Filters

Lesson 12

3CT.5 – 6

Filters

Recall: $x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$ $\quad y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$

where *h(t)* is the response due to a unit impulse function d(t)

OR

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega)$$

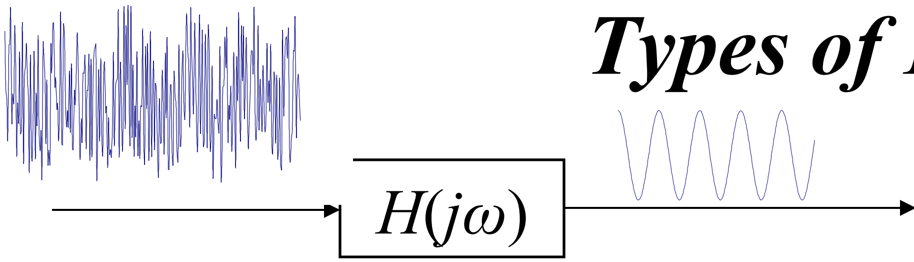
where $H(j\omega)$ is the network response in phasor form

By choosing $H(j\omega)$ or $h(t)$ we can shape the output $y(t)$ for a given $x(t)$' s

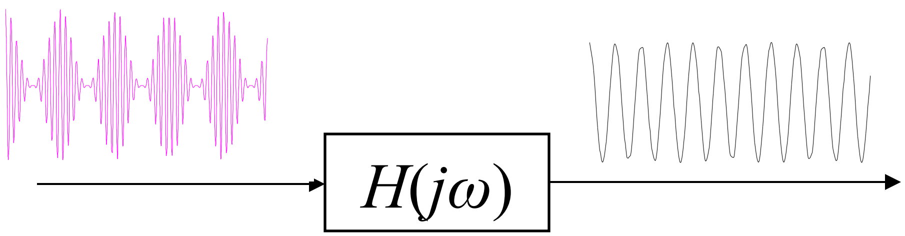
OR

in other words, we can choose $H(j\omega)$ to filter $x(t)$ to obtain a desired $y(t)$

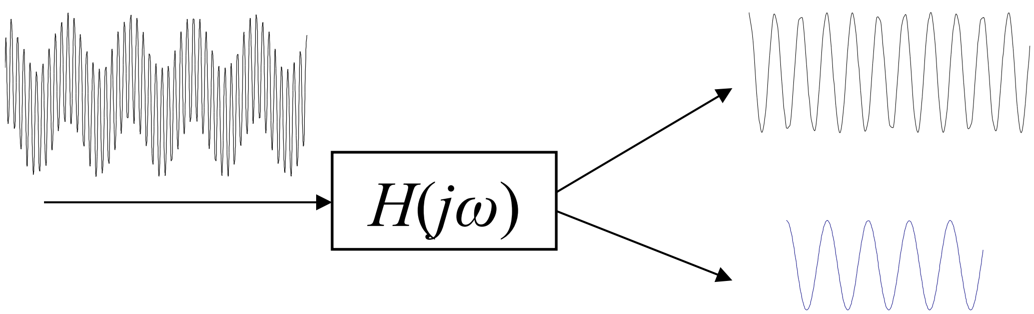
Types of Filtering



Filter out unwanted high frequencies (noise): e.g., radio signal from a satellite

