BME 333 Biomedical Signals and Systems

Some Review of Signals and Systems

Lecture #1

1.1 – 1.3

What Is this Course All About ?

- To Gain an Appreciation of the Various Types of Signals and Systems
- To Analyze The Various Types of Systems
- To Learn the Skills and Tools needed to Perform These Analyses.

What are Signals?

- A Signal is a term used to denote the information carrying property being transmitted to or from an entity such as a device, instrument, or physiological source
- Examples:
  - Radio and Television Signals
  - Telecommunications and Computer Signals
  - Biomedical Engineering Signals

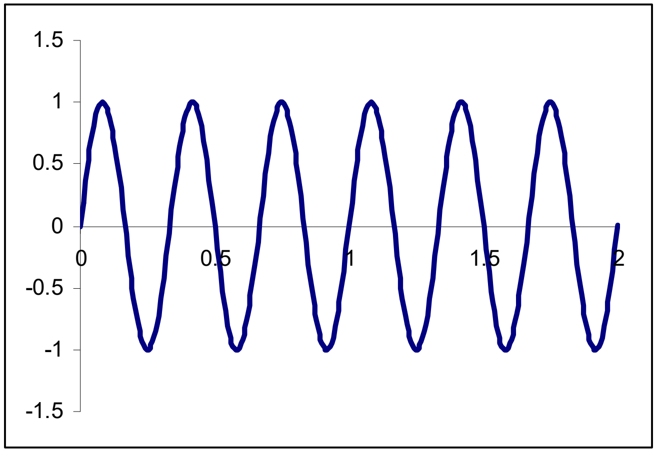
What is a System?

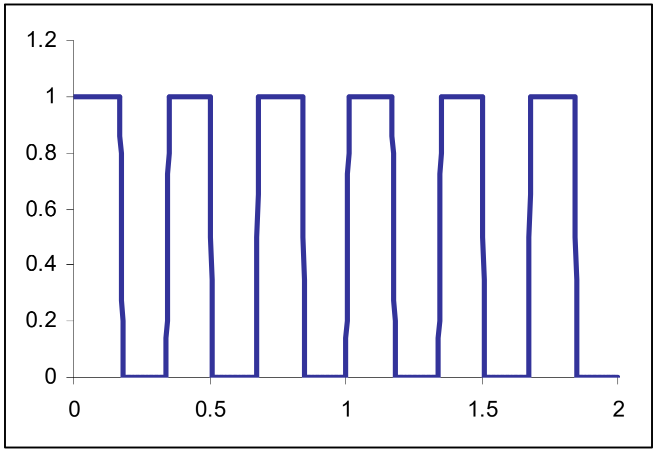
- A System is a term used to denote an entity that processes a Signal
- A System has inputs and outputs
- Examples
  - Amplifiers, Radios, Televisions
  - Telephone, Modem, Computer
  - Oscilloscopes, EKG, EEG, EMG

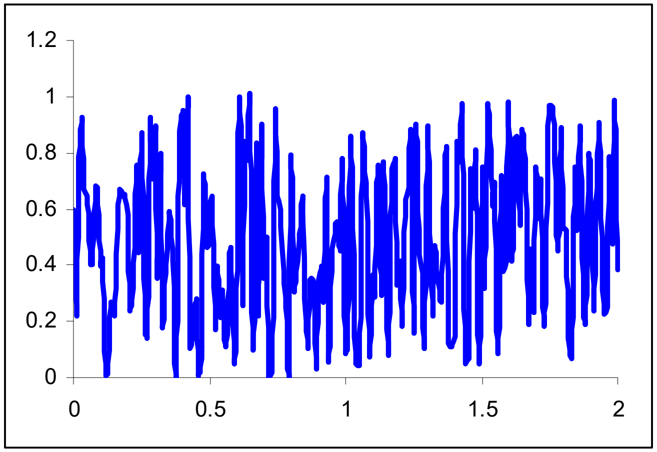
How do we describe Signals?

- Signals are associated with an independent variable(s): e.g., time, single or multivariate spatial coordinate
  - Most instrumentation signals have time as their independent variable
  - A digital photograph or image has spatial coordinates as its independent variables
- Signal Independent Variables can be either Continuous or Discrete

