Discrete Fourier Transform

Lesson 11

5DT

DFT

- Recall that for FS that if we have a continuous periodic signal in the time domain, it will have a infinite discrete values in the frequency domain
- Similarly, we can formulate the FT of a discrete signal in the time domain as having continuous periodic values in the frequency domain

RECALL: FT of a Periodic Function

FT of a Periodic Function $f(t)$

A periodic function can formulated in a FS: $f(t) = \sum_{-\infty}^{\infty} a_k e^{jk2\pi t/T}$

$$\therefore \Im[f(t)] = F(j\omega) = 2\pi \sum_{-\infty}^{\infty} a_k \delta(\omega - \frac{2\pi}{T}k)$$

$\therefore$ FT of a periodic function is a series of unit impulse functions and is discrete in the frequency domain.

Note: FT of a train of unit impulses $f(t) = \sum_{-\infty}^{\infty} A_n \delta(t - nT)$

$$\Im[f(t)] = F(j\omega) = \sum_{-\infty}^{\infty} A_n e^{-j\omega nT}$$

$\therefore$ FT of a train of unit impulses is a FS in the frequency domain and is continuous in the frequency domain

Discrete Time Fourier Transform (DTFT)

$f[n] = f(t)_{t=nT_s} = f(nT_s) \Rightarrow \sum_{-\infty}^{\infty} f(nT_s)\delta(t - nT_s)$ This is a model of a discrete signal from a continuous signal.

$$\Im[f[n]] = \int_{-\infty}^{\infty} \sum_{-\infty}^{\infty} f(nT_s)\delta(t - nT_s)e^{-j\omega t}\,dt = \sum_{-\infty}^{\infty} f(nT_s)\int_{-\infty}^{\infty} \delta(t - nT)e^{-j\omega t}\,dt$$

$$= \sum_{-\infty}^{\infty} f(nT_s)e^{-j\omega nT_s} = \sum_{-\infty}^{\infty} f[n]e^{-j\hat{\omega} n}; \text{ where } \hat{\omega} = \omega T_s \text{ and } \int_{-\infty}^{\infty} \delta(t - nT)e^{-j\omega t}\,dt = e^{-j\omega nT_s}$$

$$= F(j\hat{\omega}) = F(e^{j\hat{\omega}}) \text{ since } F(e^{j\hat{\omega}}) \text{ is periodic in } 2\pi$$

Discrete Fourier Transform (DFT)

- The DTFT yield a spectrum which is a continuous function of $\hat{\omega}$

$$F(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} f[n]e^{-j\hat{\omega}n}$$

- How do we get around this? Sample the spectrum.

When we sampled in the time domain, we replaced t by nT_s where T_s is the distance (in time) between samples. Therefore to sample in the frequency domain we replace $\omega = 2\pi f$ by $2\pi k f_\Delta$ where f_Δ is the distance (in frequency) between spectrum samples.

Note that since $\hat{\omega} = \omega T_s = \dfrac{\omega}{f_s}; \hat{\omega} = \dfrac{\omega}{f_s} \Rightarrow \dfrac{2\pi k f_\Delta}{f_s}$

$$F(e^{j\hat{\omega}=\frac{2\pi k f_\Delta}{f_s}}) = F[k] = \sum_{n=-\infty}^{\infty} x[n]e^{-j\frac{2\pi k f_\Delta}{f_s}n}$$

- Let us assume that there are only L samples for time domain and N samples for the spectrum.

$$F[k] = \sum_{n=0}^{L-1} f[n]e^{-j\frac{2\pi k f_\Delta}{f_s}n}$$

- Since f_s is the maxium frequency in the spectrum, then $f_\Delta = \dfrac{f_s}{N}$. This is just the resolution of the displaced spectrum.

$$F[k] = \sum_{n=0}^{L-1} f[n]e^{-j\frac{2\pi k f_\Delta}{f_s}n} = \sum_{n=0}^{L-1} f[n]e^{-j\frac{2\pi k f_s}{f_s N}n} = \sum_{n=0}^{L-1} f[n]e^{-j\frac{2\pi}{N}kn}$$

- This is called the Discrete Fourier Transform

Discrete Fourier Transform (DFT)

- Since the computer can only process discrete functions of finite time, we have to define a new Fourier Transform called the Discrete Fourier Transform, DFT.
  - Do not confuse this with the Discrete-time Fourier Transform, DTFT.
- It is defined as

$$F(k) = \sum_{n=0}^{L-1} f[n] e^{-j\frac{2\pi}{N}kn}$$

where there are the L samples of $x[n]$,

we evaluate the Spectrum over N frequencies, i.e., $0 \le k \le N-1$,

and each frequency is f_Δ apart and chose $f_\Delta = \dfrac{f_s}{N}$

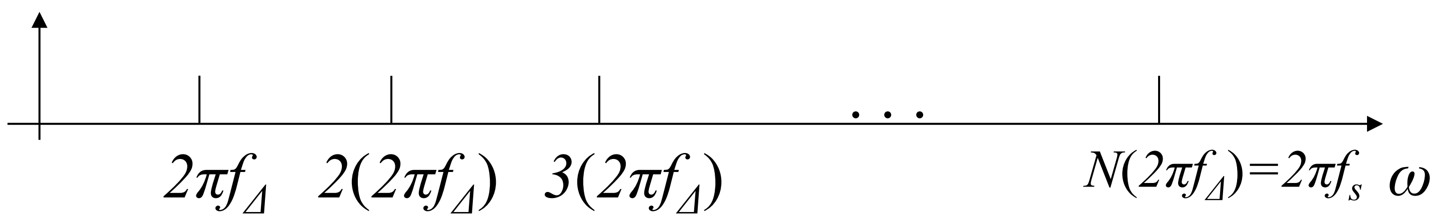
since f_s is the maximum frequency of the spectrum.

