Laplace Transforms

Lesson 18

6CT.1-4

Laplace Transform

- Up till now we have generally assumed that the quadratic content of a signal is finite (unless there are impulses) and that the signal may exist for $t<0$.
- More realistically, these assumptions are too limiting. In fact, $f(t) = 0$ for $t<0$ and we can not always guarantee the quadratic content. Therefore:

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt = \int_{0}^{\infty} f(t)e^{-j\omega t}dt$$

may not converge.

Laplace Transform #2

- Let's look at the following formulation such that the there is a real positive number σ which makes the integral converge

$$\int_0^{\infty} \left| f(t)e^{-\sigma t} \right| dt$$

even if $\int_0^{\infty} f(t)dt$ is not finite.

Laplace Transform #3

- Let's then consider:

$$\Im[f(t)e^{-\sigma t}] = \int_{-\infty}^{\infty} f(t)e^{-\sigma t}e^{-j\omega t}\,dt \quad \text{If } f(t) = 0, t < 0, \text{ then}$$

$$\Im[f(t)e^{-\sigma t}] = \int_{0}^{\infty} f(t)e^{-(\sigma + j\omega)t}\,dt = F(\sigma + j\omega) \text{ Otherwise the integral will blow up for } t < 0$$

Let's define $s = \alpha + j\omega$; $\text{Re}(s) > 0; f(t) \equiv 0, t < 0;$

then the Laplace Transform of $f(t) = F(s) = \int_{0}^{\infty} f(t)e^{-st}\,dt$

note that $F(s)$ is not the FT of $f(t)$

- *F(s)* is called the Laplace Transform of *f(t)*

$$£[f(t)]=F(s)$$

- Note that for functions which are zero for $t<0$ and have finite content $F(s)$ ➔ $F(s)|_{s=j\omega} = F(j\omega)$

An example

$$f(t) = u(t) - u(t-a)$$

$$F(s) = \int_0^\infty [u(t) - u(t-a)]e^{-st} dt$$

$$= \int_0^\infty e^{-st} dt - \int_a^\infty e^{-st} dt$$

$$= \frac{1 - e^{-sa}}{s}$$

Inverse Laplace Transform

Recall that $\Im^{-1}[F(j\omega)] = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega$

By analogy $£^{-1}[F(s)] = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{st}d\omega$; since σ is considered a constant

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\sigma + j\omega)e^{(\sigma+j\omega)t}d\omega \Rightarrow F(s) = F(\sigma + j\omega) = \int_{t=0}^{\infty} f(\tau)e^{-(\sigma+j\omega)\tau}d\tau$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\{\int_{t=0}^{\infty} f(\tau)e^{-(\sigma+j\omega)\tau}d\tau\}e^{(\sigma+j\omega)t}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{t=0}^{\infty} f(\tau)e^{\sigma(t-\tau)}e^{j\omega(t-\tau)}d\tau\, d\omega$$

$$= \int_{0}^{\infty} f(\tau)e^{\sigma(t-\tau)}\{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega(t-\tau)}d\omega\}\, d\tau$$

$$= \int_{0}^{\infty} f(\tau)e^{\sigma(t-\tau)}\{\delta(t-\tau)\}\, d\tau$$

$$= f(t)$$

Recall from lecture 10:

$$\Im^{-1}[1] = \delta(t)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} 1e^{j\omega t}\, d\omega$$

Recall: $\int_{-\infty}^{\infty} f(t)\delta(t-\tau)dt$

$= f(\tau)$

Inverse Laplace Transform

Also $$£[f(t)] = F(s) = \int_0^{\infty} f(t)e^{-st}dt$$

$$= \int_{-\infty}^{\infty} f(t)e^{-st}dt;\ \text{since}\ f(t)\ = 0\ \text{for t} < 0$$

$$= \int_{-\infty}^{\infty} f(t)e^{-\sigma t}e^{-j\omega t}dt = \int_{-\infty}^{\infty} [f(t)e^{-\sigma t}]e^{-j\omega t}dt$$

$$\therefore £[f(t)] = F(s) = \Im[f(t)e^{-\sigma t}]$$

And $\Im^{-1}[F(s)] = f(t)e^{-\sigma t}$

$$\Im^{-1}[F(\sigma + j\omega)] = f(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\sigma + j\omega)e^{j\omega t}d\omega$$

And $f(t) = e^{\sigma t}\,\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} F(\sigma + j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\sigma + j\omega)e^{(\sigma + j\omega)t}d\omega$

now substitute $s \to \sigma + j\omega \Rightarrow ds = jd\omega; -\infty < \omega < \infty \Rightarrow \sigma - j\infty < s < \sigma + j\infty$

$$f(t) = \frac{1}{2\pi j}\int_{s=\sigma - j\infty}^{s=\sigma + j\infty} F(s)e^{st}ds$$

Some Properties

- Superposition holds

$$£[f_1(t)]=F_1(s);\ £[f_2(t)]=F_2(s)$$

$$£[f_1(t)+f_2(t)]=F_1(s)+F_2(s)$$

- Differentiation

$$£[\frac{df(t)}{dt}]=\int_0^\infty \frac{df(t)}{dt}e^{-st}dt$$

Integration by parts, we get

$$udv = uv - vdu$$

$$dv = \frac{df(t)}{dt}dt; u = e^{-st}$$

$$v = f(t); du = -se^{-st}dt$$

$$\int_0^\infty \frac{df(t)}{dt}e^{-st}dt = e^{-st}f(t)\Big|_0^\infty - \int_0^\infty f(t)(-se^{-st})dt$$

$$= 0 - f(0) + s\int_0^\infty f(t)e^{-st}dt = sF(s) - f(0)$$

This seems clumsy – what is $f(0)$?