Continuous-Time Signals

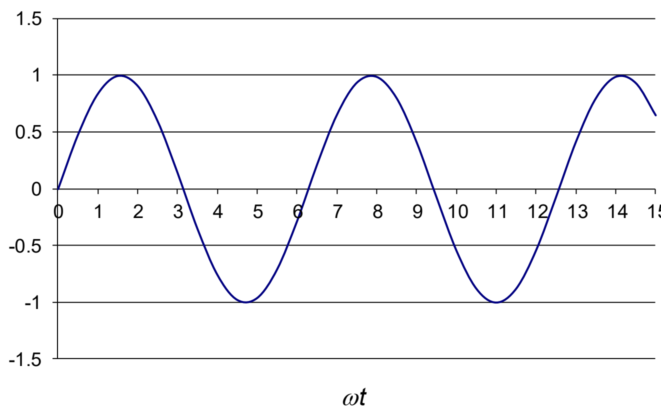


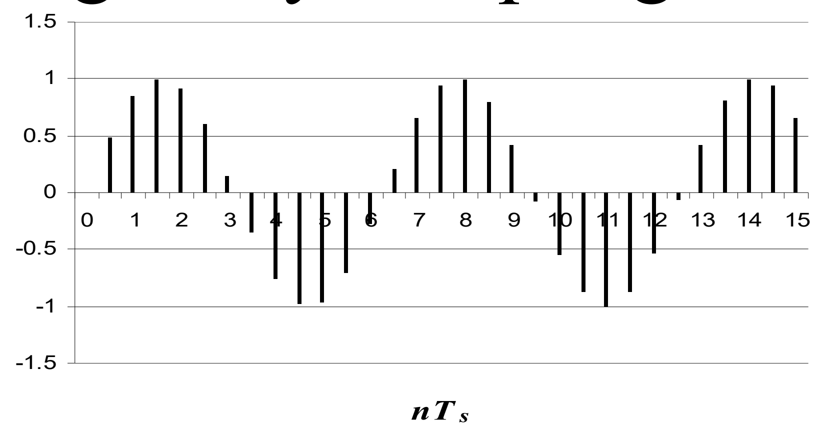




Discrete-Time Signals

A Discrete-Time Signal can be obtained from a Continuous-Time signal by Sampling.





Continuous-Time Signal

$$x(t) = \sin(\omega t)$$

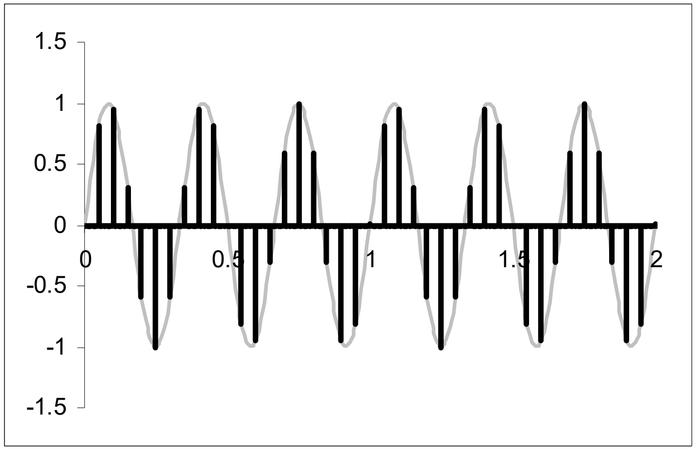
Discrete-Time Signal

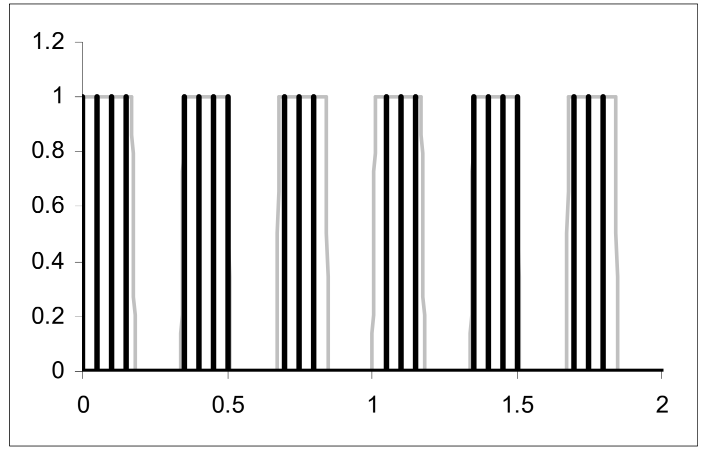
$$x(t \Rightarrow nT_s) \Rightarrow \; x(nT_s) = x[n] = \sin \; (\omega n T_s)$$

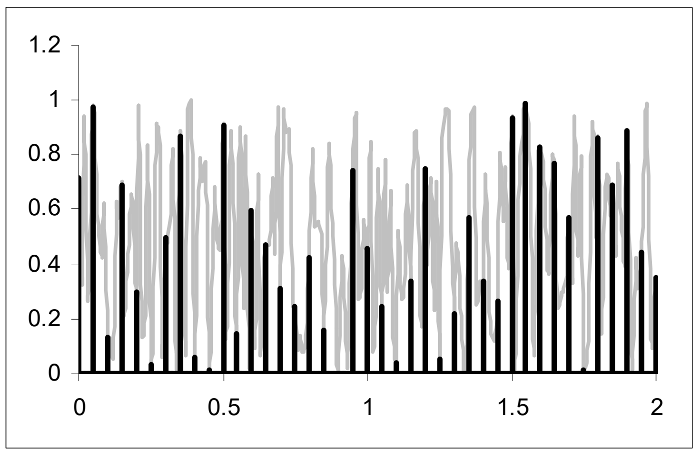
where n is an integer: $N_1 < n < N_2$

and T_s is the sampling period

Discrete-Time Signals







Discrete Spatial Signal

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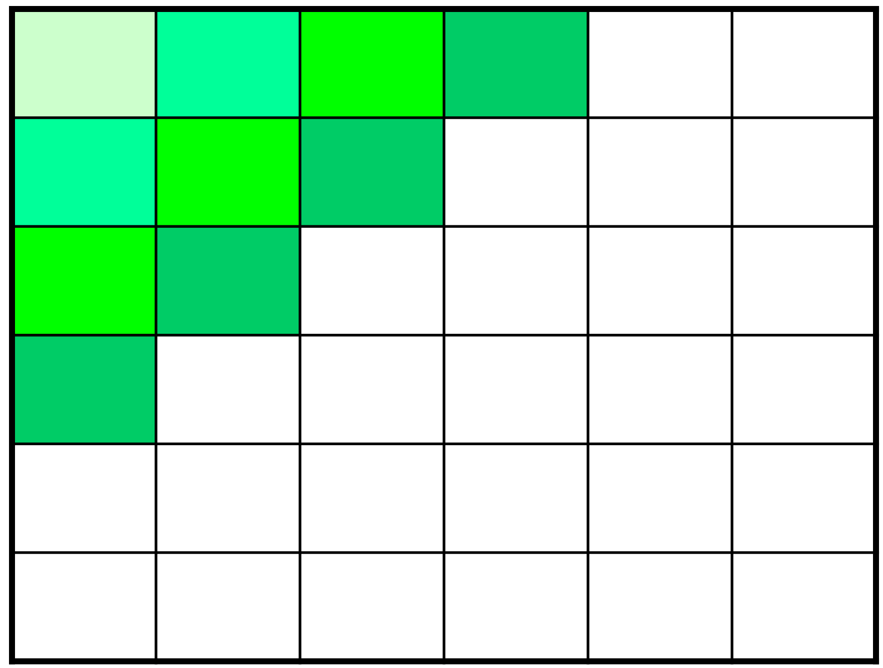