Therefore, $f_\Delta = \dfrac{f_s}{N} = \dfrac{1}{NT_s}$. We call this the resolution of the spectrum.

Discrete Fourier Transform (DFT)

Let's start with the DTFT: $X(e^{j\hat{\omega}}) = \sum_{-\infty}^{\infty} x[n]e^{-j\hat{\omega}n}; \hat{\omega} = \omega T_s$

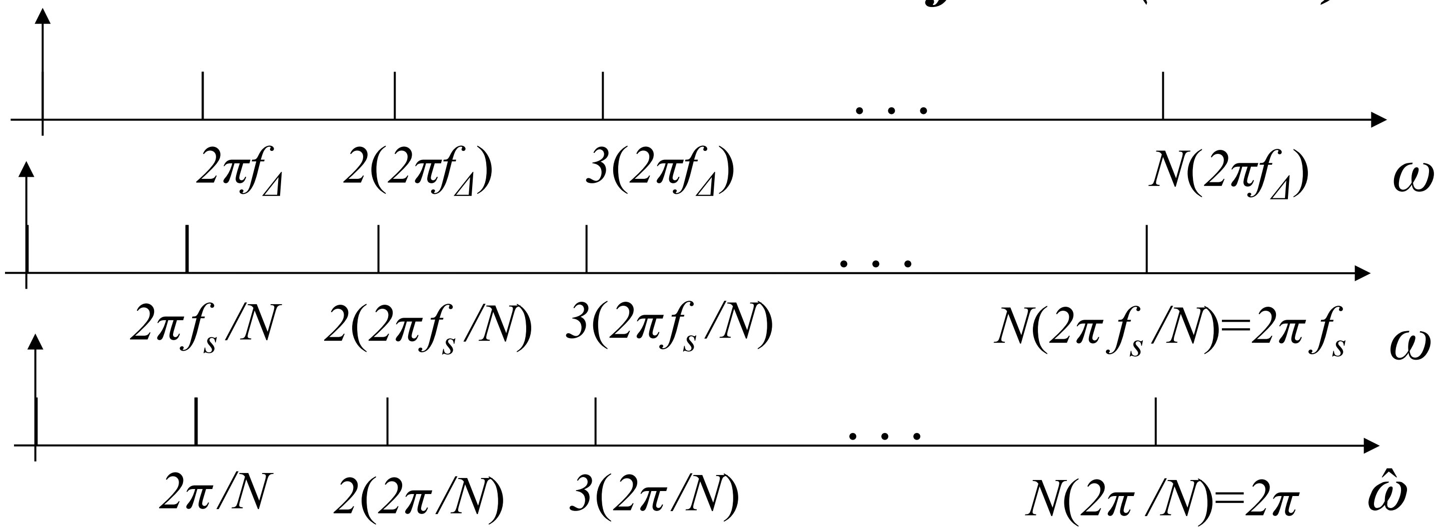
Let's divide the spectrum is into N frequencies equally spaced f_Δ Hz apart (i.e., we are sampling the spectrum).



Therefore, let's define the kth sample in the frequency domain as $\omega_k = 2\pi f_k = 2\pi k f_\Delta$ where k goes from 1 to N.

When $k = N$, the highest frequency in the spectrum is $\omega_N = \dfrac{2\pi N}{T_o} = 2\pi N f_\Delta = 2\pi f_s$.

Discrete Fourier Transform (DFT)



If f_s meets the Nyquist rate, then the spectrum of $f[n] = F(e^{j\hat{\omega}})$ must end at or below $\frac{f_s}{2}$.

Therefore, $\hat{\omega}_k = \omega_k T_s = 2\pi k f_\Delta T_s = \frac{2\pi k f_s}{N} T_s = \frac{2\pi k}{N}$.

Let's substitute $\hat{\omega}_k$ for $\hat{\omega}$ in the DTFT: $F(e^{j\hat{\omega}_k}) = \sum_{n=-\infty}^{\infty} f[n] e^{-j\hat{\omega}_k n} = \sum_{n=-\infty}^{\infty} f[n] e^{-j\frac{2\pi k}{N}n}$

This sum will only be a function of k. In addition, let's assume that there are L samples of $x[n]$.

Then, we have the Discrete Fourier Transform, DFT as $F[k] = F(e^{j\hat{\omega}_k}) = \sum_{n=0}^{L-1} f[n] e^{-j\frac{2\pi k}{N}n}$

Discrete Fourier Transform (DFT)

The spectrum now becomes

$$F(e^{j\hat{\omega}})\big|_{\hat{\omega}\Rightarrow\hat{\omega}_k} \Rightarrow F(e^{j\hat{\omega}_k}) = F(e^{j\frac{2\pi k}{N}}) = F[k] = \sum_{0}^{M-1} f[n]e^{-j\frac{2\pi k}{N}n}$$

using the M samples of the time domain yielding N samples in the frequency domain.

We usually choose $N = M$ and so $F[k] = \sum_{0}^{N-1} f[n]e^{-j\frac{2\pi k}{N}n}$

Why do we need DFT?

- From samples of $f(t)$, we can use the DFT to get a frequency spectrum which is similar to $F(j\omega)$ representation.
- This helps where we can not get an exact representation of $f(t)$
  – in the laboratory
  – In the field

Rectangular Pulse

$$f[n] = 1, |n| \le M$$
$$= 0, |n| > M$$

$$F(e^{j\hat{\omega}}) = \sum_{-\infty}^{\infty} f[n] e^{-j\hat{\omega}n} = \sum_{-M}^{M} 1 e^{-j\hat{\omega}n}$$

Substitute $m = n + M$

$$F(e^{j\hat{\omega}}) = \sum_{0}^{2M} e^{-j\hat{\omega}(m-M)} = e^{j\hat{\omega}M} \sum_{0}^{2M} e^{-j\hat{\omega}m}$$

$$= e^{j\hat{\omega}M} \frac{1 - e^{-j\hat{\omega}(2M+1)}}{1 - e^{-j\hat{\omega}}}, \hat{\omega} \ne 0, \pm 2\pi, \pm 4\pi, ...$$

$$= 2M + 1, \hat{\omega} = 0, \pm 2\pi, \pm 4\pi, ... \text{ Using L'Hopital's Rule}$$