Some Properties #2

Let's look at $du(t)/dt = \delta(t)$

$$
\begin{aligned}
£[\delta(t)] &= £\,[du(t)/dt] \\
&= s\,£\,u(t) - u(0) \\
&= s\int_0^{\infty} e^{-st}\,dt - u(0) \\
&= 1 - u(0)
\end{aligned}
$$

What is $u(0)$? If we choose $u(0)=u(0^+)=1$, ***then*** $£[\delta(t)]=0$; otherwise if we use $u(0)=u(0^-)=0$, *then* $£[\delta(t)]=1$.

This is a better choice!!! Henceforth, we use:

$$£[df(t)/dt] = sF(s) - f(0^-)$$

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt$$

Some Properties #3

- Higher Order Derivatives

$$\pounds\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2}\frac{df(t)}{dt}\Big|_{t=0} - \cdots - \frac{d^{n-1} f(t)}{dt^{n-1}}\Big|_{t=0}$$

- Integration $g(t) = \int_{-\infty}^{t} f(\tau)\,d\tau$

$$\frac{dg(t)}{dt} = f(t)$$

$$sG(s) - g(0^-) = F(s)$$

$$G(s) = \frac{F(s)}{s} + \frac{1}{s} g(0^-)$$

Initial Conditions

- To simplify this, we can assume that $f(t)=f_a(t)u(t)$ and all the initial conditions are zero.
- If we need to have systems which are not 0 at $t=0^-$, then we can add special sources (called initial condition generators) to represent these conditions.
- So

$$£[df(t)/dt]=sF(s)$$

$$£[d^n f(t)/dt^n]=s^n F(s)$$

$$£[\int f(t)\,dt]=F(s)/s$$

Impulse Response

- Recall:

$$A(p)y(t)=B(p)x(\mathrm{t})$$

$$A(p)h(t)=B(p)\delta(t)$$

$$\pounds[A(p)h(t)]=\pounds[\ B(p)\delta(t)]$$

$$A(s)H(s)=B(s)1$$

$$H(s)=B(s)\ /\ A(s)$$

$$h(t)=\ \pounds^{-1}[H(s)]$$

Some Transforms

$$£[\delta(\mathrm{t})]=1$$

$$£[u(\mathrm{t})]=\frac{1}{s}$$

$$£[tu(t)]=\frac{1}{s^2}$$

$$£[\frac{1}{2}t^2u(t)]=\frac{1}{s^3};£[t^2u(t)]=\frac{2}{s^3}$$

$$\vdots$$

$$£[\frac{1}{n!}t^nu(t)]=\frac{1}{s^{n+1}};£[t^nu(t)]=\frac{n!}{s^{n+1}}$$

Frequency Displacement

$$£[f(t)] = F(s)$$

$$£[f(t)e^{s_1 t}] = \int_0^\infty f(t)e^{-st}e^{s_1 t}\,dt = \int_0^\infty f(t)e^{-(s-s_1)t}\,dt = F(s-s_1)$$

then

$$£[e^{-\alpha t}u(t)] = \frac{1}{s+\alpha}$$

$$£[e^{\pm j\omega t}u(t)] = \frac{1}{s \mp j\omega}$$

$$£[\cos\ \omega t\ u(t)] = \frac{1}{2}\{\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\}$$

$$= \frac{s}{s^2+\omega^2}$$

$$£[\sin\ \omega t\ u(t)] = \frac{1}{2j}\{\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\}$$

$$= \frac{\omega}{s^2+\omega^2}$$

Examples

Example #1

$$£[e^{-4t}\cos\ (10t - 30^o)\ u(t)] = £[e^{-4t}\{.866\cos\ 10t + .5\sin 10t\}\ u(t)]$$

$$= \frac{.866(s+4)}{(s+4)^2+10^2} + \frac{.5\times 10}{(s+4)^2+10^2}$$

Example #2

$$F(s) = \frac{2s+4}{s^2+2s+5}$$

$$= \frac{2(s+1)+2}{(s+1)^2+4}$$

$$f(t) = e^{-t}(2\cos 2t + \sin 2t)u(t)$$

Time Displacement

$$\pounds[f(t)u(t)] = F(s)$$

$$\pounds[f(t\text{-}T)u(t\text{-}T)] = F(s)e^{-sT}$$

Proof :

$$\int_0^{\infty} f(t\text{-}T)u(t\text{-}T)dt = \int_0^{T} 0 + \int_T^{\infty} f(t\text{-}T)u(t\text{-}T)e^{-st}dt$$

