Solving Systems using Laplace Transforms

Lesson #19

6CT.5-7

Inversion of the LT by Partial Fraction Expansion

$$F(s) = \frac{N(s)}{D(s)} = \frac{F_0(s^m + b_1 s^{m-1} + \cdots + b_m)}{(s^n + a_1 s^{n-1} + \cdots + a_n)}$$

Like a System Function

$$\frac{V_2(s)}{V_1(s)} = H(s) = \frac{B(s)}{A(s)}$$

Simple Roots:

$$F(s) = \frac{N(s)}{(s-s_1)(s-s_2)\cdots(s-s_n)}; s_1 \neq s_2 \neq \cdots \neq s_n$$

$$F(s) = \frac{K_1}{(s-s_1)} + \frac{K_2}{(s-s_2)} + \cdots + \frac{K_n}{(s-s_n)}$$

$$f(t) = (K_1 e^{s_1 t} + K_2 e^{s_2 t} + \cdots + K_n e^{s_n t})u(t)$$

Finding the Ks

$$F(s)=\frac{N(s)}{D(s)}=\frac{N(s)}{(s-s_1)(s-s_2)\cdots(s-s_n)}$$

$$=\frac{K_1}{(s-s_1)}+\frac{K_2}{(s-s_2)}+\cdots+\frac{K_n}{(s-s_n)}$$

$$K_1=(s-s_1)F(s)\,|_{s=s_1}=(s-s_1)\frac{N(s)}{D(s)}\,|_{s=s_1}=(s-s_1)[\frac{K_1}{(s-s_1)}+\frac{K_2}{(s-s_2)}+\cdots+\frac{K_n}{(s-s_n)}]\,|_{s=s_1}$$

$$=K_1+\frac{(s-s_1)K_2}{(s-s_2)}+\cdots+\frac{(s-s_1)K_n}{(s-s_n)}\,|_{s=s_1}=K_1$$

$$\vdots$$

$$K_n=(s-s_n)F(s)\,|_{s=s_n}=(s-s_1)\frac{N(s)}{D(s)}\,|_{s=s_n}$$

Example

$$F(s)=\frac{2s^2+13s+17}{(s+1)(s+3)}$$

$$=2+\frac{N(s)}{(s+1)(s+3)}$$

$$\boxed{\therefore f(t)=2\delta(t)+[3e^{-t}+2e^{-3t}]u(t)}$$

$$=2+\frac{K_1}{(s+1)}+\frac{K_2}{(s+3)}$$

$$K_1=\frac{2(-1)^2+13(-1)+17}{(-1+3)}=\frac{6}{2}=3$$

$$K_2=\frac{2(-3)^2+13(-3)+17}{(-3+1)}=\frac{-4}{-2}=2$$

Complex Roots

$$F(s) = \frac{\mathbf{K_1}}{s - \mathbf{s_1}} + \frac{\mathbf{K_1^*}}{s - \mathbf{s_1^*}} + \cdots + \frac{\mathbf{K}_n}{s - \mathbf{s}_n} + \frac{\mathbf{K}_n^*}{s - \mathbf{s}_n^*}$$

Roots are complex conjugates but the coefficients must also be complex conjugates so that *f(t)* is real.

Therefore, if

$$£^{-1}[F(s)] = £^{-1}[\frac{\mathbf{K_1}}{s - \mathbf{s_1}} + \frac{\mathbf{K_1^*}}{s - \mathbf{s_1^*}}] = £^{-1}[\frac{\mathbf{K_1}}{s - \mathbf{s_1}}] + £^{-1}[\frac{\mathbf{K_1^*}}{s - \mathbf{s_1^*}}]$$

$$= (\mathbf{K_1} e^{\mathbf{s_1} t} + \mathbf{K_1^*} e^{\mathbf{s_1}^* t})u(t)$$

$$= (K_1 e^{j\gamma} e^{\sigma t + j\omega t} + K_1 e^{-j\gamma} e^{\sigma t - j\omega t})u(t);$$

where $\mathbf{K_1} = K_1 e^{j\gamma}$ and $\mathbf{s_1} = \sigma + \mathrm{j}\omega$

$$£^{-1}[F(s)] = K_1 e^{\sigma t}[e^{j(\omega t + \gamma)} + e^{-j(\omega t + \gamma)}]u(t)$$

