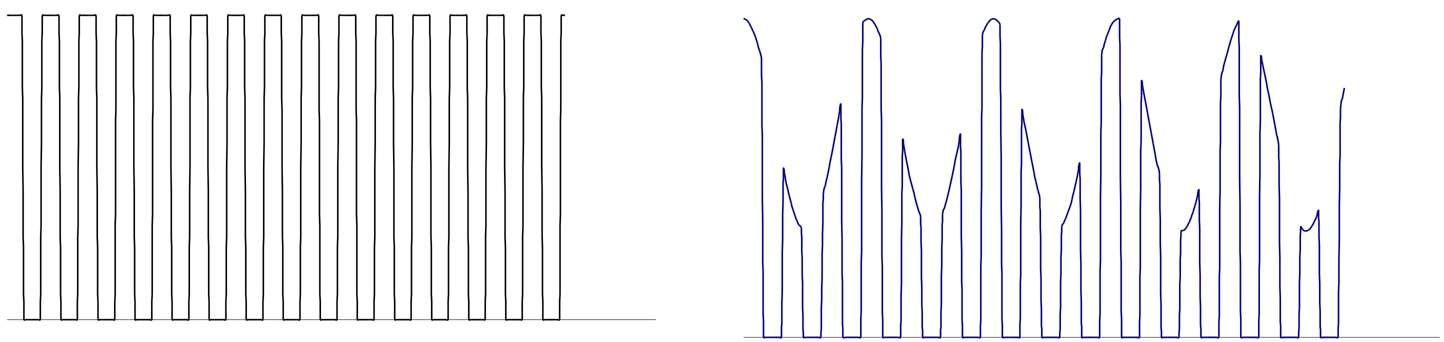
The Sampling Theorem

Lesson 14

Sec 5.3.3

Pulse Amplitude Modulation

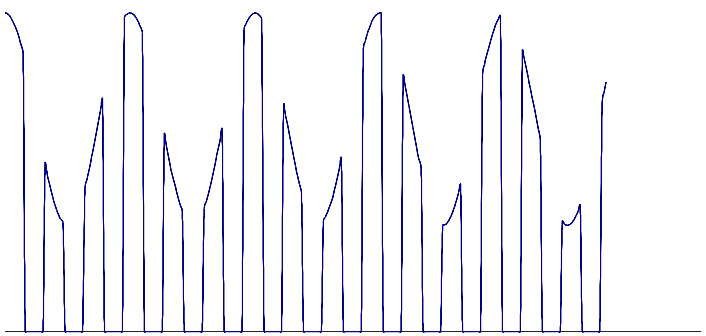
- Instead of using a cosine function as the carrier, let's use a pulse train. This is called PAM.

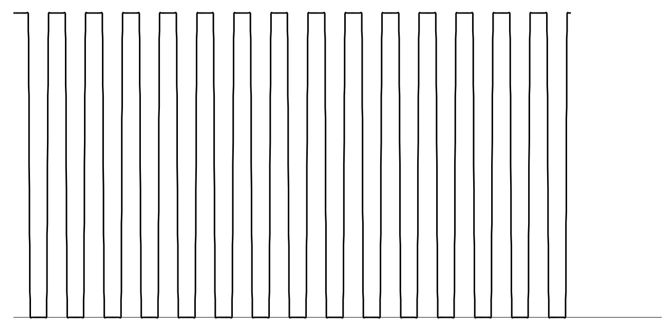


Pulse Amplitude Modulation

- Let $m(t)$ be the signal carrying information which modulates the periodic PAM signal $p(t)$ with fundamental frequency $f_o = 1/\ T_o$.

$$f(t) = m(t)\,p(t) \qquad p(t) = \sum_{-\infty}^{\infty} \mathbf{P}_k e^{jk2\pi t/T_o}, \quad f_o = \frac{1}{T_o}; \mathbf{P}_k = \frac{1}{T_o}\int_{-T_o/2}^{T_o/2} p(t)e^{-jk2\pi t/T_o}dt$$



