi}{3} - \frac{\pi}{4})} e^{j\frac{\pi}{3}n}$$

$$= 6e^{-j\frac{\pi}{12}} e^{j\frac{\pi}{3}n}$$

Using Superposition

- We can use superposition to find the outputs of more complex signals.

$$x[n] = A_0 + A_1\cos(\hat{\omega}_1 n + \theta_1)$$

$$= A_0 e^{j0n} + \frac{A_1}{2}(e^{j\theta_1}e^{j\hat{\omega}_1 n} + e^{-j\theta_1}e^{-j\hat{\omega}_1 n})$$

$$y[n] = H(e^{j\hat{\omega}})\times\{A_0 e^{j0n} + \frac{A_1}{2}(e^{j\theta_1}e^{j\hat{\omega}_1 n} + e^{-j\theta_1}e^{-j\hat{\omega}_1 n})\}$$

$$= H(e^{j0})A_0 e^{j0n} + H(e^{j\hat{\omega}_1})\frac{A_1}{2}e^{j\theta_1}e^{j\hat{\omega}_1 n} + H(e^{-j\hat{\omega}_1})\frac{A_1}{2}e^{-j\theta_1}e^{-j\hat{\omega}_1 n}$$

$$= H(e^{j0})A_0 e^{j0n} + H(e^{j\hat{\omega}_1})\frac{A_1}{2}e^{j\theta_1}e^{j\hat{\omega}_1 n} + H^*(e^{j\hat{\omega}_1})\frac{A_1}{2}e^{-j\theta_1}e^{-j\hat{\omega}_1 n}$$

$$= H(e^{j0})A_0 e^{j0n} + \frac{A_1}{2}\left|H(e^{j\hat{\omega}_1})\right|e^{j\angle H(e^{j\hat{\omega}_1})}e^{j\theta_1}e^{j\hat{\omega}_1 n} + \frac{A_1}{2}\left|H(e^{j\hat{\omega}_1})\right|e^{-j\angle H(e^{j\hat{\omega}_1})}e^{-j\theta_1}e^{-j\hat{\omega}_1 n}$$

$$= H(e^{j0})A_0 + \frac{A_1}{2}\left|H(e^{j\hat{\omega}_1})\right|\{e^{j\angle H(e^{j\hat{\omega}_1})}e^{j\theta_1}e^{j\hat{\omega}_1 n} + e^{-j\angle H(e^{j\hat{\omega}_1})}e^{-j\theta_1}e^{-j\hat{\omega}_1 n}\}$$

$$= H(e^{j0})A_0 + \frac{A_1}{2}\left|H(e^{j\hat{\omega}_1})\right|\{e^{j(\hat{\omega}_1 n + \theta_1 + \angle H(e^{j\hat{\omega}_1}))} + e^{-j(\hat{\omega}_1 n + \theta_1 + \angle H(e^{j\hat{\omega}_1}))}\}$$

$$= H(e^{j0})A_0 + A_1\left|H(e^{j\hat{\omega}_1})\right|\cos(\hat{\omega}_1 n + \theta_1 + \angle H(e^{j\hat{\omega}_1}))$$

An Example

For this system let $x[n] = 2\cos(\frac{\pi}{4}n - \frac{\pi}{2})$

since $\hat{\omega} = \frac{\pi}{4}$,

$$H(e^{j\frac{\pi}{4}}) = e^{-j\frac{\pi}{4}}(2 + 2\cos\frac{\pi}{4}) = e^{-j\frac{\pi}{4}}(2 + \sqrt{2}) = 3.41e^{-j\frac{\pi}{4}}$$

$$y[n] = 2 \times 3.41\cos(\frac{\pi}{4}n - \frac{\pi}{2} - \frac{\pi}{4})$$

$$= 6.82\cos(\frac{\pi}{4}(n-1) - \frac{\pi}{2})$$

This system is a time shifter

Superposition

- Using superposition, find the output using the frequency response function for each component of the input
- Add up the component outputs to yield the complete output solution.

