More on FT

Lecture 10

4CT.5

3CT.3-5,7,8

Higher Order Differentiation

$$\Im[\frac{d^n f(t)}{dt^n}] = (j\omega)^n F(j\omega)$$

$$\sum_{n=0}^{N} a_n \frac{d^n y(t)}{dt^n} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m}, \Rightarrow \Im\{\sum_{n=0}^{N} a_n \frac{d^n y(t)}{dt^n}\} = \Im\{\sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m}\}$$

$$\sum_{n=0}^{N} a_n (j\omega)^n Y(j\omega) = \sum_{m=0}^{M} b_m (j\omega)^m X(j\omega)$$

$$\mathbf{Y}(j\omega) = \frac{\sum_{m=0}^{M} b_m (j\omega)^m}{\sum_{n=0}^{N} a_n (j\omega)^n} \mathbf{X}(j\omega)$$

$$(a_n p^n + a_{n-1} p^{n-1} + \cdots + a_1 p^1 + a_0) y(t) = (b_m p^m + \cdots + a_0) x(t)$$

$$y(t) = \frac{B(p)}{A(p)} x(t)$$

$$\mathbf{Y}(j\omega) = \frac{\mathbf{B}(j\omega)}{\mathbf{A}(j\omega)} \mathbf{X}(j\omega) = \mathbf{H}(j\omega)\mathbf{X}(j\omega)$$

Frequency Response

- We just have seen that the FT of the differential equation describing a system yields the Frequency Response, $\mathbf{H}(j\omega)$
- We have also seen that convolving the impulse response, $h(t)$, with any input, $x(t)$ will yield its output, $y(t)$.
- Is the Frequency Response related to the impulse response?

Frequency Response and the Impulse Response

Using convolution, for only sinusoidal inputs:

$$x(t) = Ae^{j(\omega t+\phi)}$$

$$y(t) = \int h(\tau)x(t-\tau)\,d\tau = \int h(\tau)Ae^{j[\omega(t-\tau)+\phi]}\,d\tau$$

$$= \int h(\tau)Ae^{j(\omega t+\phi)}e^{-j\omega\tau}\,d\tau = Ae^{j(\omega t+\phi)}\int h(\tau)e^{-j\omega\tau}\,d\tau = x(t)\int h(\tau)e^{-j\omega\tau}\,d\tau$$

$$y(t) = x(t)\int h(\tau)e^{-j\omega\tau}\,d\tau$$

Taking the FT of both sides and noting that $\int h(\tau)e^{-j\omega\tau}\,d\tau$ is not a function of t:

$$\int y(t)e^{-j\omega t}dt = \int x(t)e^{-j\omega t}dt\int h(\tau)e^{-j\omega\tau}\,d\tau$$

$$Y(j\omega) = X(j\omega)\int h(\tau)e^{-j\omega\tau}\,d\tau \Rightarrow H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \int h(\tau)e^{-j\omega\tau}\,d\tau$$

But $\int h(\tau)e^{-j\omega\tau}\,d\tau$ is the FT of $h(t)$ and, therefore,

the frequency response is just the FT of the impulse response.

We say that $H(j\omega)$ and $h(t)$ are FT pairs.

Another Interesting Example

$$f(t) = V \text{ for } -\tau/2 < t < \tau/2;$$
$$= V[u(t+\tau/2) - u(t-\tau/2)]$$
$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$
$$= \int_{-\tau/2}^{\tau/2} Ve^{-j\omega t}\,dt = \frac{V}{-j\omega}e^{-j\omega t}\Big|_{-\tau/2}^{\tau/2} = \frac{V}{-j\omega}(e^{-j\omega\tau/2} - e^{j\omega\tau/2})$$
$$= \frac{V}{j\omega}(e^{j\omega\tau/2} - e^{-j\omega\tau/2}) = 2\frac{V}{\omega}(\frac{e^{j\omega\tau/2} - e^{-j\omega\tau/2}}{2j}) = 2\frac{V}{\omega}\sin(\omega\tau/2)$$
$$= V\tau\frac{\sin(\omega\tau/2)}{\omega\tau/2} = V\tau \text{ Sa}(\omega\tau/2)$$

- FT of rectangular pulse is the Sampling Function
- What is the FT of $u(t)$?