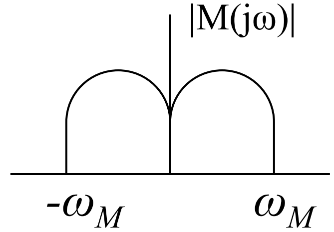


$$f(t) = m(t)\sum_{-\infty}^{\infty} \mathbf{P}_k e^{jk2\pi t/T_o} = \sum_{-\infty}^{\infty} \mathbf{P}_k m(t) e^{jk2\pi t/T_o}$$

$$\Im[f(t)] = F(j\omega) = \sum_{-\infty}^{\infty} \mathbf{P}_k \Im[m(t)e^{jk2\pi t/T_o}]$$

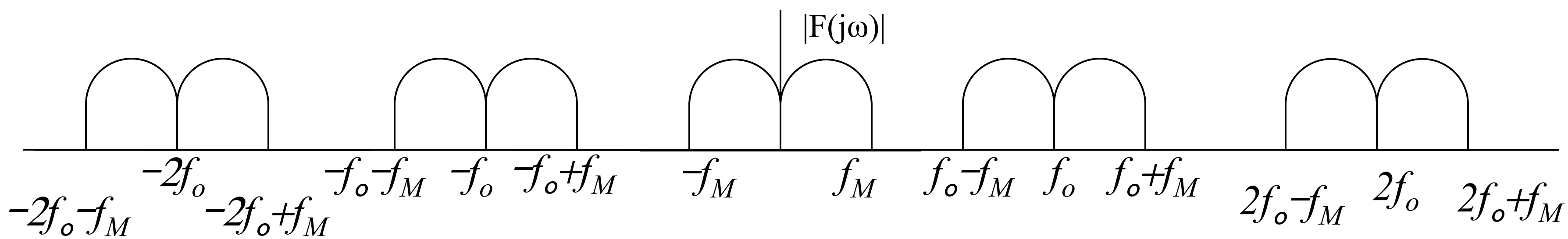
The FT of a PAM signal

$M(j\omega) = \Im[m(t)]$



$$\Im[f(t)] = F(j\omega) = \sum_{-\infty}^{\infty} \mathbf{P}_k \Im[m(t)e^{jk2\pi t/T_o}] = \sum_{-\infty}^{\infty} \mathbf{P}_k M[j(\omega - k2\pi/T_o)] = \sum_{-\infty}^{\infty} \mathbf{P}_k M[j(\omega - k2\pi f_o]$$

Note that to assure that there is no loss of information, we must have $f_o > f_M$ and the minimum value of f_o is 2 f_M or $T_o <= 1/(2 f_M)$. The rate $2f_M$ is called the Nyquist Sampling rate.



Calculation of PAM for a true Impulse Train

$$\mathbf{P}_k = \frac{1}{T_o}\int_{-T_o/2}^{T_o/2} p(t)e^{-jk2\pi t/T_o}dt$$

Let's use the rectangular unit impulse model for $p(t)$

$$p(t) = \frac{1}{2t_c},\ \text{ for } -t_c < t < t_c$$
$$= 0;\ \text{ for } |t| > t_c$$

$$\mathbf{P}_k = \frac{1}{T_o}\int_{-t_c}^{-t_c}\frac{1}{2t_c}e^{-jk2\pi t/T_o}dt = \frac{1}{2t_cT_o}\int_{-t_c}^{-t_c}e^{-jk2\pi t/T_o}dt$$

$$= \frac{1}{2t_cT_o(-jk2\pi/T_o)}e^{-jk2\pi t/T_o}\Big|_{-t_c}^{t_c} = \frac{1}{2t_cT_o(-jk2\pi/T_o)}[e^{-jk2\pi t_c/T_o} - e^{jk2\pi t_c/T_o}]$$

$$= \frac{1}{T_o(\frac{k2\pi t_c}{T_o})}[\frac{e^{jk2\pi t_c/T_o} - e^{-jk2\pi t_c/T_o}}{2j}] = \frac{1}{T_o}\frac{\sin(k2\pi t_c/T_o)}{\frac{k2\pi t_c}{T_o}}$$

$$\mathbf{P}_k = \frac{1}{T_o}Sa(k2\pi t_c/T_o)$$

Note that for a true unit impulse $t_c \to 0$,

$$\lim_{t_c\to 0} \mathbf{P}_k = \frac{1}{T_o}$$

OR

$$\mathbf{P}_k = \frac{1}{T_o}\int_{-T_o/2}^{T_o/2} p(t)e^{-jk2\pi t/T_o}dt = \frac{1}{T_o}\int_{-T_o/2}^{T_o/2}\delta(t)e^{-jk2\pi t/T_o}dt = \frac{1}{T_o}e^{-jk2\pi 0/T_o} = \frac{1}{T_o}$$