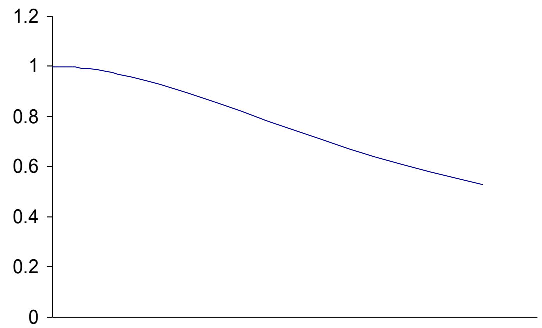


Filter out unwanted low frequencies (noise): e.g., power line disruption

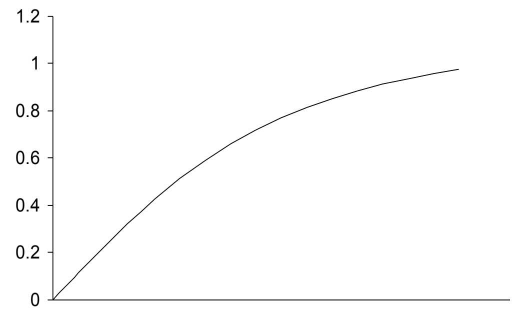


Separate signals in their frequency components: e.g., stereo recording

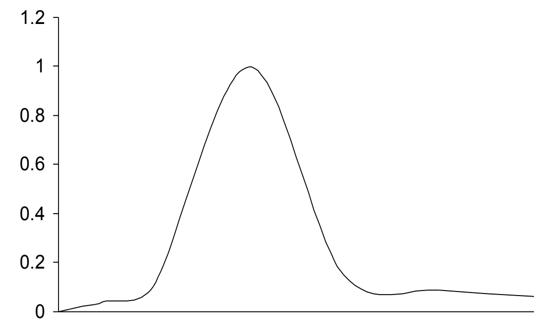
Filters



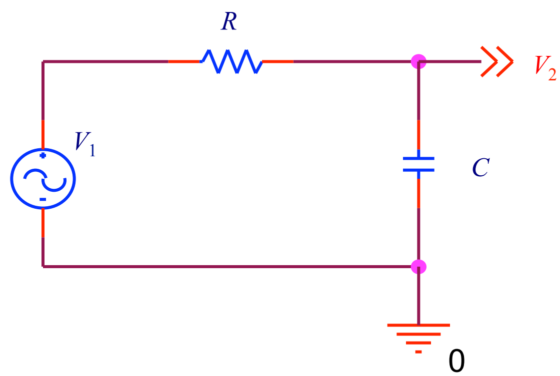
Low Pass

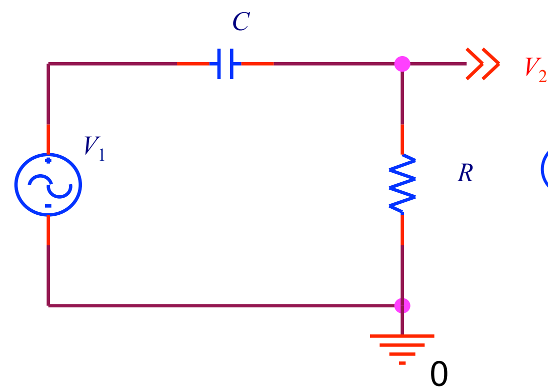


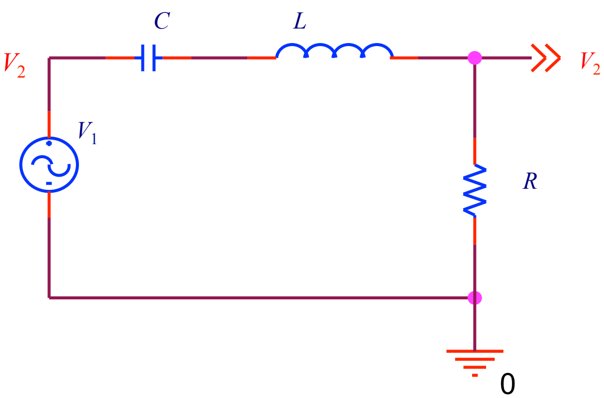
High Pass



Band Pass







Idealized Filters

- Real filters can distort the signal since the filter can treat each frequency different and, therefore, change the signal's form
- Then how do we get distortionless transmission?
  - In general, $V_2(j\omega)=H(j\omega)V_1(j\omega)$ where $H(j\omega)= |H(j\omega)| e^{j\theta(\omega)}$ where $|H(j\omega)|$ is the amplitude and $\theta(\omega)$ is the phase angle
  - To be distortionless $v_2(t)$ should have the same shape as $v_1(t)$ and can be a time shift of $v_2(t)$ or $v_2(t)=h_o v_1(t-T_d)$
  - Then, we need to have $H(j\omega)=h_o e^{-j\omega T_d}$ where the amplitude is independent of frequency and the phase angle linear with ω.