FT of the Unit Impulse

$$\Im[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t}\, dt$$

$$\text{From } f(x) = \int_{-\infty}^{\infty} f(t)\delta(t-x)\, dt$$

$$\Im[\delta(t)] = e^{-j\omega 0} = 1$$

- If $x(t)=\delta(t)$, then $y(t)=h(t)$ the response due to a unit impulse response and $H(j\omega)$ is the network response or system function in phasor form

$$Y(j\omega) = \frac{B(j\omega)}{A(j\omega)} X(j\omega); \quad X(j\omega) = 1; \quad \therefore \frac{B(j\omega)}{A(j\omega)} = H(j\omega)$$

This implies that the FT of the impulse response is the network response or frequency response

$$H(j\omega) = \Im[h(t)] = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$$

FT of δ(t) continued

We just saw that $\Im[\delta(t)] = e^{-j\omega 0} = 1$

therefore, the inverse Fourier transform of 1 must be the delta function.

$\Im^{-1}[1] = \delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 1e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} (\cos\omega t + j\sin\omega t)\, d\omega$ Using the definition of the inverse FT.

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos\omega t\, d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty} j\sin\omega t\, d\omega$$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos\omega t \;\, d\omega = \frac{1}{\pi}\int_{0}^{\infty} \cos\omega t \;\, d\omega$; using the properties of even and odd functions.

$$\delta(t) = \frac{1}{\pi}\int_{0}^{\infty} \cos\omega t \;\, d\omega$$

Odd and Even Properties of a Signal

Note that the sine is odd and $\lim_{a\to\infty}\int_{-a}^{a}\sin\omega t d\omega = \lim_{a\to\infty}\{\int_{0}^{a}\sin\omega t d\omega + \int_{-a}^{0}\sin\omega t d\omega\}$

If $\int_{0}^{a}\sin\omega t d\omega = g(a);$ then $\int_{-a}^{0}\sin\omega t d\omega = -g(a)$ since

Let $\omega = -x, \Rightarrow \int_{a}^{0}\sin(-xt)(-dx) = \int_{a}^{0}-\sin(xt)(-dx) = \int_{a}^{0}\sin(xt)dx = -\int_{0}^{a}\sin(xt)dx = -g(a)$

$\lim_{a\to\infty}\{g(a) - g(a)\} = 0$

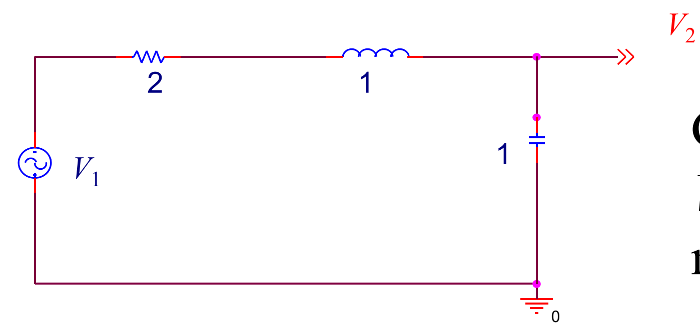
Note that the cosine is even and $\lim_{a\to\infty}\int_{-a}^{a}\cos\omega t d\omega = \lim_{a\to\infty}\{\int_{0}^{a}\cos\omega t d\omega + \int_{-a}^{0}\cos\omega t d\omega\}$

If $\int_{0}^{a}\cos\omega t d\omega = g(a);$ then $\int_{-a}^{0}\cos\omega t d\omega = g(a)$since

Let $\omega = -x, \Rightarrow \int_{a}^{0}\cos(-xt)(-dx) = \int_{a}^{0}\cos(xt)(-dx) = -\int_{a}^{0}\cos(xt)dx = \int_{0}^{a}\cos(xt)dx = g(a)$