$$f(t) = m(t)p(t)$$

$$f(t) = m(t)\sum_{-\infty}^{\infty}\frac{1}{T_o}e^{jk2\pi t/T_o} = \sum_{-\infty}^{\infty}\frac{1}{T_o}m(t)e^{jk2\pi t/T_o}$$

$$\Im[f(t)] = F(j\omega) = \sum_{-\infty}^{\infty}\frac{1}{T_o}\Im[m(t)e^{jk2\pi t/T_o}]$$

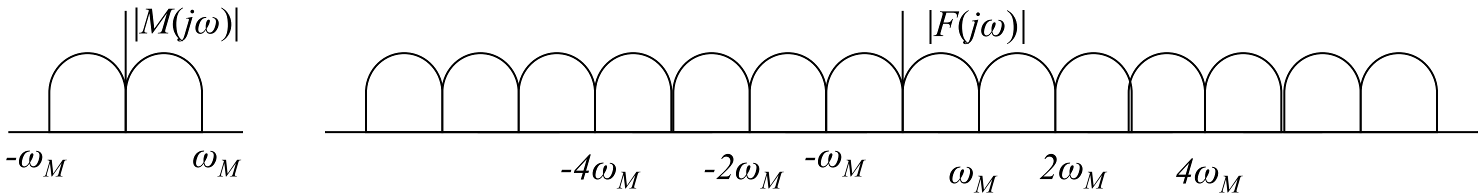
$$= \frac{1}{T_o}\sum_{-\infty}^{\infty}M[j(\omega - k2\pi/T_o)]$$

For the Nyquist rate $T_o = \frac{1}{2f_M}$

$$F(j\omega) = 2f_M\sum_{-\infty}^{\infty}M[j(\omega - k2\omega_M)]$$

Shannon's Sampling Theory

- A BL signal which has no spectral components above the frequency f_M is uniquely specified by its values at uniform intervals of $1/(2f_M)$ seconds.



Assuming $\mathbf{P}_k = \frac{1}{T_o} = f_o$ and for sampling at the Nyquist rate $f_o = 2f_M$

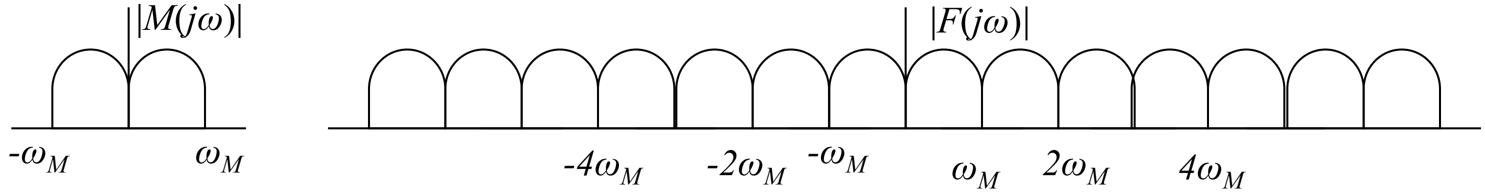
Looks like a Fourier Series in the frequency domain with coefficents, $\mathbf{F}_k$

$$F(j\omega) = \sum_{-\infty}^{\infty} \mathbf{P}_k M[j(\omega - k2\pi f_o)] = 2f_M \sum_{-\infty}^{\infty} M[j(\omega - k2\omega_M)] = \sum_{k=-\infty}^{\infty} \mathbf{F}_k e^{jk2\pi\omega/2\omega_M}$$

$$\mathbf{F}_k = \frac{1}{2\omega_M} \int_{-\omega_M}^{\omega_M} F(j\omega) e^{-jk2\pi\omega/2\omega_M} d\omega = \frac{1}{2\omega_M} \int_{-\omega_M}^{\omega_M} M(j\omega) e^{-j\frac{k2\pi}{2\omega_M}\omega} d\omega$$

$$m(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} M(j\omega) e^{j\omega t} d\omega \underset{\text{BL Signal}}{=} \frac{1}{2\pi} \int_{-\omega_M}^{\omega_M} M(j\omega) e^{j\omega t} d\omega$$