$$= 2K_1 e^{\sigma t}\cos(\omega t + \gamma)u(t)$$

$$= 2\,\mathrm{Re}[\mathbf{K_1} e^{\mathbf{s_1} t}]u(t)$$

Multiple Roots

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s-s_1)^P D_1(s)}$$

where $D_1(s) = (s-s_1)(s-s_2)\cdots(s-s_n)$

$$= \frac{M_0}{(s-s_1)^P} + \frac{M_1}{(s-s_1)^{P-1}} + \cdots + \frac{M_{P-1}}{(s-s_1)} + \frac{R(s)}{D_1(s)}$$

$$f(t) = \{\frac{M_0}{(P-1)!}t^{P-1} + \frac{M_1}{(P-2)!}t^{P-2} + M_{P-1}\}e^{-s_1 t}u(t) + f_1(t)$$

$$M_0 = (s-s_1)^P F(s)|_{s=s_1}$$

$$M_1 = \frac{d}{ds}\{(s-s_1)^P F(s)\}|_{s=s_1} = \frac{d}{ds}\{(s-s_1)^P \frac{M_0}{(s-s_1)^P} + \frac{M_1}{(s-s_1)^{P-1}} + \cdots + \frac{M_{P-1}}{(s-s_1)} + \frac{R(s)}{D_1(s)}\}|_{s=s_1}$$

$$= \frac{d}{ds}\{M_0 + (s-s_1)M_1 + \cdots + (s-s_1)^{P-1}M_{P-1} + (s-s_1)^P \frac{R(s)}{D_1(s)}\}|_{s=s_1}$$

$$= 0 + M_1 + \cdots + (P-1)(s-s_1)^{P-2}M_{P-1} + (P)(s-s_1)^{P-1}\frac{R(s)}{D_1(s)}\}|_{s=s_1} = M_1$$

$$\vdots$$

$$M_k = \frac{1}{k!}\frac{d^k}{ds^k}\{(s-s_1)^P F(s)\}|_{s=s_1}$$

An Example

$$F(s) = \frac{3s^2 + 39s + 138}{(s+4)^2(s+2)}$$

$$= \frac{M_o}{(s+4)^2} + \frac{M_1}{(s+4)} + \frac{K_1}{(s+2)}$$

$$K_1 = (s+2)F(s)\,|_{s=-2} = \frac{3(-2)^2 + 39(-2) + 138}{(-2+4)^2} = \frac{72}{4} = 18$$

$$M_0 = (s+4)^2 F(s)\,|_{s=-4} = \frac{3(-4)^2 + 39(-4) + 138}{(-4+2)} = \frac{30}{-2} = -15$$

$$M_1 = \frac{d}{ds}\{(s+4)^2 F(s)\}\,|_{s=-4} = \frac{d}{ds}\{\frac{3s^2 + 39s + 138}{(s+2)}\}\,|_{s=-4} =$$

$$= \{\frac{6s+39}{s+2} - \frac{3s^2 + 39s + 138}{(s+2)^2}\}\,|_{s=-4}$$

$$= \frac{15}{-2} - \frac{30}{4} = -15$$

$$\therefore f(t) = [-15(t+1)e^{-4t} + 18e^{-2t}]u(t)$$

Another Example

$$F(s)=\frac{3s^2+17s+47}{(s+2)(s^2+4s+29)}=\frac{3s^2+17s+47}{(s+2)(s+2-j5)(s+2+j5)}$$

$$=\frac{K_1}{(s+2)}+\frac{\mathbf{K_2}}{(s+2-j5)}+\frac{\mathbf{K_2^*}}{(s+2+j5)}$$

$$K_1=(s+2)F(s)|_{s=-2}=\frac{3(-2)^2+17(-2)+47}{[(-2)^2+4(-2)+29]}=\frac{25}{25}=1$$

$$\mathbf{K_2}=(s+2-j5)F(s)|_{s=-2+j5}$$

$$=\frac{3(-2+j5)^2+17(-2+j5)+47}{(-2+j5+2)(-2+j5+2+j5)}$$

$$=\frac{(-63-j60)+(-34+j85)+47}{(j5)(j10)}$$

$$=\frac{(-50+j25)}{(-50)}=1-j0.5$$

$$=1.116\angle-0.46$$

$$\mathbf{K_2^*}=1+j.5=1.116\angle 0.46$$

$$\therefore f(t)=[1+2.232\cos(5t-0.46)]e^{-2t}u(t)$$

One More Example

$$F(s) = \frac{s^2}{(s^2+1)^2} = \frac{s^2}{(s-j)^2(s+j)^2}$$

$$= \frac{\mathbf{M_o}}{(s-j)^2} + \frac{\mathbf{M_o}^*}{(s+j)^2} + \frac{\mathbf{M}_1}{(s-j)} + \frac{\mathbf{M}_1^*}{(s+j)}$$