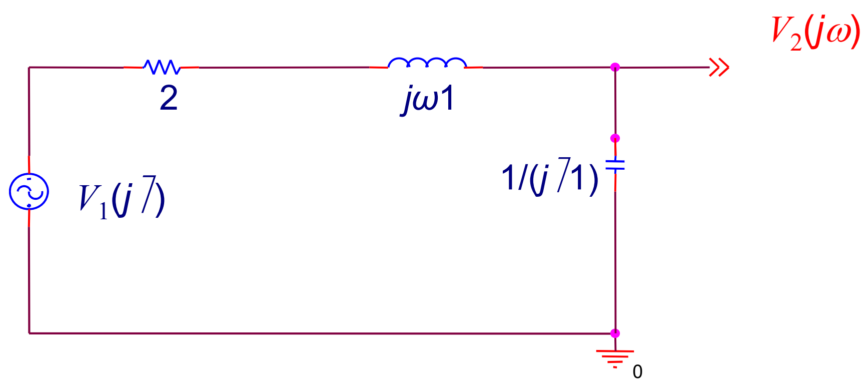
$\lim_{a\to\infty}\{g(a) + g(a)\} = \lim_{a\to\infty}\{2g(a)\} = \lim_{a\to\infty} 2\int_{0}^{a}\cos\omega t d\omega$

Calculation of the Frequency Response



Given that $V_1(t)=\delta(t)$ and it is known that $V_2(t)=te^{-t}\,u(t)$ what is FT of [$te^{-t}\,u(t)$] which must equal be $H(j\omega)$?

We can calculate FT of [$te^{-t}\,u(t)$] by evaluating the integral of $te^{-t}\,u(t)\,e^{-j\omega t}$ or calculate the system response function:



$$V_2(j\omega) = \frac{1/j\omega}{2 + j\omega + 1/j\omega} V_1(j\omega)$$

$$H(j\omega) = \frac{1/j\omega}{2 + j\omega + 1/j\omega} = \frac{1}{2j\omega - \omega^2 + 1}$$

$$= \frac{1}{(1 + j\omega)^2} = \Im[te^{-t}u(t)]$$

Fourier Transform of A Constant

$$\Im[1] = \int_{-\infty}^{\infty} e^{-j\omega t} dt = \int_{-\infty}^{\infty} (\cos \omega t - j \sin \omega t) dt$$

Since a constant is an even function over ∞:

$$= \int_{-\infty}^{\infty} \cos \omega t dt$$

But we said that $\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos \omega t d\omega$

$$\therefore \Im[1] = 2\pi\delta(\omega)$$

- Our first Duality:
  - A constant in the time domain is a unit impulse function in the frequency domain
  - A unit impulse function in the time domain is a constant in the frequency domain

FT of a Unit Step Function

$$\Im[u(t)] = \int_{-\infty}^{\infty} u(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-j\omega t}dt = \frac{e^{-j\omega t}}{-j\omega}\Big|_0^{\infty}$$

$$= \frac{1}{j\omega}(1 - e^{-j\omega\infty}), \text{ which is undefined}$$

Let's use another technique:

$$\Im[\frac{df(t)}{dt}] = j\omega F(j\omega)$$

$$\Im[\delta(t)] = 1 = \Im[\frac{du(t)}{dt}] = j\omega\Im[u(t)]$$

$$\therefore \Im[u(t)] = \frac{1}{j\omega}, \text{ for } \omega \neq 0$$

Time & Frequency Displacement

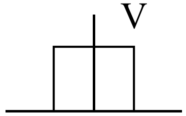
Time Displacement

If $\Im[f(t)] = F(j\omega)$, then what is $\Im[f(t - T_d)]$

$$= \int_{-\infty}^{\infty} f(t - T_d) e^{-j\omega t}\, dt, \text{ let } x = t - T_d$$