Shannon's Sampling Theory



$$\mathbf{F}_k = \frac{1}{2\omega_M}\int_{-\omega_M}^{\omega_M} M(j\omega)e^{-j\frac{k2\pi}{2\omega_M}\omega}\,d\omega$$

$$m(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} M(j\omega)e^{j\omega t}\,d\omega \underset{\text{BL Signal}}{=} \frac{1}{2\pi}\int_{-\omega_M}^{\omega_M} M(j\omega)e^{j\omega t}\,d\omega$$

let us substitute for $t \Rightarrow \frac{k2\pi}{2\omega_M}$, then $\frac{1}{2\pi}\int_{-\omega_M}^{\omega_M} M(j\omega)e^{-j\frac{k2\pi}{2\omega_M}\omega}\,d\omega = m(-\frac{k2\pi}{2\omega_M})$

$$\mathbf{F}_k = \frac{1}{2\omega_M}\int_{-\omega_M}^{\omega_M} M(j\omega)e^{-j\frac{k2\pi}{2\omega_M}\omega}\,d\omega = \frac{\pi}{\omega_M}m(-\frac{k2\pi}{2\omega_M})$$

$m(-\frac{k2\pi}{2\omega_M}) = m(-\frac{k2\pi}{2\times 2\pi f_M}) = m(-\frac{k}{2f_M}) = m(-\frac{k}{f_S}) = m(-kT_s)$ which are the samples of $m(t)$

$$\therefore \mathbf{F}_k = \frac{\pi}{\omega_M}m(-kT_s), \mathbf{F}_{-k} = \frac{\pi}{\omega_M}m(kT_s)$$

That is, the $\mathbf{F}_k$ are specified from the samples of $m(t)$

Shannon's Sampling Theory Continued

- These latter equations state:

1. The signal $m(t)$ is sampled at the Nyquist rate which is represented by $f(t) = m(t)p(t)$
2. This signal, $f(t)$, which contains the samples of $m(t)$ is transmitted.
3. The spectrum of the transmitted signal $f(t) \rightarrow F(j\omega)$ can be shown to be periodic in the FREQUENCY DOMAIN where its fundamental shape is $M(j\omega)$ the spectrum of $m(t)$.
4. Since $F(j\omega)$ is periodic, it can be represented by a Fourier Series, where $\mathbf{F}_k$ is the Fourier coefficients of this series.
5. We showed that the $\mathbf{F}_k$ *'s* are specified by the samples of $m(t)$

$$\Im[f(t) = p(t)m(t)] = F(j\omega) = 2f_M \sum_{-\infty}^{\infty} M[j(\omega - k2\omega_M)] = \sum_{k=-\infty}^{\infty} \mathbf{F}_k e^{jk2\pi\omega/2\omega_M}$$

$$\mathbf{F}_k = \frac{\pi}{\omega_M} m(-kT_s), \mathbf{F}_{-k} = \frac{\pi}{\omega_M} m(kT_s)$$