Signals have Properties

- Take on Real or Complex values
- Periodic or non-periodic
- Symmetries
- Bounded or Unbounded

Complex Signals

- Continuous Signals have to be solutions of differential equations they can be in the form:

$$x(t) = (A_1 + B_1 t + \cdots)e^{s_1 t} + (A_2 + B_2 t + \cdots)e^{s_2 t} + \cdots$$

- Discrete Signals have to be solutions of difference equations they can be in the form:

$$x[n] = (A_1 + B_1 n + \cdots)z_1^n + (A_2 + B_2 n + \cdots)z_2^n + \cdots$$

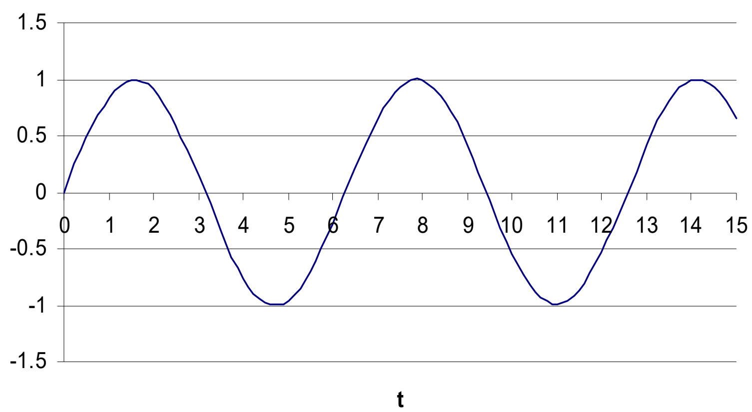
where A_i, B_i, etc., s_i and z_i can be complex numbers with real and imaginary parts.

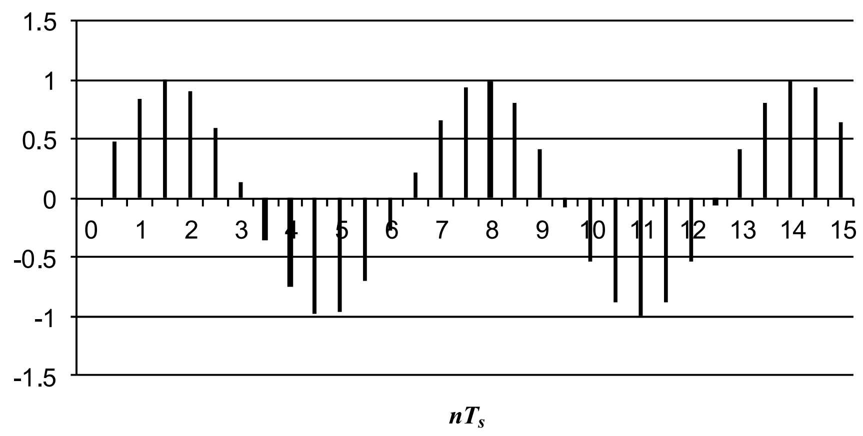
Periodic or non-periodic

- Periodic signals are those which satisfy

$$x(t+T) = x(t) \quad \text{for all } t$$

and T is called the Period.





Sinusoidal Continuous Signals

- Sinusoidal Signals are **periodic** functions which are based on the sine or cosine function from trigonometry.
- The general form of a Sinusoidal Signal