NOTE:

$$\sum_{0}^{N-1} a^n = 1 + a + a^2 + \cdots a^{N-1}$$

$$\frac{1}{1-a} = 1 + a + a^2 + \cdots a^{N-1} + \frac{a^N}{1-a}$$

$$\frac{1-a^N}{1-a} = 1 + a + a^2 + \cdots a^{N-1}$$

$$\sum_{0}^{N-1} a^n = \frac{1-a^N}{1-a}$$

Rectangular Pulse Continued

$$F(e^{j\hat{\omega}}) = e^{j\hat{\omega}M}\frac{1-e^{-j\hat{\omega}(2M+1)}}{1-e^{-j\hat{\omega}}}, \hat{\omega} \neq 0, \pm 2\pi, \pm 4\pi, \ldots$$

$$= 2M+1, \hat{\omega} = 0, \pm 2\pi, \pm 4\pi, \ldots$$

$$\frac{1-e^{-j\hat{\omega}(2M+1)}}{1-e^{-j\hat{\omega}}} = \frac{e^{-j\hat{\omega}(2M+1)/2}(e^{+j\hat{\omega}(2M+1)/2} - e^{-j\hat{\omega}(2M+1)/2})}{e^{-j\hat{\omega}/2}(e^{j\hat{\omega}/2} - e^{-j\hat{\omega}/2})}$$

$$= e^{j\hat{\omega}M}\frac{e^{-j\hat{\omega}(2M+1)/2}\sin[\hat{\omega}(2M+1)/2]}{e^{-j\hat{\omega}/2}\sin[\hat{\omega}/2]}$$

$$F(e^{j\hat{\omega}}) = \frac{\sin[\hat{\omega}(2M+1)/2]}{\sin[\hat{\omega}/2]}, \hat{\omega} \neq 0, \pm 2\pi, \pm 4\pi, \ldots$$

$$= 2M+1, \hat{\omega} = 0, \pm 2\pi, \pm 4\pi, \ldots$$

Note: $\frac{\sin[\hat{\omega}(2M+1)/2]}{\sin[\hat{\omega}/2]} = 2M+1$, when $\hat{\omega} = 0$

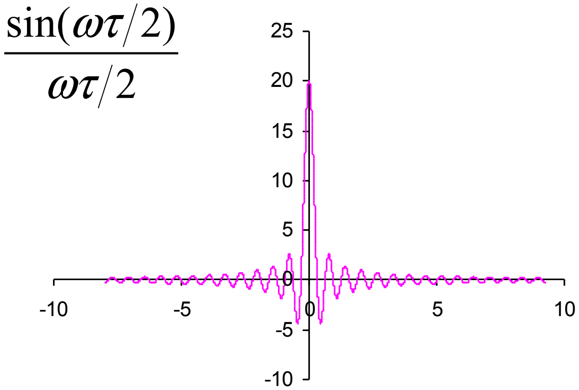
using L'Hopital's rule

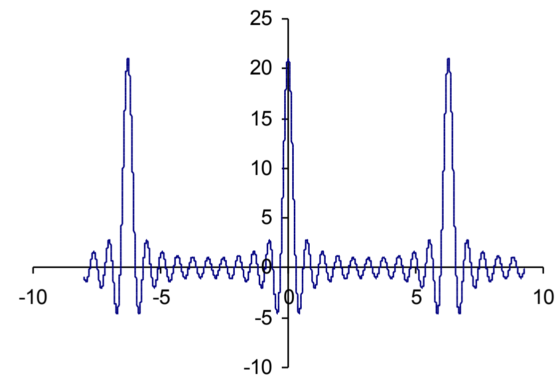
$$F(e^{j\hat{\omega}}) = \frac{e^{-j\hat{\omega}}\sin[\hat{\omega}(2M+1)/2]}{\sin[\hat{\omega}/2]}$$

Recall for a continuous rectangular pulse

$$M(j\omega) = V\tau Sa(\omega\tau/2)$$

$$= V\tau\frac{\sin(\omega\tau/2)}{\omega\tau/2}$$





Rectangular Pulse

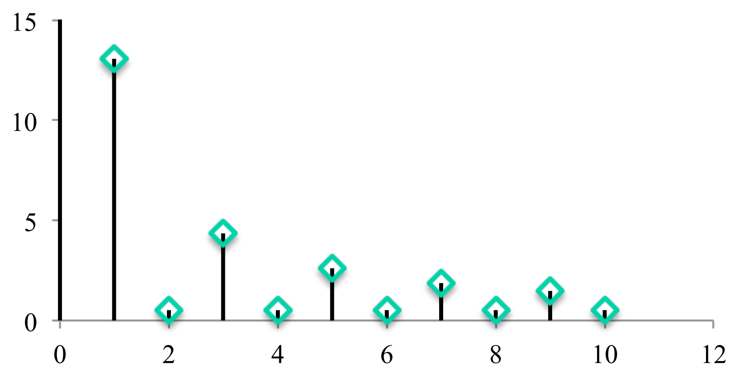
$$f[n] = 1, |n| \le M$$
$$= 0, M < |n| \le P$$
$$F[k] = \sum_{-P}^{P} f[n] e^{-j\frac{2\pi k}{2P+1}n} = \sum_{-M}^{M} 1 e^{-j\frac{2\pi k}{2P+1}n}$$