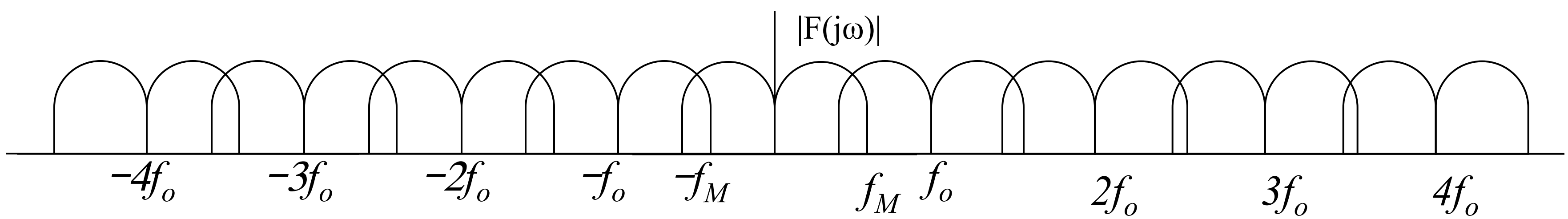
Shannon's Sampling Theory Continued

6. Therefore when we transmit $f(t)$, which are the samples of $m(t)$, we can extract $m(t)$ from it since
   a. The spectrum of $f(t)$, $F(j\omega)$, contains the spectrum of $m(t)$, $M(j\omega)$
   b. $F(j\omega)$ which is periodic can be represented by a Fourier Series with coefficients $\mathbf{F}_k$ $\quad F(j\omega) = 2f_M \sum_{-\infty}^{\infty} M[j(\omega - k2\omega_M)] = \sum_{k=-\infty}^{\infty} \mathbf{F}_k e^{jk2\pi\omega/2\omega_M}$
   c. We can calculate $\mathbf{F}_k$ since it is specified by the samples of $m(t)$ (which is just $f(t)$). $\quad \mathbf{F}_k = \frac{\pi}{\omega_M} m(-kT_s), \mathbf{F}_{-k} = \frac{\pi}{\omega_M} m(kT_s)$
   d. Once we have $\mathbf{F}_k$ we can get $F(j\omega)$ and hence $M(j\omega)$.
   e. From $M(j\omega)$ we then can get $m(t)$ and, therefore we get $m(t)$ from the samples of $m(t)$.

$$m(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} M(j\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\omega_M}^{\omega_M} M(j\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\omega_M}^{\omega_M} F(j\omega) e^{j\omega t} d\omega$$

Shannon's Sampling Theory Continued

- The rate $1/(2f_M)$ is called the Nyquist Sampling rate
- Note the Spectrum if we sample at less than the Nyquist rate:



Reconstruction of f(t) From the Nyquist Samples of f(t)

$$m(t)=\frac{1}{2\pi}\int_{-\omega_M}^{\omega_M} M(j\omega)e^{j\omega t}\,d\omega=\frac{1}{2\pi}\int_{-\omega_M}^{\omega_M} F(j\omega)e^{j\omega t}\,d\omega$$

FS in the frequency domain

$$F(j\omega)=\sum_{k=-\infty}^{\infty}\mathbf{F}_k e^{jk2\pi\omega/2\omega_M}$$

$$=\sum_{k=\infty}^{-\infty}\mathbf{F}_{-k} e^{-jk2\pi\omega/2\omega_M}$$

$$=\sum_{k=\infty}^{-\infty}\frac{\pi}{\omega_M}m(k\pi/\omega_M)e^{-jk2\pi\omega/2\omega_M}$$

$$=\sum_{k=-\infty}^{\infty}\frac{\pi}{\omega_M}m(k\pi/\omega_M)e^{-jk2\pi\omega/2\omega_M}$$

$$m(t)=\frac{1}{2\pi}\int_{-\omega_M}^{\omega_M} F(j\omega)e^{j\omega t}\,d\omega$$

$$=\frac{1}{2\pi}\int_{-\omega_M}^{\omega_M}\sum_{k=-\infty}^{\infty}\frac{\pi}{\omega_M}m(k\pi/\omega_M)e^{-jk2\pi\omega/2\omega_M}e^{j\omega t}\,d\omega$$

$$=\sum_{k=\infty}^{-\infty}m(k\pi/\omega_M)\int_{-\omega_M}^{\omega_M}\frac{e^{j(t-k\pi/\omega_M)\omega}}{2\omega_M}\,d\omega$$

$$=\sum_{k=\infty}^{-\infty}m(k\pi/\omega_M)\frac{\sin[\omega_M(t-k\pi/\omega_M)]}{[\omega_M(t-k\pi/\omega_M)]}=\sum_{k=-\infty}^{\infty}m(k\pi/\omega_M)Sa[\omega_M(t-k\pi/\omega_M)]$$

- Now we know why we call *Sa(x)* the sampling function.
- Also recall that the unit impulse response of a LPF is the *Sa*(*x*).
- So apply the samples, $m(k\pi/\omega_M)$, to a LPF and we will get back $m(t)$.