Let $x = t\text{-}T$

$$= \int_0^{\infty} f(x)e^{-s(x+T)}dx$$

$$= e^{-sT}\int_0^{\infty} f(x)e^{-sx}dx$$

$$= e^{-sT}F(s)$$

Example

$$v(t) = \frac{t}{T}u(t) - \frac{(t-T)}{T}u(t-T)$$

$$V(s) = \frac{1}{T}[\frac{1}{s^2} - \frac{e^{-sT}}{s^2}]$$

Convolution

$$£[\int_{\tau=-\infty}^{t} f_1(\tau)f_2(t-\tau)d\tau] = £[\int_{\tau=0}^{t} f_1(\tau)f_2(t-\tau)d\tau]$$

$$= \int_{t=0}^{\infty}[\int_{\tau=0}^{\infty} f_1(\tau)f_2(t-\tau)d\tau]e^{-st}dt$$

since $0 < t < \infty;\ 0 < \tau < t \Rightarrow\ 0 < \tau < \infty$

$$= \int_{\tau=0}^{\infty} f_1(\tau)[\int_{t=0}^{\infty} f_2(t-\tau)e^{-st}dt]d\tau$$

Let $x = t - \tau$

$$= \int_{\tau=0}^{\infty} f_1(\tau)[\int_{t=0}^{\infty} f_2(x)e^{-s(x+\tau)}dx]d\tau$$

$$= \int_{\tau=0}^{\infty} f_1(\tau)e^{-s\tau}[\int_{t=0}^{\infty} f_2(x)e^{-sx}dx]d\tau = \int_{\tau=0}^{\infty} f_1(\tau)e^{-s\tau}[F_2(s)]d\tau$$

$$= F_1(s)F_2(s)$$

Convolution

$$£[f_1(t)f_2(t)] = \int_{\tau=0}^{t} f_1(t)f_2(t)e^{-st}\,dt =$$

$$= \int_{\tau=0}^{t} [\frac{1}{2\pi j}\int_{w=\sigma-j\infty}^{w=\sigma+j\infty} F_1(w)e^{wt}\,dw]f_2(t)e^{-st}\,dt$$

$$= \frac{1}{2\pi j}\int_{w=\sigma-j\infty}^{w=\sigma+j\infty} [F_1(w)][\int_{\tau=0}^{t} f_2(t)e^{-(s-w)t}\,dt]dw$$

$$= \frac{1}{2\pi j}\int_{w=\sigma-j\infty}^{w=\sigma+j\infty} [F_1(w)][F_2(s-w)]dw$$

De-Convolution Example

$$\sin(t)u(t) = \int_{-\infty}^{t} f(\tau)e^{-(t-\tau)}u(t-\tau)d\tau = \int_{0}^{t} f(\tau)e^{-(t-\tau)}d\tau$$

What is $f(t)$? Note the above equation is equivalent to the convolution of $f(t)$ with $e^{-t}u(t)$.

$$£[\sin(t)u(t)] = \frac{1}{s^2+1} \text{ and } £[e^{-t}u(t)] = \frac{1}{s+1}$$

and the LT of the convolution is:

$$= F(s) \times \frac{1}{s+1}$$

$$\frac{1}{s^2+1} = F(s) \times \frac{1}{s+1}$$

$$F(s) = \frac{s+1}{s^2+1}$$

$$f(t) = \{\cos t + \sin t\}u(t)$$

Homework

- Problems: 3.1a,b,

3.1a, b The following model for mean aterial pressure following an IV infusion of sodium nitorprusside (SNP) (developed by Slate and Sheppard) us

$$\Delta MAP(s) = \frac{-K_p e^{-s\delta_p}}{\tau_p s + 1} Q(s)$$

where ΔMAP is the SMP - induced change in MAP, Q is the specific SNP IV infusion rate in mg/kg patient weight/min. K_p is the patient's response constant, δ_p is the delay time in minutes to ΔMAP and τ_p is the time constant.

Let $K_p = 1.0$, $\tau_p = 0.75$ min, $\delta_p = 0.5$ min

a) Plot the impulse response, i.e., $Q(t) = \delta(t)$

b)Plot the response due to an unit step $U(t) = u(t)$

Homework

- Problems: 3.4a,

The unit impulse response is given as

$$h(t) = (0.7e^{-5t} + 0.2e^{-t} + 0.1e^{-0.1t})u(t)$$

Find the transfer function $H(s)$ and its poles and zeroes.

- 3.12a,b

A LTI system is described as

$$\dot{y} + 3y = x(t)$$

Find the transfer function $H(s)$ and impulse response : $h(t)$

Homework

- Problems: 13a,b

A LTI system is described as

$$\ddot{y} + 8\dot{y} + 15y = 5x(t)$$

Find the transfer function $H(s)$ and impulse response : $h(t)$

- Sketch $f(t) = tu(t)u(1\text{-}t)$ and find $F(s)$
- Find $f(t)$, if $F(s) = (1\text{-}e^{-2s})/s^2$. Repeat for $F(s) = (1\text{-}s+e^{-2s})/s^3$
- 6CT.2.1
- 6CT.2.2