Substitute $m = n + M$

$$F[k] = \sum_{0}^{2M} e^{-j\frac{2\pi k}{2P+1}(n-M)} = \sum_{0}^{2M} e^{-j\frac{2\pi k}{2P+1}n} e^{j\frac{2\pi k}{2P+1}M}$$
$$= e^{j\frac{2\pi k}{2P+1}M} \sum_{0}^{2M} e^{-j\frac{2\pi k}{2P+1}n}$$
$$= e^{j\frac{2\pi M}{2P+1}k} \frac{1 - e^{-j\frac{2\pi}{2P+1}(2M+1)k}}{1 - e^{-j\frac{2\pi}{2P+1}k}} = e^{j\frac{2\pi M}{2P+1}k} \frac{e^{-j\frac{2\pi}{2P+1}\frac{(2M+1)}{2}k}\left(e^{+j\frac{2\pi}{2P+1}\frac{(2M+1)}{2}k} - e^{-j\frac{2\pi}{2P+1}\frac{(2M+1)}{2}k}\right)}{e^{-j\frac{2\pi}{2P+1}\frac{1}{2}k}\left(e^{+j\frac{2\pi}{2P+1}\frac{1}{2}k} - e^{-j\frac{2\pi}{2P+1}\frac{1}{2}k}\right)}$$
$$= e^{j\frac{2\pi M}{2P+1}k} \frac{e^{-j\frac{2\pi}{2P+1}\frac{(2M)}{2}k} \sin\left(\frac{2\pi}{2P+1}\frac{(2M+1)}{2}k\right)}{\sin\left(\frac{2\pi}{2P+1}\frac{1}{2}k\right)}$$
$$= \frac{\sin\left(\frac{2\pi}{2P+1}\frac{(2M+1)}{2}k\right)}{\sin\left(\frac{2\pi}{2P+1}\frac{1}{2}k\right)}; k \ne 0$$
$$= 2M+1, k = 0$$

NOTE :

$$\sum_{0}^{N-1} a^n = 1 + a + a^2 + \cdots a^{N-1}$$
$$\frac{1}{1-a} = 1 + a + a^2 + \cdots a^{N-1} + \frac{a^N}{1-a}$$
$$\frac{1-a^N}{1-a} = 1 + a + a^2 + \cdots a^{N-1}$$
$$\sum_{0}^{N-1} a^n = \frac{1-a^N}{1-a}$$



FT of discrete signals Properties

- To assure a FT, convergence is required for the quadratic content of the discrete signal:

$$\sum_{n=-\infty}^{\infty} |f[n]|^2 < \infty$$

- To assure a FT, convergence is required for the discrete signal:

$$\sum_{n=-\infty}^{\infty} |f[n]| < \infty$$

- Periodicity $F(e^{j\hat{\omega}}) = F(e^{j(\hat{\omega}-2\pi)})$
- Linearity $af_1[n] + bf_2[n] \Rightarrow aF_1(e^{j\hat{\omega}}) + bF_2(e^{j\hat{\omega}})$

FT of discrete signals Properties Continued

- Time Shifting $f[n-n_o] \rightarrow e^{-j\hat{\omega}n_o}F(j\hat{\omega})$

- Frequency Shifting $e^{j\omega_o n}f[n] \rightarrow F[j(\hat{\omega}-\omega_o)]$

- Parseval's Theorem $\sum_{n=-\infty}^{\infty}|f[n]|^2 = \frac{1}{2\pi}\int_{2\pi}|F(e^{j\hat{\omega}})|^2\,d\hat{\omega}$

Homework

- Problems 6.2-4

6.2 Find the DFT of the sequence, $f[n]=\sin(\frac{n\pi}{N})$ for n = 0,1,2,...N-1.

Hint express the sinusoid in exponential form.

6.3a)Find the DFT of the sequence $f[n]=e^{-bnT}$ in closed form. Use the identity $\sum_{n=0}^{N-1}\rho^n \equiv \frac{1-\rho^N}{1-\rho} \quad 0<\rho<1;$

Note that $N \rightarrow \infty; \sum_{n=0}^{N-1}\rho^n \rightarrow \frac{1}{1-\rho}$

b) Express the DFT in polar form as a function of *k*.

c) Plot the DFT in terms of its magnitude and its phase angle for T = .2, b = 1 and N = 8 for k = 0,1,2,..7

6.4a) Repeat the 6.3 for b = 10; T = .01, N = 20

6.6 Find the DFT for the following signals both are sampled at 1*m*sec intervals:

a) $f(nT_s)=f[n]=1; 0\le n\le 4$

Number of samples = 5

b) $f(nT_s)=f[n]=1; 0\le n\le 4$

$=0;\ 9>n>4$

Number of samples = 10

Homework

- Using Matlab and its FFT function calculate and plot the time signal and spectrum for the following single.
  - A single sine wave of frequency 200 Hz
  - A single square wave of frequency 200 Hz
  - Two simultaneous sine wave of frequency 200 and 200/3 Hz
  - Two sequential sine wave of frequency 200 and 200/3 Hz
  - Compare the spectrum of the latter two cases.

Filters

Lesson 12

3CT.5 – 6

Filters

Recall: $x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$ $\quad y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$

where *h(t)* is the response due to a unit impulse function d(t)

OR

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega)$$

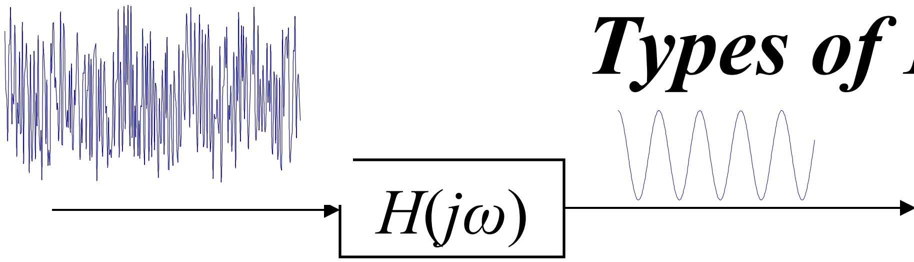
where $H(j\omega)$ is the network response in phasor form