$$x(t)=A\cos(\omega_o t+\phi)$$

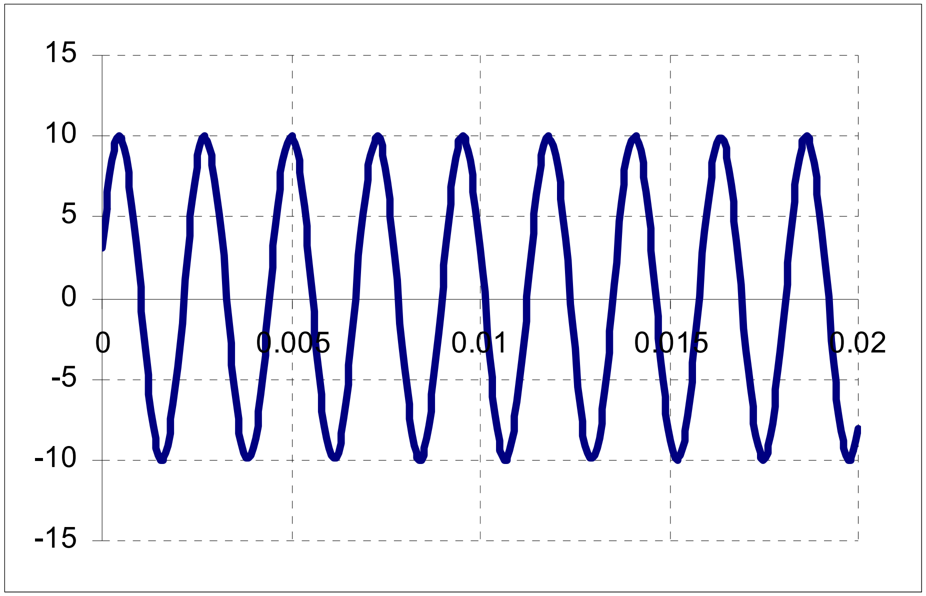
Or

$$x(t)=A\cos(2\pi f_o t+\phi)$$

- where cos (·) represent the cosine function
  - We can also use sin(·), the sine function
- $\omega_o t+\phi$ or $2\pi f_o t+\phi$ is angle (in radians) of the cosine function
  - Since the angle depends on time, it makes $x(t)$ a signal
- ω_o is the radian frequency of the sinusoidal signal
  - f_o is called the cyclical frequency of the sinusoidal signal
- ϕ is the phase shift or phase angle
- A is the amplitude of the signal

Example

$$x(t)=10\cos(2\pi(440)t - 0.4\pi)$$

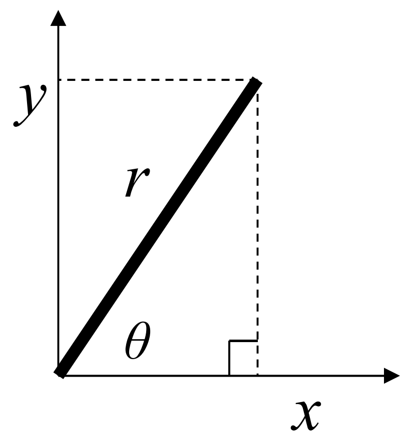


One cycle takes 1/440 = .00227 seconds

This is called the period, T, of the sinusoid and is equal to the inverse of the frequency, *f*

Sine and Cosine Functions

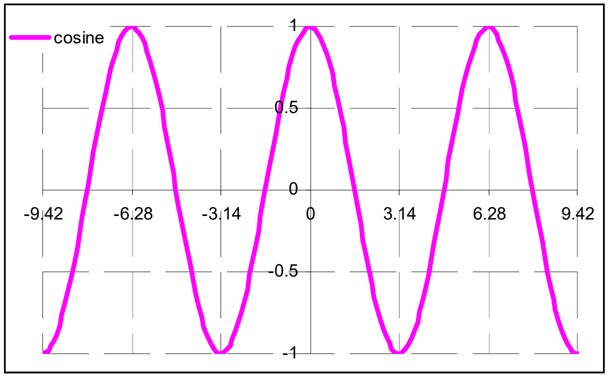
- Definition of sine and cosine

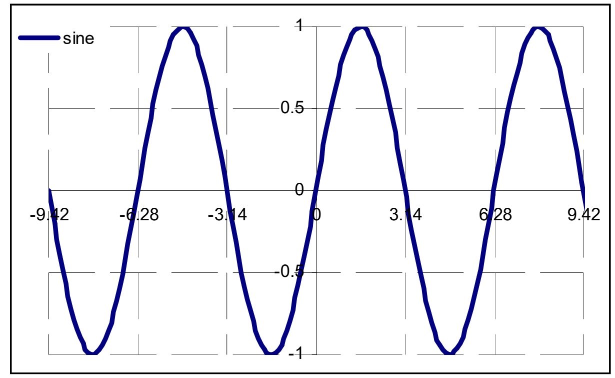


$$\sin\theta = \frac{y}{r}$$
$$\Rightarrow y = r\sin\theta$$
$$\cos\theta = \frac{x}{r}$$
$$\Rightarrow x = r\cos\theta$$

- Depending upon the quadrant of θ the sine and cosine function changes
  - As the θ increases from 0 to $\pi/2$, the cosine decreases from 1 to 0 and the sine increases from 0 to 1
  - As the θ increases beyond $\pi/2$ to π, the cosine decreases from 0 to -1 and the sine decreases from 1 to 0
  - As the θ increases beyond π to $3\pi/2$, the cosine increases from -1 to 0 and the sine decreases from 0 to -1
  - As the θ increases beyond $3\pi/2$ to 2π, the cosine increases from 0 to 1 and the sine increases from -1 to 0