Another Example

For this system let $x[n] = 4 + 3\cos(\frac{\pi}{3}n - \frac{\pi}{2}) + 3\cos(\frac{7\pi}{8}n)$

for $\hat{\omega} = 0,\ \frac{\pi}{3},\ \frac{7\pi}{8}$

$$H(e^{j0}) = e^{-j0}(2 + 2\cos 0) = 4$$

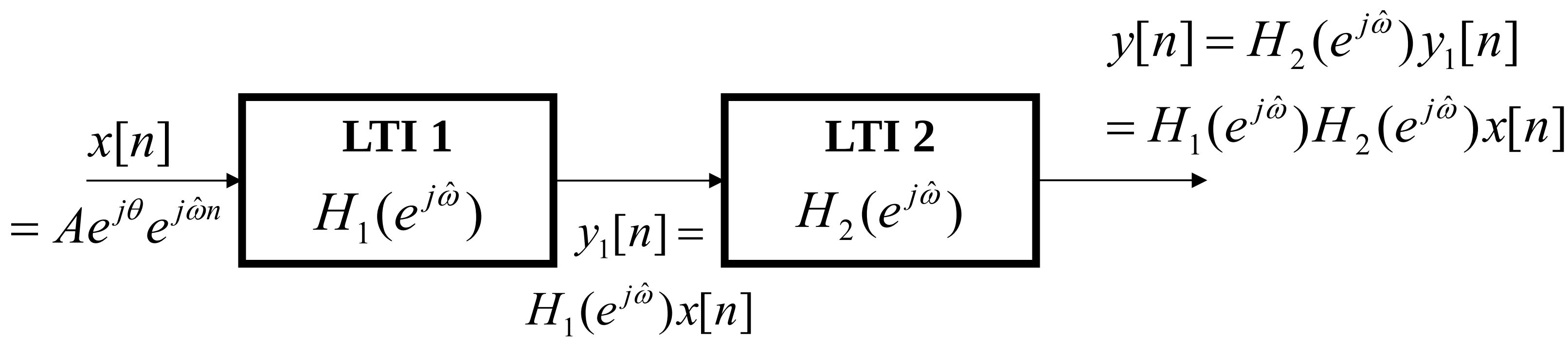
$$H(e^{j\frac{\pi}{3}}) = e^{-j\frac{\pi}{3}}(2 + 2\cos\frac{\pi}{3}) = 3e^{-j\frac{\pi}{3}}$$

$$H(e^{j\frac{7\pi}{8}}) = e^{-j\frac{7\pi}{8}}(2 + 2\cos\frac{7\pi}{8}) = 0.1522e^{-j\frac{7\pi}{8}}$$

$$y[n] = 4 \times 4 + 3 \times 3\cos(\frac{\pi}{3}(n-1) - \frac{\pi}{2}) + 0.1522 \times 3\cos(\frac{7\pi}{8}(n-1))$$

Cascaded System

- Two systems where the output of the first is the input to the second.



- This states that a cascaded system can be replaced a single system with frequency response,

$$H(e^{j\hat{\omega}}) = H_1(e^{j\hat{\omega}})H_2(e^{j\hat{\omega}})$$

Summary

- The Frequency Response is associated a FIR system with complex exponential signals as inputs
- The Frequency Response can be used to calculate the output signal of a system
- Superposition can be used to find the output signal of each component and the complete response can be obtained by adding the outputs of the components
- This type of analysis is called frequency domain analysis and we can work exclusively in the frequency
  - Solving for the output in the time domain (the difference equation or impulse response) is not necessary.