$$\mathbf{M_o} = (s-j)^2 F(s)|_{s=j} = \frac{j^2}{(j+j)^2} = \frac{-1}{(2j)^2} = \frac{1}{4}$$

$$\mathbf{M}_1 = \frac{d}{ds}(s-j)^2 F(s)|_{s=j} = \frac{d}{ds}\frac{s^2}{(s+j)^2}|_{s=j}$$

$$= \{\frac{2s}{(s+j)^2} - \frac{2s^2}{(s+j)^3}\}|_{s=j} = \frac{2j}{(j+j)^2} - \frac{2j^2}{(j+j)^3}$$

$$= \frac{2j}{-4} - \frac{-2}{-8j} = -\frac{j}{2} - \frac{1}{4j} = -\frac{j}{2} + \frac{j}{4} = -\frac{j}{4}$$

$$f(t) = 2\frac{1}{4}[t\cos t + \cos(t - \frac{\pi}{2})]u(t) = \frac{1}{2}[t\cos t + \cos(t - \frac{\pi}{2})]u(t)$$

Solving a System with LT

$$V(t) = i(t)R + L\frac{di(t)}{dt}$$

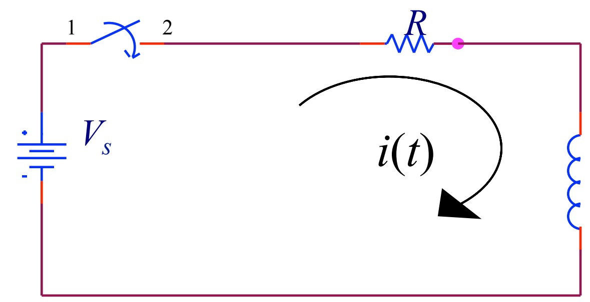
$$\frac{di}{dt} + \frac{R}{L}i = \frac{V(t)}{L} = \frac{Vu(t)}{L}$$

$$\pounds[\frac{di}{dt} + \frac{R}{L}i] = \pounds[\frac{V(t)}{L}] = \pounds[\frac{Vu(t)}{L}]$$

$$sI(s) + \frac{R}{L}I(s) = \frac{V/s}{L}$$

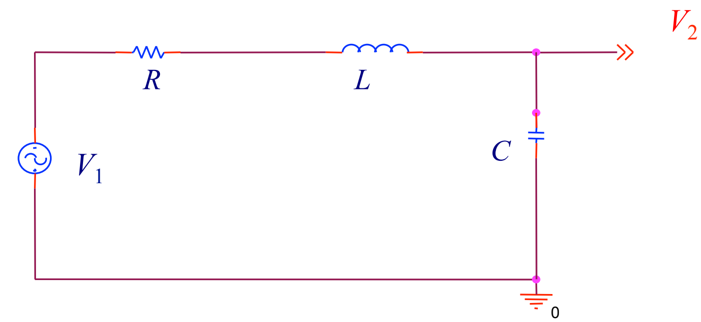
$$I(s) = \frac{V/sL}{s + R/L} = \frac{V/L}{s(s + R/L)}$$

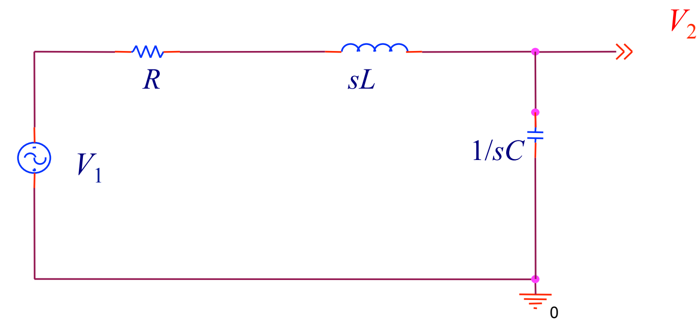
$$I(s) = \frac{V/R}{s} - \frac{V/R}{(s + R/L)}$$



$$i(t) = \frac{V}{R}(1 - e^{-\frac{R}{L}t})u(t)$$

Replacing Circuit Element With Their LT Equivalents





Note based on the previous example, we can replace circuit elements with their LT equivalents