By choosing $H(j\omega)$ or $h(t)$ we can shape the output $y(t)$ for a given $x(t)$' s

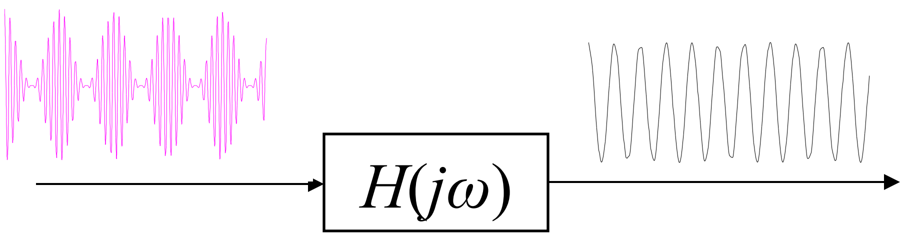
OR

in other words, we can choose $H(j\omega)$ to filter $x(t)$ to obtain a desired $y(t)$

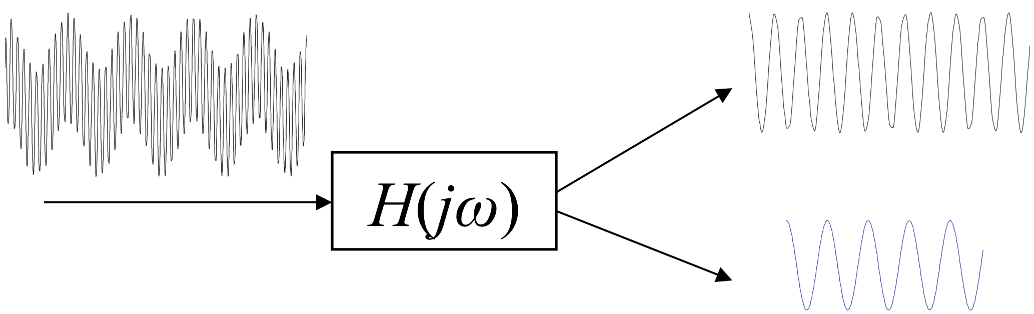
Types of Filtering



Filter out unwanted high frequencies (noise): e.g., radio signal from a satellite