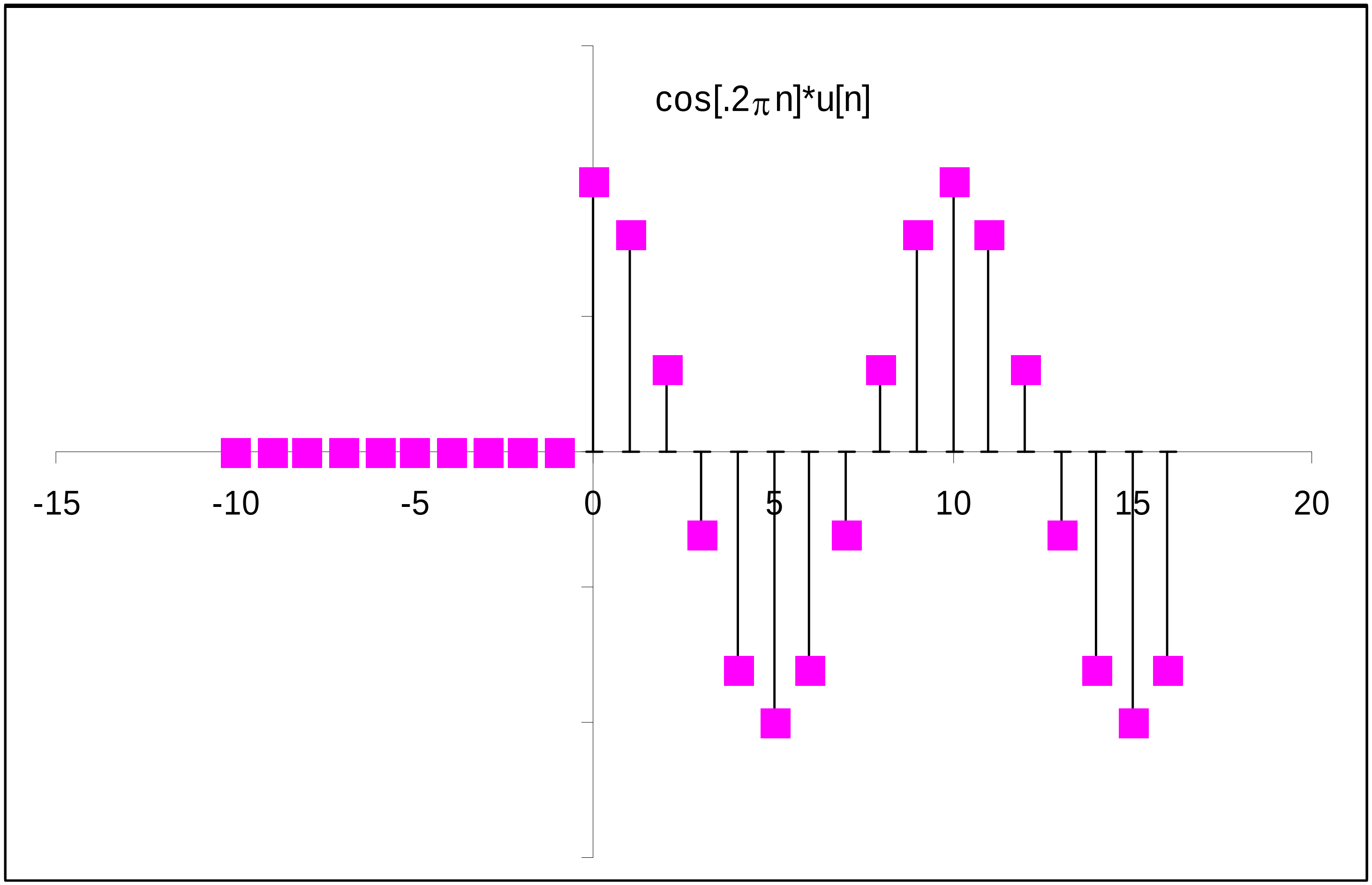
The Unit-Step Signal

- Before we continue, let's define another important sequence called the unit step signal:

$$u[n] = \begin{cases} 0 & \text{for } n < 0 \\ 1 & \text{for } n \geq 0 \end{cases}$$

- This signal can be used represent real world signals since signals start as some point in time.
- A signal like $x[n]=\cos(.2\pi n)$ is defined for all $-\infty < n < \infty$, while $x[n]=\cos(.2\pi n)\, u[n]$ is more realistic since it is defined only for $n \geq 0$

Unit Step Example



Steady-State and Transient Response

- Let’s examine how systems responds to signals which begin at a particular time.
- We will see that the output show two phases:
  - A Transient Response or a period of time where the signal “builds up” from its starting value (usually zero) to its final “state”
  - A Steady State Response or the period where the transient response has “completed”
- Examples:
  - Turning on a light or a motor
  - Starting a car

FIR System Transient and Steady-State Response

- The input to the system is:

$$x[n] = Xe^{j\hat{\omega}n}u[n] = \begin{cases} Xe^{j\hat{\omega}n} & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