Idealized Filters Continued

$$H(j\omega) = h_o e^{-j\omega T_d}$$

$$v_2(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h_o e^{-j\omega T_d} V_1(j\omega) e^{+j\omega t} d\omega$$

$$= \frac{h_o}{2\pi} \int_{-\infty}^{\infty} V_1(j\omega) e^{j\omega(t-T_d)} d\omega$$

Replacing $t - T_d$ with x

$$v_2(x + T_d) = \frac{h_o}{2\pi} \int_{-\infty}^{\infty} V_1(j\omega) e^{j\omega x} d\omega = h_o v_1(x)$$

or

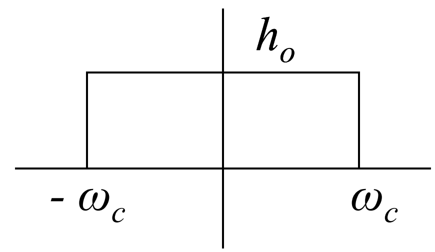
$$v_2(t) = h_o v_1(t - T_d)$$

Idealized Filters Formulation

Low Pass with cutoff frequency ω_c

$H_{lp}(j\omega) = h_o e^{-j\omega T_d}$ for $-\omega_c < \omega < \omega_c$

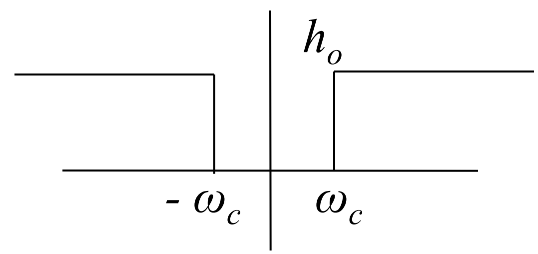
0 elsewhere



High Pass with cutoff frequency ω_c

$H_{hp}(j\omega) = h_o e^{-j\omega T_d}$ for $|\omega| > \omega_c$

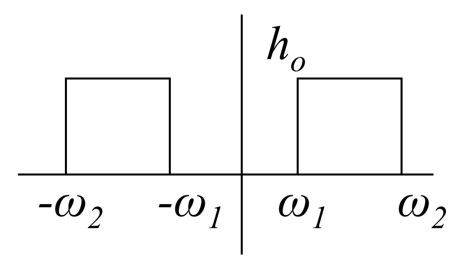
0 elsewhere



Band Pass with cutoff frequencies ω_1 and ω_2

$H_{bp}(j\omega) = h_o e^{-j\omega T_d}$ for $\omega_2 > |\omega| > \omega_1$

0 elsewhere



Response of an Ideal Low Pass Filter to a Unit Impulse

$$v_2(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} V_2(j\omega)e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)V_1(j\omega)e^{j\omega t}d\omega$$

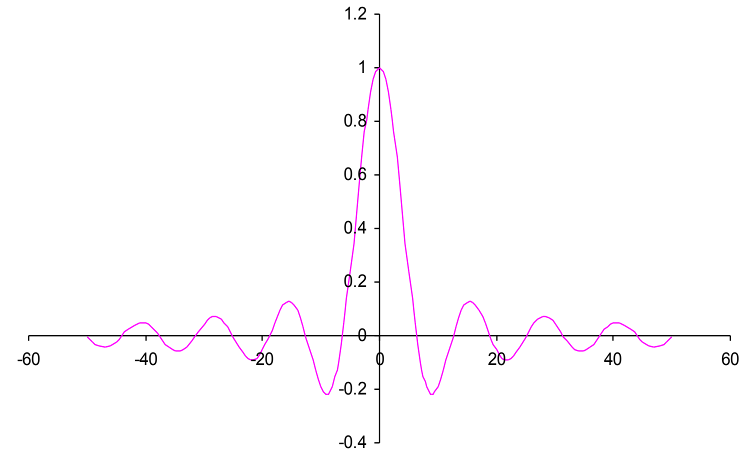
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)1e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} h_o e^{-j\omega T_d}e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} h_o e^{j\omega(t-T_d)}d\omega$$

$$= \frac{h_o}{2\pi}\frac{e^{j\omega(t-T_d)}}{j(t-T_d)}\Big|_{-\omega_c}^{\omega_c}$$