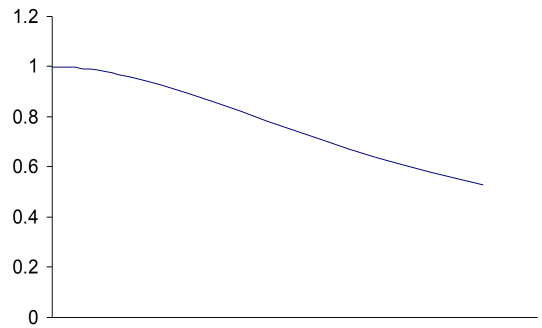


Filter out unwanted low frequencies (noise): e.g., power line disruption

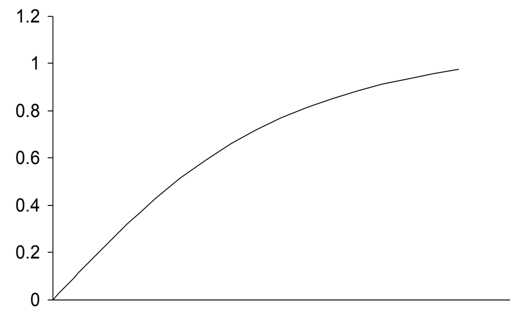


Separate signals in their frequency components: e.g., stereo recording

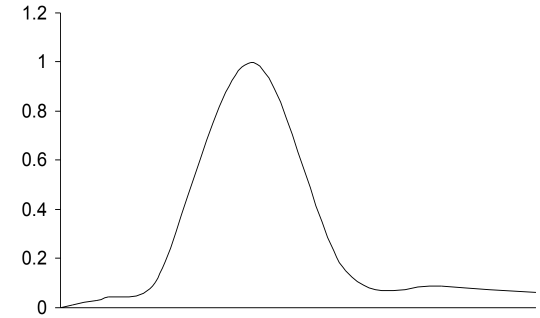
Filters



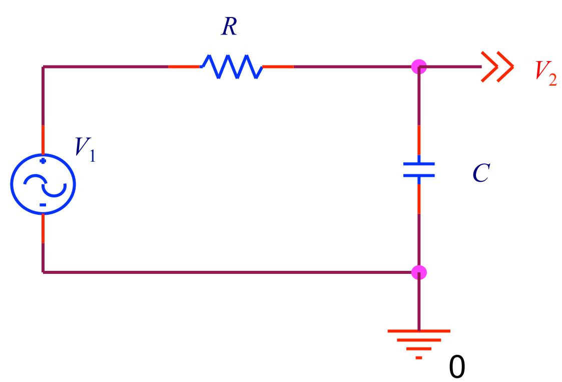
Low Pass

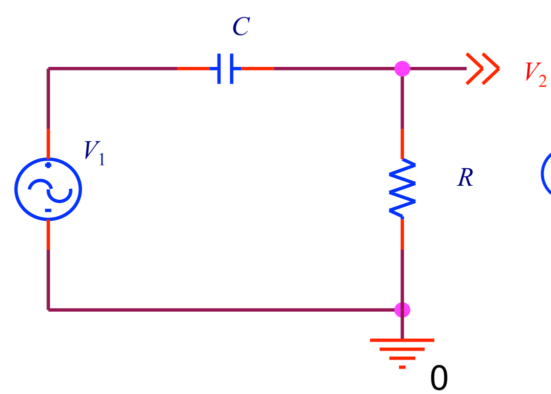


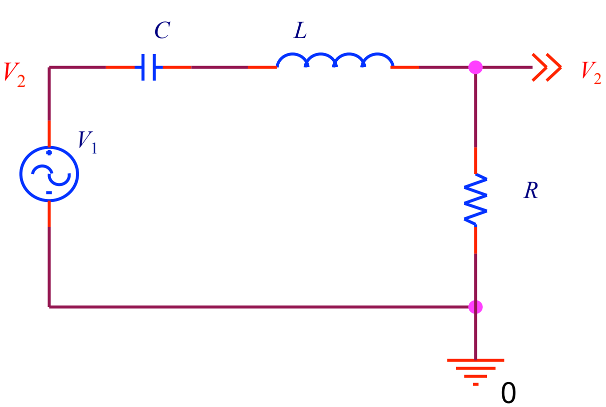
High Pass



Band Pass







Idealized Filters

- Real filters can distort the signal since the filter can treat each frequency different and, therefore, change the signal's form
- Then how do we get distortionless transmission?
  - In general, $V_2(j\omega)=H(j\omega)V_1(j\omega)$ where $H(j\omega)= |H(j\omega)| e^{j\theta(\omega)}$ where $|H(j\omega)|$ is the amplitude and $\theta(\omega)$ is the phase angle
  - To be distortionless $v_2(t)$ should have the same shape as $v_1(t)$ and can be a time shift of $v_2(t)$ or $v_2(t)=h_o v_1(t-T_d)$
  - Then, we need to have $H(j\omega)=h_o e^{-j\omega T_d}$ where the amplitude is independent of frequency and the phase angle linear with ω.

Idealized Filters Continued

$$H(j\omega) = h_o e^{-j\omega T_d}$$

$$v_2(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h_o e^{-j\omega T_d} V_1(j\omega) e^{+j\omega t} d\omega$$

$$= \frac{h_o}{2\pi} \int_{-\infty}^{\infty} V_1(j\omega) e^{j\omega(t-T_d)} d\omega$$

Replacing $t - T_d$ with x

$$v_2(x + T_d) = \frac{h_o}{2\pi} \int_{-\infty}^{\infty} V_1(j\omega) e^{j\omega x} d\omega = h_o v_1(x)$$

or

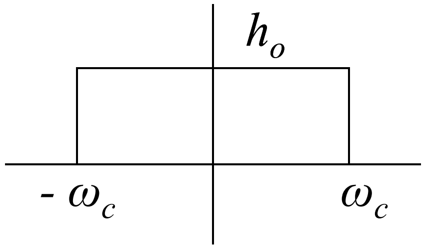
$$v_2(t) = h_o v_1(t - T_d)$$

Idealized Filters Formulation

Low Pass with cutoff frequency ω_c

$$H_{lp}(j\omega) = h_o e^{-j\omega T_d} \quad \text{for } -\omega_c < \omega < \omega_c$$

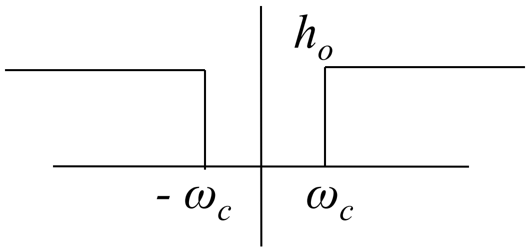
0 elsewhere



High Pass with cutoff frequency ω_c

$$H_{hp}(j\omega) = h_o e^{-j\omega T_d} \quad \text{for } |\omega| > \omega_c$$

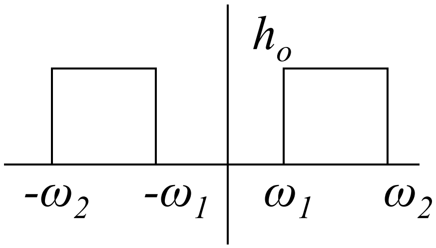
0 elsewhere



Band Pass with cutoff frequencies ω_1 and ω_2

$$H_{bp}(j\omega) = h_o e^{-j\omega T_d} \quad \text{for } \omega_2 > |\omega| > \omega_1$$

0 elsewhere



Response of an Ideal Low Pass Filter to a Unit Impulse

$$v_2(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} V_2(j\omega)e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)V_1(j\omega)e^{j\omega t}d\omega$$

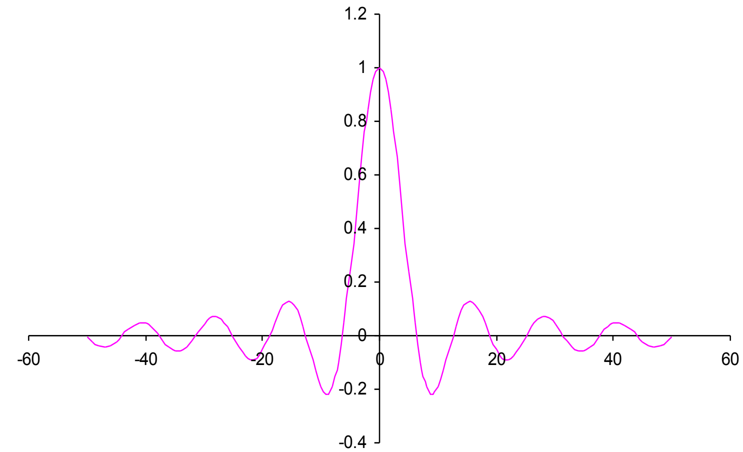
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)1e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} h_o e^{-j\omega T_d}e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} h_o e^{j\omega(t-T_d)}d\omega$$

$$= \frac{h_o}{2\pi}\frac{e^{j\omega(t-T_d)}}{j(t-T_d)}\Big|_{-\omega_c}^{\omega_c}$$

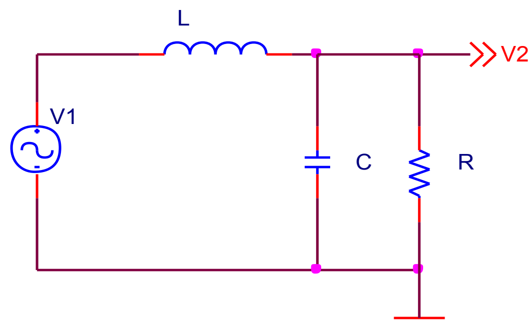
$$= \frac{h_o\omega_c}{\pi}Sa[\omega_c(t-T_d)]$$



This shows the peak of Sa is proportional to the cutoff frequency and that $v_2(t)$ is nonzero for $t<0$….Ooops

Ideal Filters are not realizable but are still a useful mathematical tool!