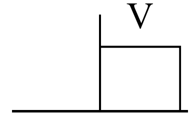
$$= \int_{-\infty}^{\infty} f(x) e^{-j\omega(x + T_d)}\, dx = e^{-j\omega T_d} \int_{-\infty}^{\infty} f(x) e^{-j\omega x}\, dx$$

$$= F(j\omega) e^{-j\omega T_d}$$



$$V_1(j\omega) = V\tau Sa(\omega\tau/2)$$

$$= V\tau \frac{\sin(\omega\tau/2)}{\omega\tau/2}$$



$$V_2(j\omega) = V_1(j\omega) e^{-j\omega\tau/2}$$

$$= V\tau Sa(\omega\tau/2) e^{-j\omega\tau/2}$$

$$= V\tau \frac{\sin(\omega\tau/2)}{\omega\tau/2} e^{-j\omega\tau/2}$$

$$= V \frac{(1 - e^{-j\omega\tau})}{j\omega}$$

Time & Frequency Displacement

Frequency Displacement

If $\Im[f(t)] = F(j\omega)$, then what is $F(\alpha + j\omega)$

Assume there exist a duality

between Time and Frequency displacement and try

$$\Im[f(t)e^{-\alpha t}] = \int_{-\infty}^{\infty} f(t)e^{-\alpha t}e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} f(t)e^{-(\alpha + j\omega)t}dt = F(\alpha + j\omega)$$

- Our next Duality:
  - A shift in the time domain equals a multiplication by $e^{-j\omega T}$ the frequency domain
  - A shift in the frequency domain equals a multiplication by $e^{-\alpha t}$ the time domain

FT Symmetries

Note: $F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$

And: $F(-j\omega) = \int_{-\infty}^{\infty} f(t)e^{j\omega t}\,dt$

And: $F^*(j\omega) = \int_{-\infty}^{\infty} f(t)e^{j\omega t}\,dt$

Which means: $F^*(j\omega) = F(-j\omega)$

$\therefore$ Re$[F(j\omega)]$ is even, Im $[F(j\omega)]$ is odd

Proof:

$F(j\omega) = A(\omega) + jB(\omega)$

$F^*(j\omega) = A(\omega) - jB(\omega)$

$F(-j\omega) = A(-\omega) + jB(-\omega)$

Since $F^*(j\omega) = F(-j\omega)$ then

$A(\omega) - jB(\omega) = A(-\omega) + jB(-\omega)$

and $A(\omega) = A(-\omega)$ & $-B(\omega) = B(-\omega)$

Some Interesting FTs

Exponential Functions

$$\Im[e^{-\alpha t}u(t)] = \frac{1}{\alpha + j\omega}, \text{since } \Im[u(t)] = \frac{1}{j\omega}$$

$$\Im[e^{-\alpha t}tu(t)] = \frac{1}{(\alpha + j\omega)^2}, \text{since } \Im[tu(t)] = \frac{1}{(j\omega)^2},$$

$$\text{since } \Im[\frac{dtu(t)}{dt}] = j\omega\Im[tu(t)] = \Im[u(t)] = \frac{1}{j\omega}$$

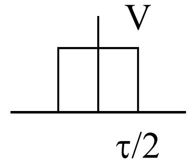
More Interesting FTs

$f(t) = m(t)\cos\omega_0 t$ This a called amplitude modulation

$$= \frac{m(t)}{2} e^{j\omega_0 t} + \frac{m(t)}{2} e^{-j\omega_0 t}$$

$$\Im[f(t)] = \tfrac{1}{2} M[j(\omega - \omega_0)] + \tfrac{1}{2} M[j(\omega + \omega_0)]$$

This says that we have made two copies of the spectrum of $m(t)$ one shifted to the right of the $\omega=0$ axis and one to the left. Both of these copies are reduced by ½.

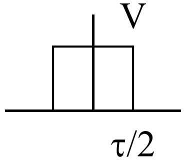


$$M(j\omega) = V\tau Sa(\omega\tau/2)$$

$$= V\tau \frac{\sin(\omega\tau/2)}{\omega\tau/2}$$

Let's look at the case where $m(t)$ is a pulse and has the spectrum shown above.