- Our FIR system: $H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$

- The output: $y[n] = \sum_{k=0}^{M} b_k x[n-k] = \sum_{k=0}^{M} b_k X e^{j\hat{\omega}(n-k)} u[n-k]$

FIR System Transient and Steady-State Response

- Let's study the output:

$$y[n]=\sum_{k=0}^{M} b_k X e^{j\hat{\omega}(n-k)}u[n-k]$$

$$=X\{b_0 e^{j\hat{\omega}(n-0)}u[n-0]+b_1 e^{j\hat{\omega}(n-1)}u[n-1]+\cdots$$

$$\cdots+b_{l-1}e^{j\hat{\omega}(n-(l-1))}u[n-(l-1)]+b_l e^{j\hat{\omega}(n-l)}u[n-l]+b_{l+1}e^{j\hat{\omega}(n-(l+1))}u[n-(l+1)]+\cdots$$

$$\cdots+b_M e^{j\hat{\omega}(n-M)}u[n-M]\}$$

$$y[n<0]=X\{b_0 e^{j\hat{\omega}(n-0)}u[n-0]+b_1 e^{j\hat{\omega}(n-1)}u[n-1]+\cdots$$

$$\cdots+b_{l-1}e^{j\hat{\omega}(n-(l-1))}u[n-(l-1)]+b_l e^{j\hat{\omega}(n-l)}u[n-l]+b_{l+1}e^{j\hat{\omega}(n-(l+1))}u[n-(l+1)]+\cdots$$

$$\cdots+b_M e^{j\hat{\omega}(n-M)}u[n-M]\}$$

$$=X\{b_0 e^{j\hat{\omega}(n-0)}0+b_1 e^{j\hat{\omega}(n-1)}0+\cdots$$

$$\cdots+b_{l-1}e^{j\hat{\omega}(n-(l-1))}0+b_l e^{j\hat{\omega}(n-l)}0+b_{l+1}e^{j\hat{\omega}(n-(l+1))}0+\cdots$$

$$\cdots+b_M e^{j\hat{\omega}(n-M)}0\}=0$$

FIR System Transient and Steady-State Response

$$y[0 \le n = l < M] = X\{b_0 e^{j\hat{\omega}(n-0)}u[n] + b_1 e^{j\hat{\omega}(n-1)}u[n-1] + \cdots$$

$$\cdots + b_{l-1}e^{j\hat{\omega}(n-(l-1))}u[n-(l-1)]$$

$$+ b_l e^{j\hat{\omega}(n-l)}u[n-l]$$

$$+ b_{l+1}e^{j\hat{\omega}(n-(l+1))}u[n-(l+1)] + \cdots$$

$$\cdots + b_M e^{j\hat{\omega}(n-M)}u[n-M]\}$$

$$= X\{b_0 e^{j\hat{\omega}(n-0)}1 + b_1 e^{j\hat{\omega}(n-1)}1 + \cdots$$

$$\cdots + b_{l-1}e^{j\hat{\omega}(n-(l-1))}1 + b_l e^{j\hat{\omega}(n-l)}1 + b_{l+1}e^{j\hat{\omega}(n-(l+1))}0 + \cdots$$

$$\cdots + b_M e^{j\hat{\omega}(n-M)}0\} = \sum_{k=0}^{n} b_k X e^{-j\hat{\omega}k} e^{j\hat{\omega}n}$$

This is the transient response!!! It is mainly due to starting the input at $n = 0$

FIR System Transient and Steady-State Response

$$y[n \geq M] = X\{b_0 e^{j\hat{\omega}(n-0)} u[n-0] + b_1 e^{j\hat{\omega}(n-1)} u[n-1] + \cdots$$

$$\cdots + b_{l-1} e^{j\hat{\omega}(n-(l-1))} u[n-(l-1)] + b_l e^{j\hat{\omega}(n-l)} u[n-l] + b_{l+1} e^{j\hat{\omega}(n-(l+1))} u[n-(l+1)] + \cdots$$