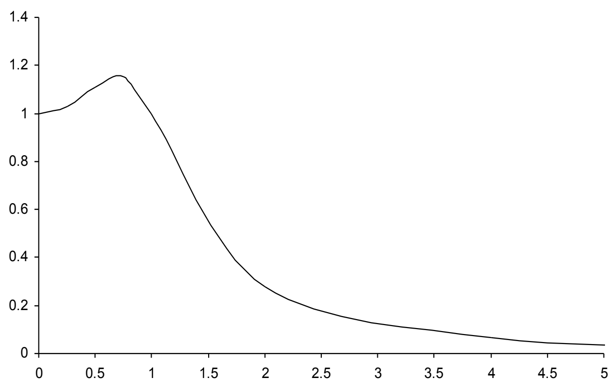
Response to a Real LP Filter

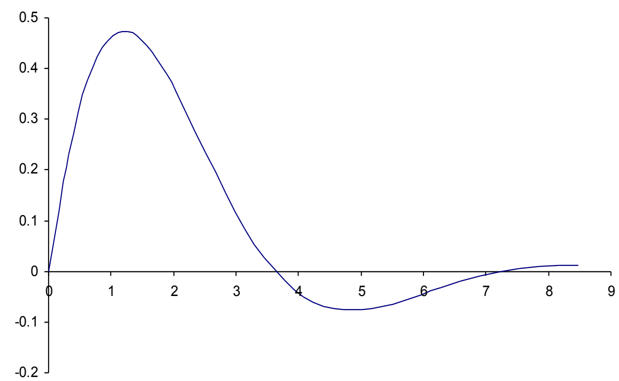


$$R = \sqrt{\frac{L}{C}};\ \omega_o^2 = \frac{1}{LC}$$

$$v_2(t) = \frac{2}{\sqrt{3}}\omega_o e^{-\omega_o t/2}\sin(\frac{\sqrt{3}}{2}\omega_o t)u(t)$$

$$H(j\omega) = \omega_o^2 \frac{1}{\omega_o^2 - \omega^2 + j\omega_o\omega}$$





Although the shapes of the spectrum and time plots are similar to the ideal filter, there is no sharp cutoff and no anticipatory time function. The advantage to studying ideal filters is that analysis is simpler.

Response to a Real LP Filter

$$R=\sqrt{\frac{L}{C}};\ \omega_o^{\,2}=\frac{1}{LC}$$

$$H(j\omega)=\frac{R\,||\,\frac{1}{j\omega}}{R\,||\,\frac{1}{j\omega}+j\omega L}$$

$$=\frac{R}{R+j\omega L+(j\omega)^2RLC}=\frac{1}{1+j\omega\frac{L}{R}+(j\omega)^2LC}=\frac{1}{1+j\frac{\omega}{\omega_o}+(j\frac{\omega}{\omega_o})^2}=\frac{1}{(j\frac{\omega}{\omega_o})^2+j\frac{\omega}{\omega_o}+(\frac{1}{2})^2+1-(\frac{1}{2})^2}$$

$$=\frac{1}{(j\frac{\omega}{\omega_o})^2+j\frac{\omega}{\omega_o}+(\frac{1}{2})^2+\frac{3}{4}}=\frac{1}{[j\frac{\omega}{\omega_o}+\frac{1}{2}]^2+\frac{3}{4}}$$

Note

$$\Im^{-1}\frac{a}{(j\omega)^2+a^2}=[\sin(at)]u(t)$$

$$\Im^{-1}\frac{2}{\sqrt{3}}\frac{\frac{\sqrt{3}}{2}}{[j\frac{\omega}{\omega_o}+\frac{1}{2}]^2+\frac{3}{4}}=\Im^{-1}\frac{2}{\sqrt{3}}\omega_o\frac{\frac{\sqrt{3}}{2}\omega_o}{[j\omega+\frac{1}{2}\omega_o]^2+\frac{3}{4}\omega_o^{\,2}}=\frac{2}{\sqrt{3}}\omega_o e^{-\omega_o t/2}\sin(\frac{\sqrt{3}}{2}\omega_o t)u(t)$$

$$H(j\omega)=\omega_o^{\,2}\frac{1}{\omega_o^{\,2}-\omega^2+j\omega_o\omega}$$

Response to a Real LP Filter

$$R = \sqrt{\frac{L}{C}};\ \omega_o^2 = \frac{1}{LC}$$

$$H(j\omega) = \frac{R \,\|\, \frac{1}{j\omega}}{R \,\|\, \frac{1}{j\omega} + j\omega L} = \frac{R}{R + j\omega L + (j\omega)^2 RLC} = \frac{\omega_o^2}{\omega_o^2 - \omega^2 + j\omega_o\omega}$$

$$\left|H(j\omega)\right| = \frac{\omega_o^2}{\sqrt{[\omega_o^2 - \omega^2]^2 + [\omega_o\omega]^2}}$$

$$\left|H(j\omega)\right|_{\omega=0} = \frac{\omega_o^2}{\sqrt{[\omega_o^2]^2}} = 1$$

$$\left|H(j\omega)\right|_{\omega\to\infty} \to \frac{\omega_o^2}{\sqrt{-\omega^4}} \to 0$$

$$\left|H(j\omega)\right|_{\omega=\omega_o} = \frac{\omega_o^2}{\sqrt{[\omega_o^2 - \omega_o^2]^2 + [\omega_o^2]^2}} = 1$$