Properties of Sinusoids

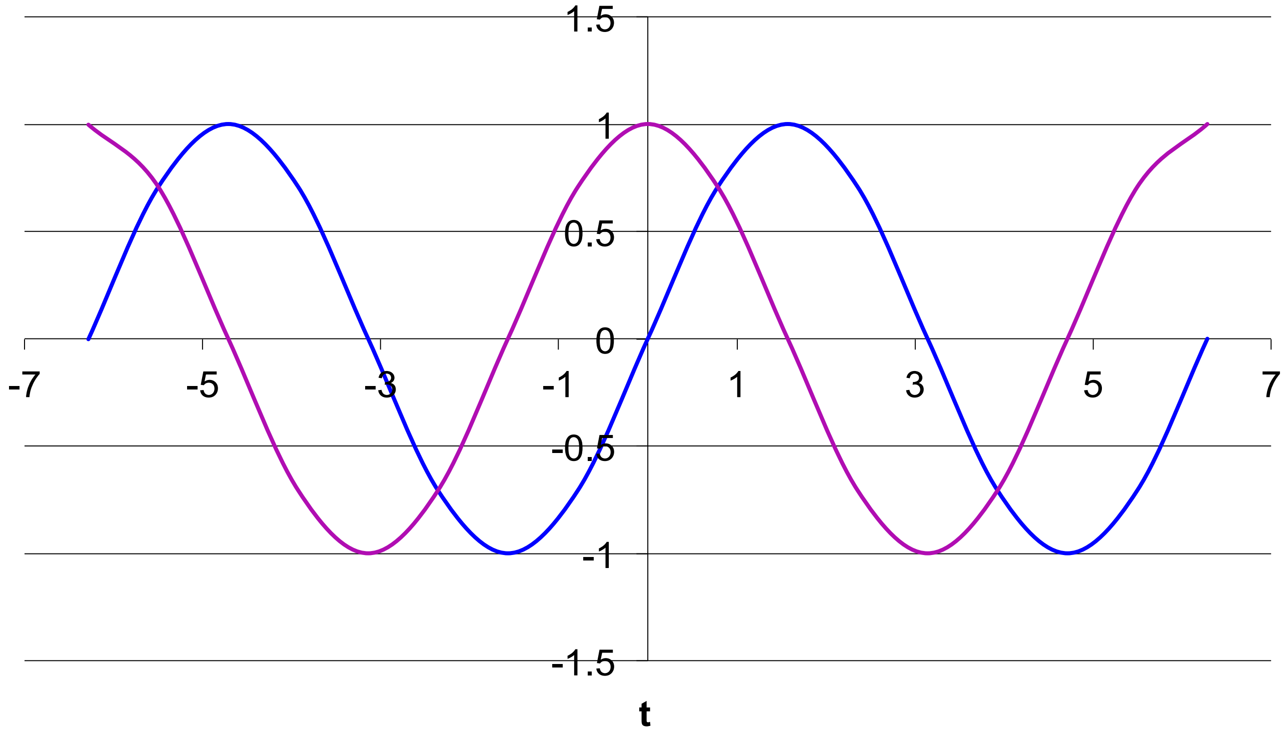




Property	Equation
Equivalence	$\sin\theta = \cos(\theta - \pi/2)$ or $\cos\theta = \sin(\theta + \pi/2)$
Periodicity	$\cos(\theta + 2\pi k) = \cos\theta$ or $\sin(\theta + 2\pi k) = \sin\theta$ where k is an integer
Evenness of cosine	$\cos\theta = \cos(-\theta)$
Oddness of sine	$\sin\theta = -\sin(-\theta)$
Zeros of sine	$\sin \pi k = 0$, when k is an integer
Zeros of cosine	$\cos[\pi(k+1)/2] = 0$, when k is an even integer; odd multiples of $\pi/2$
Ones of the cosine	$\cos 2\pi k = 1$, when k is an integer; even multiples of π
Ones of the sine	$\sin[\pi(k+1/2)] = 1$, when k is an even integer; alternate odd multiples of $\pi/2$
Negative ones of the cosine	$\cos[2\pi(k+1)/2] = -1$, when k is an integer; odd multiples of π
Negative ones of the sine	$\sin[\pi(k+1/2)] = -1$, when k is an odd integer; alternate odd multiples of $3\pi/2$

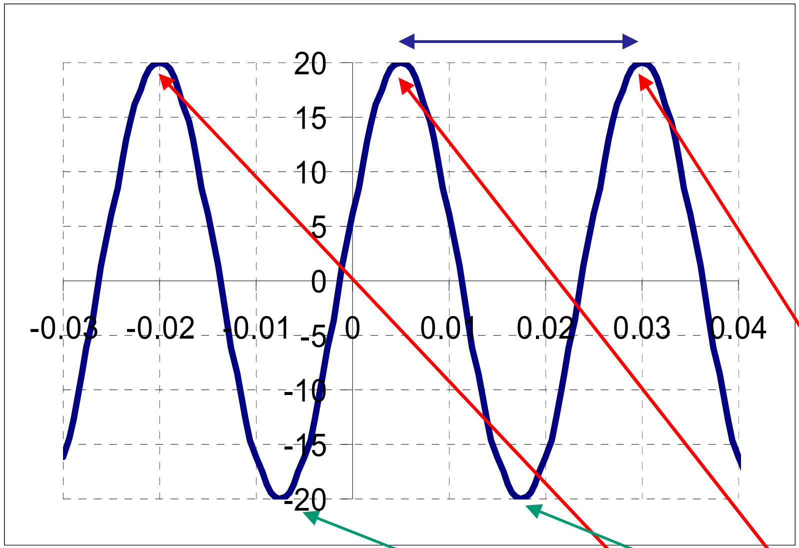
Signal Symmetries

- Even Signals are defined as $x_e(t) = x_e(-t)$
- Odd Signals are defined as $x_o(t) = -x_o(-t)$



Sinusoidal Signals

$x(t)=20\cos(2\pi(40)t - 0.4\pi)$



$A = 20,\ \omega_o = 2\pi(40),\ f_o = 40,\ \theta = -0.4\pi$

Maxima at $2\pi(40)t-0.4\pi = 2\pi k$ *or when* $t = \ldots, -0.02, 0.005, 0.03, \ldots$