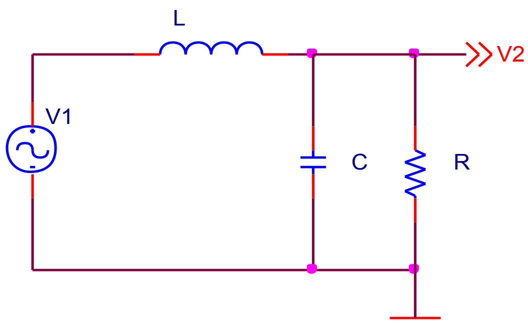
$$= \frac{h_o\omega_c}{\pi}Sa[\omega_c(t-T_d)]$$



This shows the peak of Sa is proportional to the cutoff frequency and that $v_2(t)$ is nonzero for $t<0$….Ooops

Ideal Filters are not realizable but are still a useful mathematical tool!

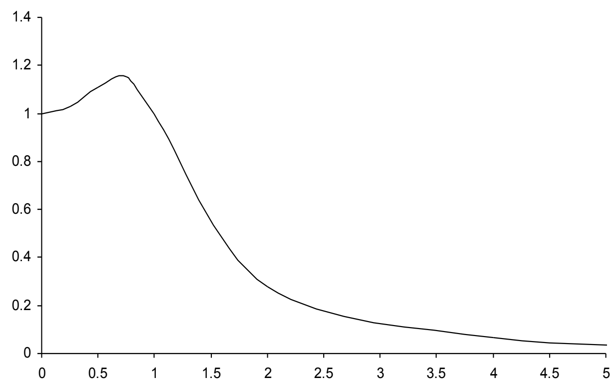
Response to a Real LP Filter

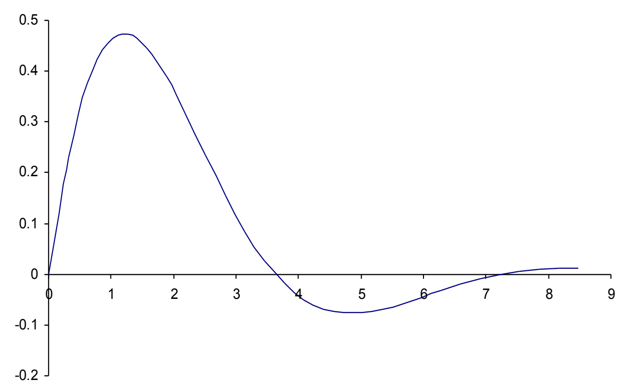


$$R = \sqrt{\frac{L}{C}};\ \omega_o^{\ 2} = \frac{1}{LC}$$

$$v_2(t) = \frac{2}{\sqrt{3}}\omega_o e^{-\omega_o t/2}\sin(\frac{\sqrt{3}}{2}\omega_o t)u(t)$$

$$H(j\omega) = \omega_o^{\ 2}\frac{1}{\omega_o^{\ 2} - \omega^2 + j\omega_o\omega}$$





Although the shapes of the spectrum and time plots are similar to the ideal filter, there is no sharp cutoff and no anticipatory time function. The advantage to studying ideal filters is that analysis is simpler.

Response to a Real LP Filter

$$R = \sqrt{\frac{L}{C}};\ \omega_o^2 = \frac{1}{LC}$$

$$H(j\omega) = \frac{R \parallel \frac{1}{j\omega}}{R \parallel \frac{1}{j\omega} + j\omega L}$$

$$= \frac{R}{R + j\omega L + (j\omega)^2 RLC} = \frac{1}{1 + j\omega \frac{L}{R} + (j\omega)^2 LC} = \frac{1}{1 + j\frac{\omega}{\omega_o} + (j\frac{\omega}{\omega_o})^2} = \frac{1}{(j\frac{\omega}{\omega_o})^2 + j\frac{\omega}{\omega_o} + (\frac{1}{2})^2 + 1 - (\frac{1}{2})^2}$$

$$= \frac{1}{(j\frac{\omega}{\omega_o})^2 + j\frac{\omega}{\omega_o} + (\frac{1}{2})^2 + \frac{3}{4}} = \frac{1}{[j\frac{\omega}{\omega_o} + \frac{1}{2}]^2 + \frac{3}{4}}$$