$$\cdots + b_M e^{j\hat{\omega}(n-M)} u[n-M]\}$$

$$= X\{b_0 e^{j\hat{\omega}(n-0)} 1 + b_1 e^{j\hat{\omega}(n-1)} 1 + \cdots$$

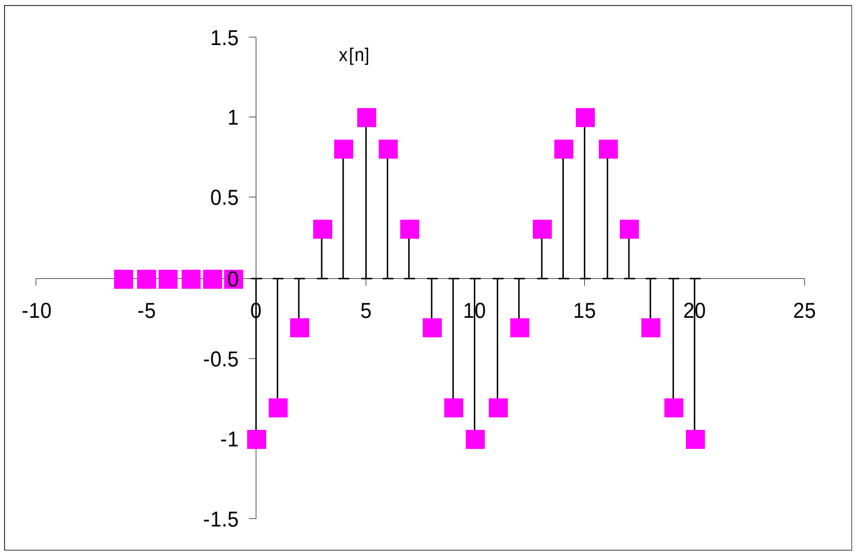
$$\cdots + b_{l-1} e^{j\hat{\omega}(n-(l-1))} 1 + b_l e^{j\hat{\omega}(n-l)} 1 + b_{l+1} e^{j\hat{\omega}(n-(l+1))} 1 + \cdots$$

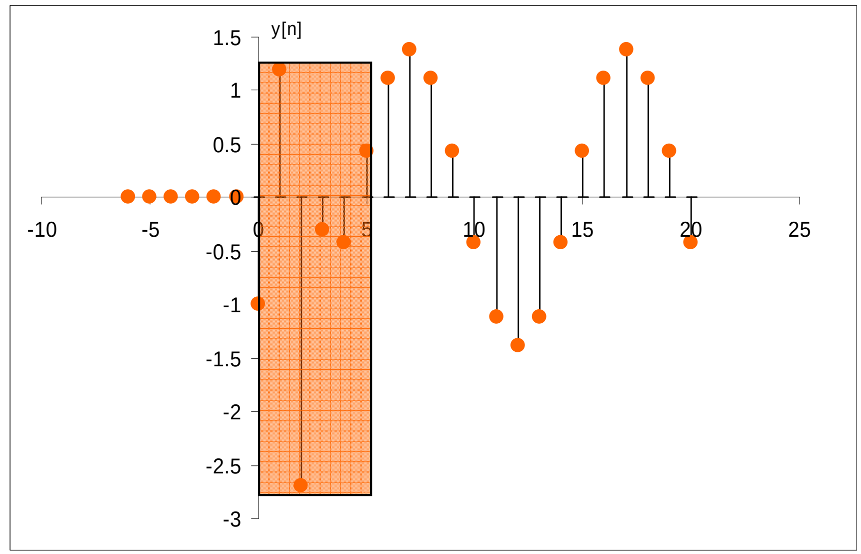
$$\cdots + b_M e^{j\hat{\omega}(n-M)} 1\} = \sum_{k=0}^{M} b_k X e^{-j\hat{\omega}k} e^{j\hat{\omega}n}$$

This is the steady-state response!!!

(Looks like the response due to the input when the input is non-zero for all *n*.)

An Example of the Transient and Steady State Response





Homework

- Exercises:
  - 6.1
- Problems:
  - 6.1, 6.2,
  - 6.4, 6.6 Use Matlab to plot the Frequency Response; show your code
  - 6.7,
  - 6.12 Use Matlab to plot the Frequency Response; show your code
  - 6.13