$R \rightarrow R$

$L \rightarrow sL$

$C \rightarrow 1/(sC)$

And we get, solving for $V_2(s)/V_1(s)$

$$\frac{V_2(s)}{V_1(s)} = \frac{1/sC}{R + sL + 1/sC}$$

$$= \frac{1/LC}{s^2 + (R/L)s + 1/LC}$$

Example Continued

Let's choose R=5, L=1h, C=1/6f, and let's solve for $h(t)$, [i.e., $v_1(t)=\delta(t)$] and then $v_1(t)=u(t)$

$$H(s)=\frac{6}{s^2+5s+6}=\frac{6}{(s+3)(s+2)}$$

$$=\frac{K_1}{s+3}+\frac{K_2}{s+2}$$

$$K_1=\frac{6}{(s+2)}\Big|_{s=-3}=\frac{6}{(-3+2)}=-6$$

$$K_2=\frac{6}{(s+3)}\Big|_{s=-2}=\frac{6}{(-2+3)}=6$$

$$h(t)=6[e^{-2t}-e^{-3t}]u(t)$$

$$V_2(s)=\frac{6}{s^2+5s+6}V_1(s)=\frac{6}{s(s+3)(s+2)}$$

$$=\frac{K_1}{s+3}+\frac{K_2}{s+2}+\frac{K_3}{s}$$

$$K_1=\frac{6}{s(s+2)}\Big|_{s=-3}=\frac{6}{(-3)(-3+2)}=2$$

$$K_2=\frac{6}{s(s+3)}\Big|_{s=-2}=\frac{6}{(-2)(-2+3)}=-3$$

$$K_3=\frac{6}{(s+2)(s+3)}\Big|_{s=0}=\frac{6}{(2)(3)}=1$$

$$v_2(t)=[1+2e^{-3t}-3e^{-2t}]u(t)$$

Poles of the Response

$$\frac{V_2(s)}{V_1(s)} = \frac{B(s)}{A(s)}$$

$$V_2(s) = \frac{B(s)}{A(s)} V_1(s) = \frac{B(s)}{A(s)} \frac{V_1^N(s)}{V_1^D(s)}$$

The poles of $H(s) = B(s)/A(s)$ (zeros of $A(s)$) are associated with the solutions to the source free response.

The poles of $V_1(s)$ (zeros of $V_1^D(s)$) are associated with the solutions to the source response.

Homework

- Problems: 3.3,

  A LTI system is described as

  $$H(s) = 5.263 \frac{s + 1.9}{(s + 10)(s + 1)}$$

  Find the impulse response : $h(t)$

  Draw the Bode plot for $0.01 \le f \le 100$cycles/min

- 3.4b,

  The unit impulse response is given as

  $$h(t) = (0.7e^{-5t} + 0.2e^{-t} + 0.1e^{-0.1t})u(t)$$

  Find the response due to a unit step function.

Homework

- Problems: 3.12c

  A LTI system is described as

  $\dot{y} + 3y = x(t)$

  Find the response due to $(1 + 2e^{-t})u(t)$

- 3.13c

  A LTI system is described as

  $\ddot{y} + 8\dot{y} + 15y = 5x(t)$

  Find the response due to a unit step : $x(t) = u(t)$

Homework

- Problems: 3.18a&c

  A LTI system is described as

  $$Y(s) = \frac{2}{s(s^2 + 3s + 2)}$$

  Find the poles of the system

  Find $y(t)$

- 3.23

  A LTI system is described as

  $$H(s) = \frac{10}{(s+4)(s+2)^3}$$

  Find $h(t)$

Homework

- Find the inverse transforms for:

$$(a)\ \frac{1-e^{-3s}}{s(s+2)};(b)\ \frac{-s^2+9s+10}{s(s+2)(s+5)}$$

$$(c)\ \frac{90s-800}{(s^2+100)(s+4)};(d)\ \frac{s+2}{[(s+1)^2+1]^2}$$

$$(e)\ \frac{s^3+8s^2+18s+12}{(s^2+2s+2)(s^2+6s+10)};(f)\ \frac{s^2+4s+3}{(s+2)(s+4)}$$