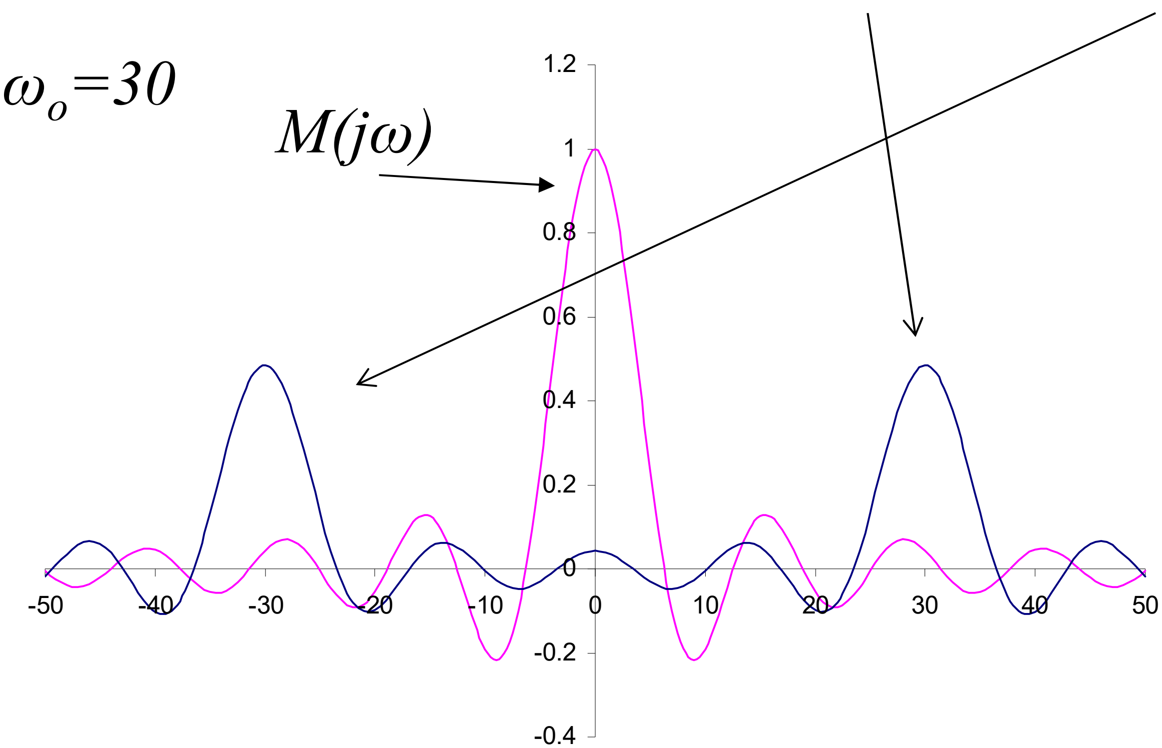
Frequency Shifting Example



$$M(j\omega) = V\tau Sa(\omega\tau/2)$$
$$= V\tau \frac{\sin(\omega\tau/2)}{\omega\tau/2}$$

$$F(j\omega) = \frac{1}{2}M[j(\omega-\omega_0)] + \frac{1}{2}M[j(\omega+\omega_0)]$$
$$= \frac{V\tau}{2}\left\{\frac{\sin[(\omega-\omega_0)\tau/2]}{(\omega-\omega_0)\tau/2} + \frac{\sin[(\omega+\omega_0)\tau/2]}{(\omega+\omega_0)\tau/2}\right\}$$