Note: Looking this up in a book to get the inverse FT.

$$\Im^{-1} \frac{a}{(j\omega)^2 + a^2} = [\sin(at)]u(t)$$

$$\Im^{-1} \frac{2}{\sqrt{3}} \frac{\frac{\sqrt{3}}{2}}{[j\frac{\omega}{\omega_o} + \frac{1}{2}]^2 + \frac{3}{4}} = \Im^{-1} \frac{2}{\sqrt{3}} \omega_o \frac{\frac{\sqrt{3}}{2}\omega_o}{[j\omega + \frac{1}{2}\omega_o]^2 + \frac{3}{4}\omega_o^2} = \frac{2}{\sqrt{3}} \omega_o e^{-\omega_o t/2} \sin(\frac{\sqrt{3}}{2}\omega_o t) u(t)$$

$$H(j\omega) = \omega_o^2 \frac{1}{\omega_o^2 - \omega^2 + j\omega_o\omega}$$

Response to a Real LP Filter

$$R = \sqrt{\frac{L}{C}};\ \omega_o^2 = \frac{1}{LC}$$

$$H(j\omega) = \frac{R \| \frac{1}{j\omega}}{R \| \frac{1}{j\omega} + j\omega L} = \frac{R}{R + j\omega L + (j\omega)^2 RLC} = \frac{\omega_o^2}{\omega_o^2 - \omega^2 + j\omega_o\omega}$$

$$\left|H(j\omega)\right| = \frac{\omega_o^2}{\sqrt{[\omega_o^2 - \omega^2]^2 + [\omega_o\omega]^2}}$$

$$\left|H(j\omega)\right|_{\omega=0} = \frac{\omega_o^2}{\sqrt{[\omega_o^2]^2}} = 1$$

$$\left|H(j\omega)\right|_{\omega\to\infty} \to \frac{\omega_o^2}{\sqrt{-\omega^4}} \to 0$$

$$\left|H(j\omega)\right|_{\omega=\omega_o} = \frac{\omega_o^2}{\sqrt{[\omega_o^2 - \omega_o^2]^2 + [\omega_o^2]^2}} = 1$$

$$\left|H(j\omega)\right|_{\omega=\frac{\omega_o}{\sqrt{2}}} = \frac{\omega_o^2}{\sqrt{[\omega_o^2 - \frac{\omega_o^2}{2}]^2 + [\frac{\omega_o}{\sqrt{2}}^2]^2}} = \frac{\omega_o^2}{\sqrt{\frac{\omega_o^4}{4} + \frac{\omega_o^4}{2}}} = \frac{1}{\sqrt{\frac{3}{4}}} = 1.15$$

$$\left|H(j\omega)\right| = \omega_o^2\{[\omega_o^2 - \omega^2]^2 + [\omega_o\omega]^2\}^{-\frac{1}{2}} = \omega_o^2\{[\omega_o^4 - \omega^2\omega_o^2 + \omega^4]\}^{-\frac{1}{2}}$$

$$\frac{d\left|H(j\omega)\right|}{d\omega} = \omega_o^2 - \frac{1}{2}\{[\omega_o^2 - \omega^2]^2 + [\omega_o\omega]^2\}^{-\frac{3}{2}}\{-2\omega_o^2\omega + 4\omega^3\} = 0$$