Response of an Ideal Low Pass Filter to a Unit Impulse

$$v_2(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} V_2(j\omega)e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)V_1(j\omega)e^{j\omega t}d\omega$$

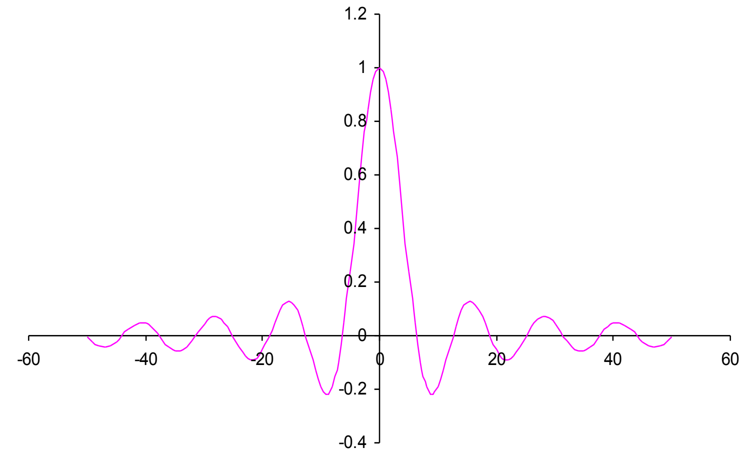
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)1e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} h_o e^{-j\omega T_d}e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} h_o e^{j\omega(t-T_d)}d\omega$$

$$= \frac{h_o}{2\pi}\frac{e^{j\omega(t-T_d)}}{j(t-T_d)}\Big|_{-\omega_c}^{\omega_c}$$

$$= \frac{h_o\omega_c}{\pi}Sa[\omega_c(t-T_d)]$$



This shows the peak of Sa is proportional to the cutoff frequency and that $v_2(t)$ is nonzero for $t<0$….Ooops

Ideal Filters are not realizable but are still a useful mathematical tool!

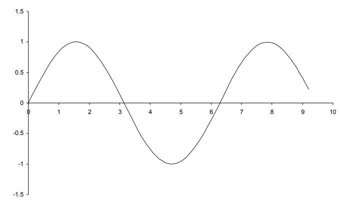
Sampling Theorem Conclusion

- If we have BL signal, don't send the whole signal.
- Sample it at a rate of greater than or equal to $1/(2f_M)$ where f_M is the highest frequency
- Send the samples
- To recover the transmitted samples, pass them through a LPF.

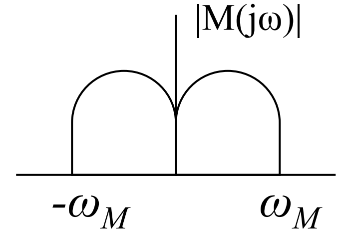
- This leads up to "new" products: Digital TV, DVDs, CD music, Digital Photography
- This also leads to even greater transmission opportunities: Time Division Multiplexing or TDM.