$V=\tau=1, \omega_o=30$



Fourier Transform and Fourier Series

	FT	FS
Time	Aperiodic	Periodic
Frequency	Continuous	Discrete

Now what about FT of a periodic function

Recall:

$$\Im[\delta(t)] = 1$$

$$\Im[1] = 2\pi\delta(\omega)$$

$$\Im[e^{j\omega_o t}] = \Im[1e^{j\omega_o t}] = 2\pi\delta(\omega - \omega_o) \text{ (Frequency Shift)}$$

$$\Im[\cos\omega_o t] = \Im[\frac{1}{2}(e^{j\omega_o t} + e^{-j\omega_o t})] = \pi[\delta(\omega - \omega_o) + \delta(\omega + \omega_o)]$$

$$\Im[\sin\omega_o t] = \Im[\frac{1}{2j}(e^{j\omega_o t} - e^{-j\omega_o t})] = -j\pi[\delta(\omega - \omega_o) - \delta(\omega + \omega_o)]$$

FT of a Periodic Function

FT of a Periodic Function $f(t)$

A periodic function can formulated in a FS: $f(t) = \sum_{-\infty}^{\infty} a_k e^{jk2\pi t/T}$

$$\therefore \Im[f(t)] = F(j\omega) = 2\pi \sum_{-\infty}^{\infty} a_k \delta(\omega - \frac{2\pi}{T}k)$$

$\therefore$ FT of a periodic function is a series of unit impulse functions and is discrete in the frequency domain.

Note: FT of a train of unit impulses $f(t) = \sum_{-\infty}^{\infty} A_n \delta(t - nT)$

$$\Im[f(t)] = F(j\omega) = \sum_{-\infty}^{\infty} A_n e^{-j\omega nT}$$

$\therefore$ FT of a train of unit impulses is a FS in the frequency domain and is continuous in the frequency domain

Energy of f(t)

- Energy dissipated by, say, a 1 ohm resistance over a period T is $\int_{-T/2}^{T/2} v(t)^2 dt$

  We also called this the quadratic content of a signal

- It can be shown that the quadratic content of a signal in terms of its FT is:

$$\int_{-T/2}^{T/2} f(t)^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)F^*(j\omega)d\omega$$

- And

$$\int_{-T/2}^{T/2} | f(t) |^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} | F(j\omega) |^2 \, d\omega$$

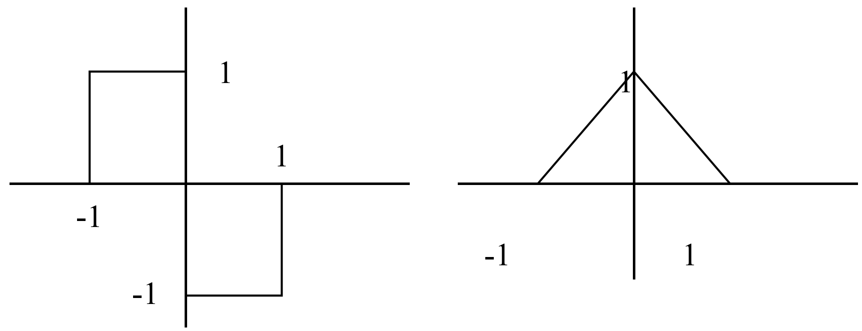
- The latter equation is called Parseval's Theorem

Homework

- Problem (1)
  - Show that $F(j\omega)$ of an even function $f(t)=f(-t)$ is real
  - Show that $F(j\omega)$ of an odd function $f(t)=-f(-t)$ is imaginary
- Problem (2)
  - Find $F(j\omega)$ for the following functions and sketch their amplitude and phase spectra:
    - $f_a(t)=e^{-\alpha t}\, u(t)$;
    - $f_b(t)= e^{\alpha t}\, u(-t)+e^{-\alpha t}\, u(t)$;
    - $f_c(t)= -e^{\alpha t}\, u(-t)+e^{-\alpha t}\, u(t)$;
    - $f_d(t)= e^{t} \sin 10t\, u(-t)$

Homework

- Problem (3)
  - Calculate the $F(j\omega)$ for the waveforms below. Note that the second is the integral of the first.



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- 3CT.5.1, 3CT.5.2, 3CT.5.3