$$\left|H(j\omega)\right|_{\omega=\frac{\omega_o}{\sqrt{2}}} = \frac{\omega_o^2}{\sqrt{[\omega_o^2 - \frac{\omega_o^2}{2}]^2 + [\frac{\omega_o}{\sqrt{2}}^2]^2}} = \frac{\omega_o^2}{\sqrt{\frac{\omega_o^4}{4} + \frac{\omega_o^4}{2}}} = \frac{1}{\sqrt{\frac{3}{4}}} = 1.15$$

$$\left|H(j\omega)\right| = \omega_o^2\{[\omega_o^2 - \omega^2]^2 + [\omega_o\omega]^2\}^{-\frac{1}{2}} = \omega_o^2\{[\omega_o^4 - \omega^2\omega_o^2 + \omega^4]\}^{-\frac{1}{2}}$$

$$\frac{d\left|H(j\omega)\right|}{d\omega} = \omega_o^2 - \frac{1}{2}\{[\omega_o^2 - \omega^2]^2 + [\omega_o\omega]^2\}^{-\frac{3}{2}}\{-2\omega_o^2\omega + 4\omega^3\} = 0$$

$$\omega^2 = \frac{\omega_o^2}{2};\omega = \frac{\omega_o}{\sqrt{2}}$$

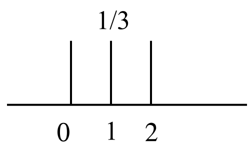
Discrete Signal Filters

Similar to Continuous Signal Filters

Smoothing Function

$$y[n]=\frac{1}{3}(x[n]+x[n-1]+x[n-2])$$

$$h[n]=\frac{1}{3}(\delta[n]+\delta[n-1]+\delta[n-2])$$

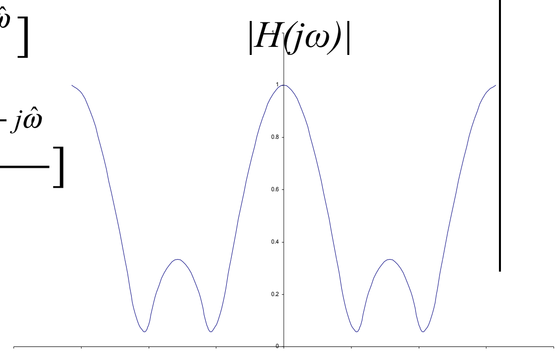


$$H(j\hat{\omega})=\sum_{n=-\infty}^{\infty}h[n]e^{-j\hat{\omega}n}=\frac{1}{3}[1+e^{-j\omega}+e^{-j2\hat{\omega}}]$$

$$=\frac{1}{3}e^{-j\omega}[e^{j\omega}+1+e^{-j\hat{\omega}}]$$

$$=\frac{1}{3}e^{-j\omega}[1+2\frac{e^{j\omega}+e^{-j\hat{\omega}}}{2}]$$

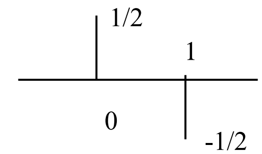
$$=(\frac{1}{3}+\frac{2}{3}\cos\hat{\omega})e^{-j\omega}$$



Edge "Differentiator"

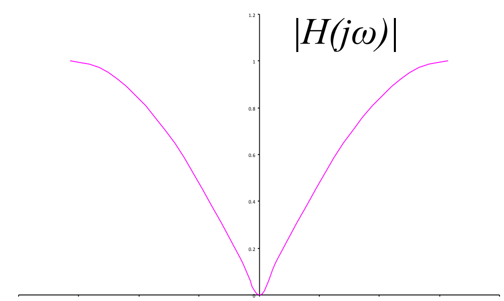
$$y(n)=\frac{1}{2}(x[n]-x[n-1])$$

$$h(n)=\frac{1}{2}(\delta[n]-\delta[n-1])$$



$$H(j\hat{\omega})=\sum_{n=-\infty}^{\infty}h[n]e^{-j\hat{\omega}n}=\frac{1}{2}[1-e^{-j\hat{\omega}}]$$

$$=je^{j\hat{\omega}/2}\sin\hat{\omega}/2$$



Homework

- Find the response to the $\delta(t)$ for
  - Bandpass filter with lower frequency f_1 and upper frequency $f_2=2\,f_1$
  - Band-elimination filter:

$$H(jw)=h_o e^{-j\omega Td},\ 0 < |\omega| < \omega_1 \text{ and } |\omega| > \omega_2$$
$$= 0,\ \omega_1 < |\omega| < \omega_2$$

- Using Matlab build an ideal bandpass filter and calculate the output of the filter for a square wave input. Show cases where the filter removes a single harmonic (e.g., 1^{st}, 2^{nd} and 3^{rd}) and removes 2 or more harmonics. Plot both the time domain and frequency domain for these signals.
  - Do this calculation in the frequency domain Hint: calculate the frequency response of the filter, multiply this with input using its spectrum to get the output spectrum, and invert the output spectrum to get the time domain plot of the output. (You may need to use fft and ifft.)
  - Repeat this calculation in the time domain using the fdatool.

Homework

- Prove that for the impulse response to a real filter

$$Given: R = \sqrt{\frac{L}{C}};\ \omega_o{}^2 = \frac{1}{LC}$$

$$1) H(j\omega) = \frac{\omega_o{}^2}{\omega_o{}^2 - \omega^2 + j\omega_o\omega}$$

$$2) \Im[v_2(t)] = \Im[\frac{2}{\sqrt{3}}\omega_o e^{-\omega_o t/2}\sin(\frac{\sqrt{3}}{2}\omega_o t)u(t)] = H(j\omega)$$

- Use Matlab to calculate a single pole low pass filter and a single pole high pass filter. Plot the Bode plots for each and then apply them to a square wave. Plot the input and output signal spectrum and time signal.