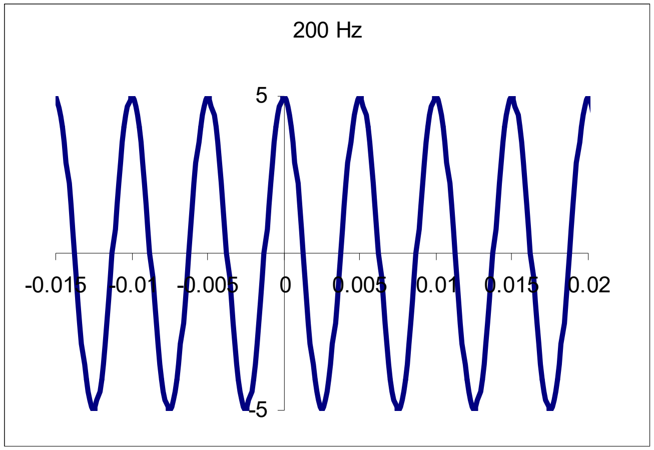
Minima at $2\pi(40)t-0.4\pi = 2\pi(k+1)0.5$ *or when* $t = \ldots -0.0075, 0.0175$

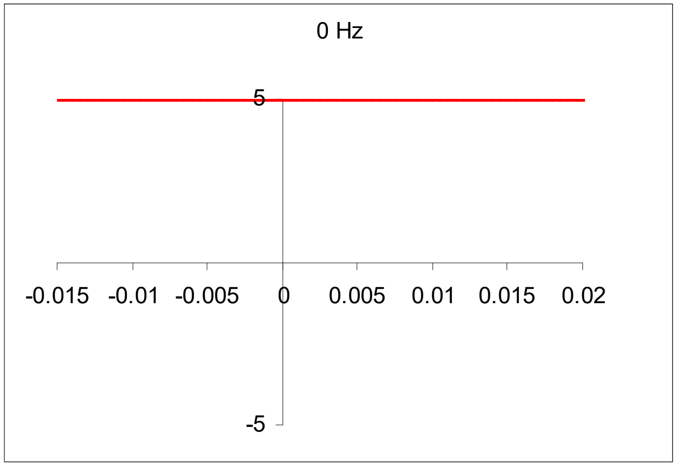
Time Period ($1/f_o$) between = $0.005 - (-0.02) = 0.025$ *sec*

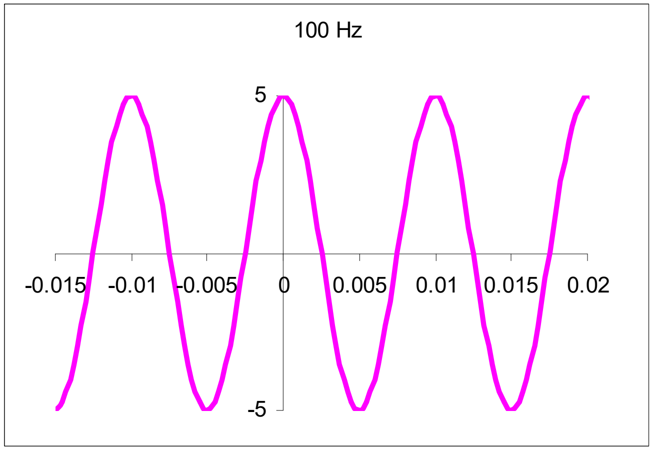
Frequencies

$$A\cos(2\pi f_o t + \theta)$$

for 200 Hz, 100 Hz, 0 Hz,







Relation of Period to Frequency

- Period of a sinusoid, T_o, is the length of one cycle and

$$T_o = 1/f_o$$

- The following relationship must be true for all Signals which are periodic (not just sinusoids)

$$x(t + T_o) = x(t)$$

- So

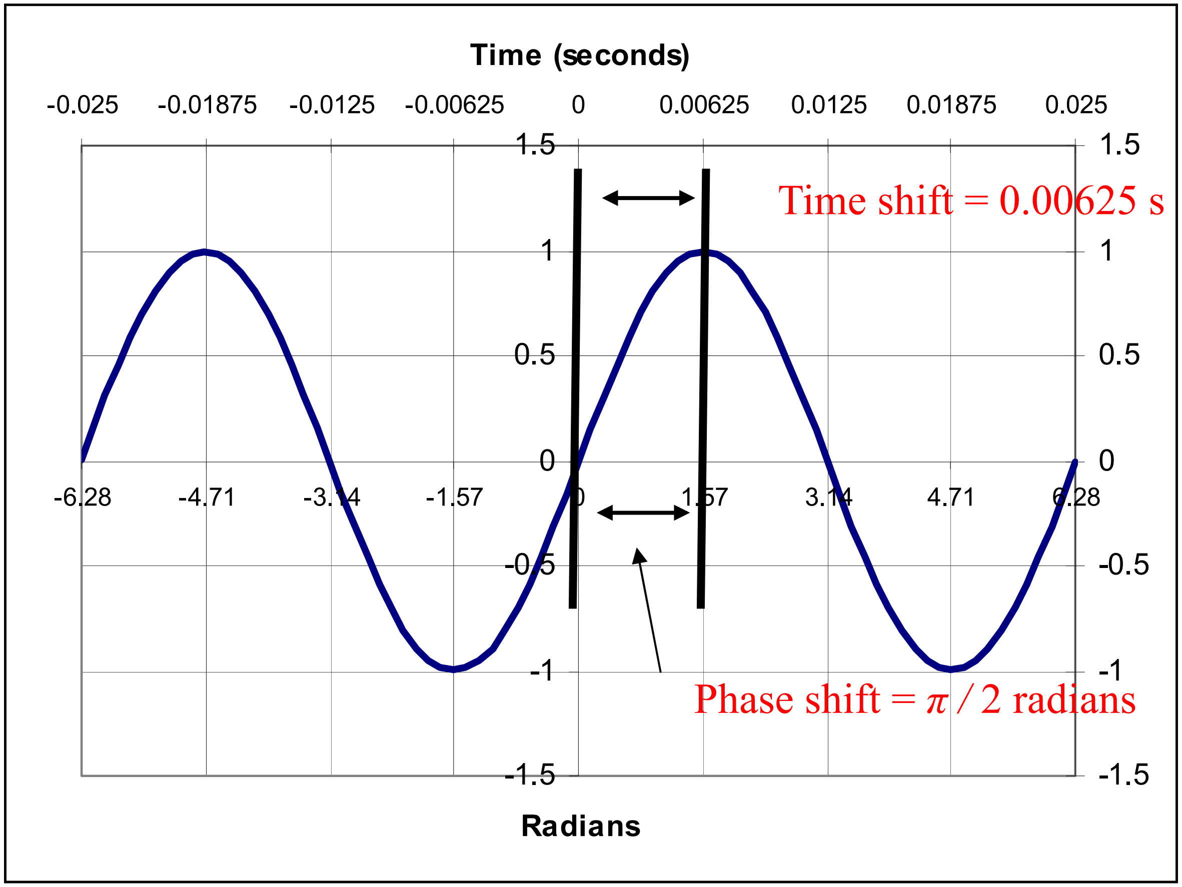
$$A\cos(\omega_o(t + T_o) + \theta) = A\cos(\omega_o t + \omega_o T_o + \theta)$$