$$\omega^2 = \frac{\omega_o^2}{2}; \omega = \frac{\omega_o}{\sqrt{2}}$$

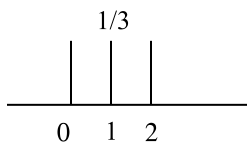
Discrete Signal Filters

Similar to Continuous Signal Filters

Smoothing Function

$$y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2])$$

$$h[n] = \frac{1}{3}(\delta[n] + \delta[n-1] + \delta[n-2])$$

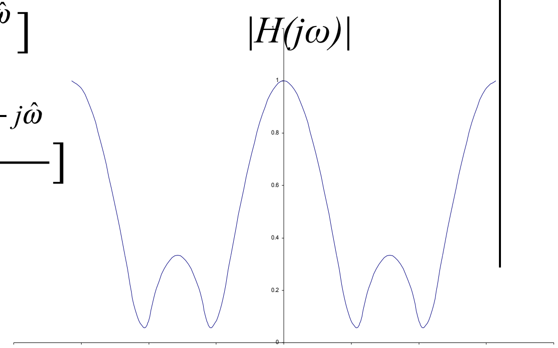


$$H(j\hat{\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\hat{\omega}n} = \frac{1}{3}[1 + e^{-j\omega} + e^{-j2\hat{\omega}}]$$

$$= \frac{1}{3}e^{-j\omega}[e^{j\omega} + 1 + e^{-j\hat{\omega}}]$$

$$= \frac{1}{3}e^{-j\omega}[1 + 2\frac{e^{j\omega} + e^{-j\hat{\omega}}}{2}]$$

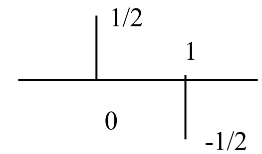
$$= (\frac{1}{3} + \frac{2}{3}\cos\hat{\omega})e^{-j\omega}$$



Edge "Differentiator"

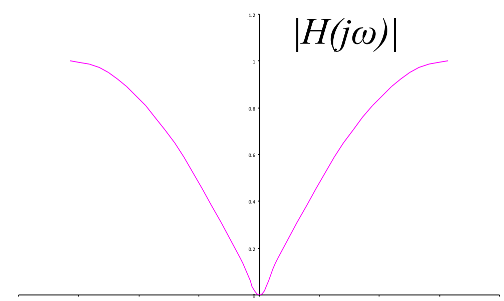
$$y(n) = \frac{1}{2}(x[n] - x[n-1])$$

$$h(n) = \frac{1}{2}(\delta[n] - \delta[n-1])$$



$$H(j\hat{\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\hat{\omega}n} = \frac{1}{2}[1 - e^{-j\hat{\omega}}]$$

$$= je^{j\hat{\omega}/2}\sin \hat{\omega}/2$$



Homework

- Find the response to the $\delta(t)$ for
  - Bandpass filter with lower frequency f_1 and upper frequency $f_2=2\,f_1$
  - Band-elimination filter:

    $H(jw)=h_o e^{-j\omega Td},\ 0 < |\omega| < \omega_1 \text{ and } |\omega| > \omega_2$

    $= 0,\ \omega_1 < |\omega| < \omega_2$
- Using Matlab build an ideal bandpass filter and calculate the output of the filter for a square wave input. Show cases where the filter removes a single harmonic (e.g., 1^{st}, 2^{nd} and 3^{rd}) and removes 2 or more harmonics. Plot both the time domain and frequency domain for these signals.
  - Do this calculation in the frequency domain Hint: calculate the frequency response of the filter, multiply this with input using its spectrum to get the output spectrum, and invert the output spectrum to get the time domain plot of the output. (You may need to use fft and ifft.)
  - Repeat this calculation in the time domain using the fdatool.

Homework

- Prove that for the impulse response to a real filter

$$Given: R = \sqrt{\frac{L}{C}};\ \omega_o^{\,2} = \frac{1}{LC}$$

$$1) H(j\omega) = \frac{\omega_o^{\,2}}{\omega_o^{\,2} - \omega^2 + j\omega_o\omega}$$

$$2) \Im[v_2(t)] = \Im[\frac{2}{\sqrt{3}}\omega_o e^{-\omega_o t/2}\sin(\frac{\sqrt{3}}{2}\omega_o t)u(t)] = H(j\omega)$$

- Use Matlab to calculate a single pole low pass filter and a single pole high pass filter. Plot the Bode plots for each and then apply them to a square wave. Plot the input and output signal spectrum and time signal.