Summary

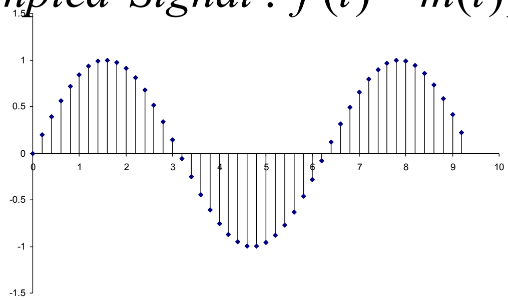
Signal : $m(t)$



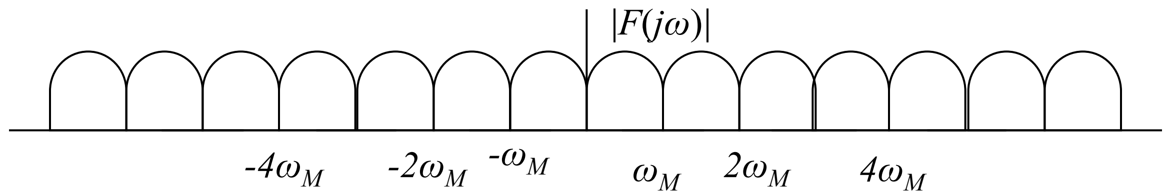
Signal Spectrum



Sampled Signal : $f(t) = m(t)\,p(t)$

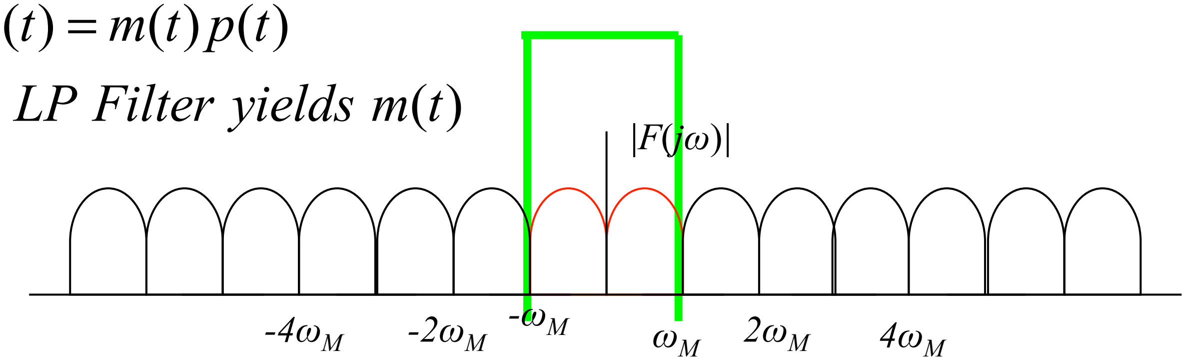


Sampled Signal Spectrum which is transmitted



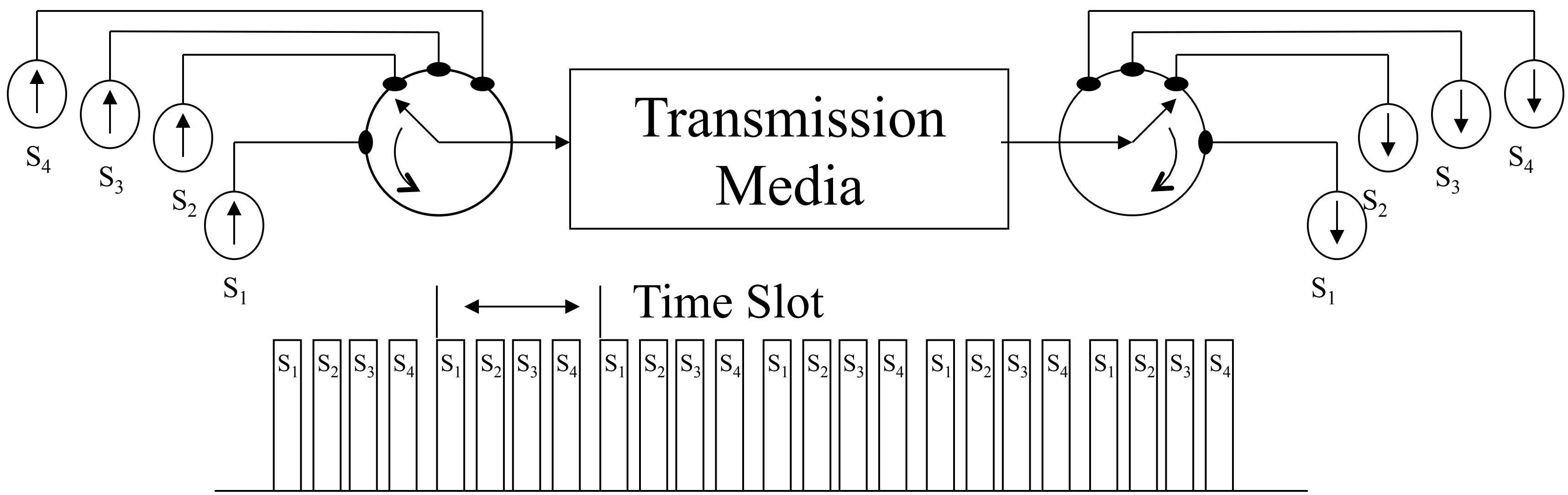
Sampled Signal : $f(t) = m(t)\,p(t)$

Filtered with Ideal LP Filter yields $m(t)$



Time Division Multiplexing

- Send several signals at the “same” time over the same transmission medium.



- This concept has spawned Telecommunications

Homework

- Problem (1)
  - A BL signal with maximum frequency 1000 Hz is sampled at rate of 1000 samples per second, 2000 samples per second, and 4000 samples per second. Draw the sampled spectrum for each and describe whether the samples are sufficient to reconstruct the original signal
- Problem (2)
  - 20 BL signals (1000 Hz) are sampled at the Nyquist Rate. Calculate the pulse width of each sample to support the multiplex of these 20 signals. Calculate the pulse rate of the aggregate multiplex signal. Repeat for 200 signals.
- Problem (3)
  - Consider N signals, each BL (1 Hz). If a transmission system can handle 40 pulses per second, how many messages can be sent? Repeat for 100 pulses per second.