$$A\cos(\omega_o t + \omega_o T_o + \theta) = A\cos(2\pi f_o t + 2\pi f_o T_o + \theta)$$

$$A\cos(2\pi f_o t + 2\pi f_o T_o + \theta) = A\cos(2\pi f_o t + 2\pi + \theta)$$

$$A\cos(2\pi f_o t + 2\pi + \theta) = A\cos(2\pi f_o t + \theta) = A\cos(\omega_o t + \theta)$$

Phase shift and Time Shift



$$x(t) = \cos(2\pi 40t - \frac{\pi}{2})$$

$$f = 40Hz;$$

$$T = \frac{1}{40} = 0.025 \text{ sec}$$

phase shift:

$$\theta = -\frac{\pi}{2}$$

time shift:

$$t_s = -\frac{-\frac{\pi}{2}}{2\pi 40} = \frac{1}{160} = 0.00625 \text{ sec}$$

$$x(t) = \cos(2\pi 40(t - 0.00625))$$

Phase Shift and Time Shift

- The phase shift parameter θ (with frequency) determines the time locations of the maxima and minima of the sinusoid.
- When $\theta = 0$, then for positive peak at $t = 0$.
- When $\theta \neq 0$, then the phase shift determines how much the maximum is shifted from $t = 0$.
- However, delaying a signal by t_1 seconds, also shifts its waveform.

$$x(t-t_1) = A\cos(\omega_o(t-t_1)) = A\cos(\omega_o t-\omega_o t_1)$$
$$\omega_o t-\omega_o t_1 = \omega_o t + \theta$$
$$-\omega_o t_1 = \theta$$
$$t_1 = -\theta / \omega_o = -\theta / 2\pi f_o$$

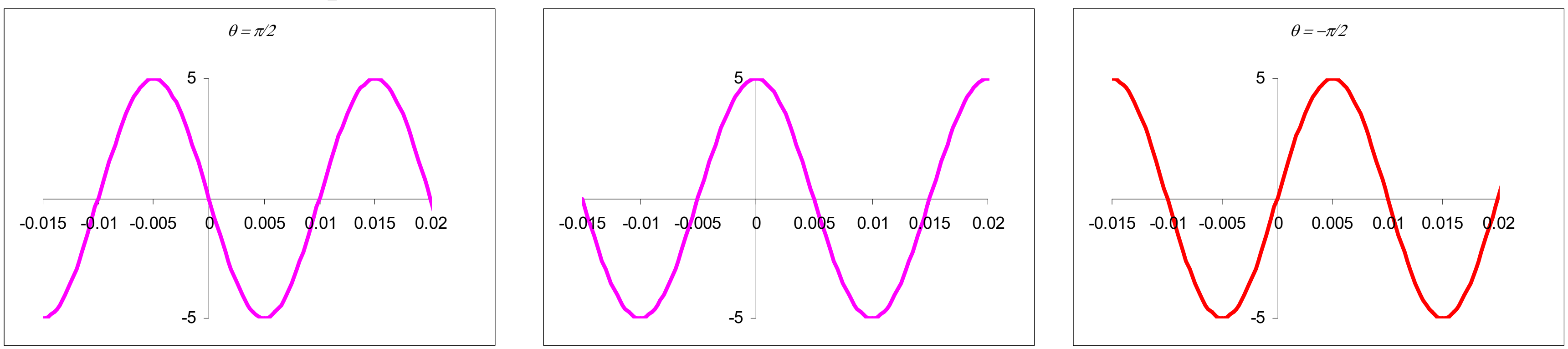
$$\theta = -2\pi f_o t_1 = -2\pi(t_1 / T_o)$$

- Note that a positive (negative) value of t_1 equates to a delay (advance)
- And a a positive (negative) value of θ equates to an advance (delay)

Phase and Time Shift

- Note that a positive (negative) value of t_1 equates to a delay (advance)
- And a a positive (negative) value of θ equates to an advance (delay)

$$x(t) = 5\cos(2\pi\, 50t + \theta)$$
$$\theta = \pi / 2; -\pi / 2$$
$$t_1 = -\pi / 2 / (2\pi\ 50) = -.005 \text{ sec}; +.005 \text{ sec}$$

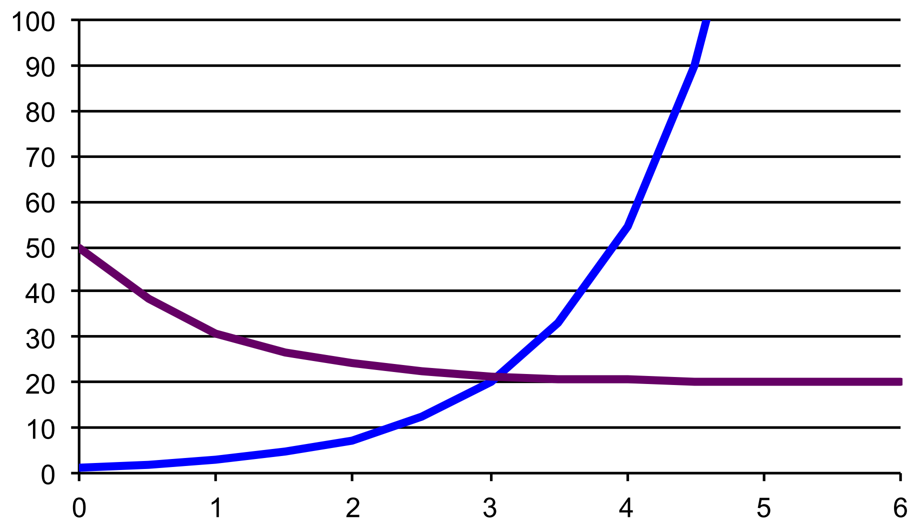


Identities and Derivatives

Number	Equation
1	$\sin^2\theta + \cos^2\theta = 1$
2	$\cos 2\theta = \cos^2\theta - \sin^2\theta$
3	$\sin 2\theta = 2 \sin\theta \cos\theta$
4	$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$
5	$\cos(a \pm b) = \cos a \cos b \quad \sin a \sin b$
6	$\cos a \cos b = [\cos(a+b) + \cos(a-b)]/2$
7	$\sin a \sin b = [\cos(a-b) - \cos(a+b)]/2$
8	$\cos^2\theta = [1 + \cos 2\theta]/2$
9	$\sin^2\theta = [1 - \cos 2\theta]/2$
10	$d \sin\theta / d\theta = \cos\theta$
11	$d \cos\theta / d\theta = -\sin\theta$

Bounded or Unbounded

- For Bounded Signals $\int |f(t)|\,dt$ approaches a constant value as $t \to \pm\infty$
- Unbounded Signals approach infinity as $t \to \pm\infty$



Euler's Formula

Basic Formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

(Note that: $e^{\theta} \neq \cos\theta + j\sin\theta$)

Also:

$$e^{j\omega t} = \cos\omega t + j\sin\omega t$$

And

$$e^{j(\omega t+\phi)} = \cos(\omega t + \phi) + j\sin(\omega t + \phi)$$

And one more where $s = \alpha + j\omega$

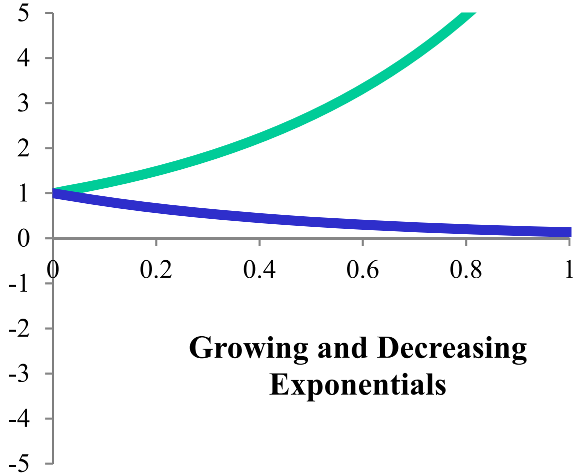
$$e^{st} = e^{(\alpha + j\omega)t} = e^{\alpha t}\cos(\omega t + \phi) + je^{\alpha t}\sin(\omega t + \phi)$$

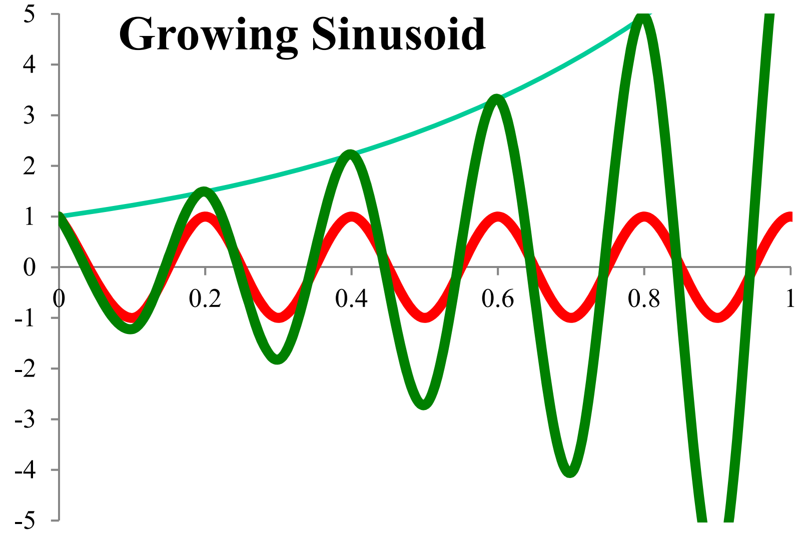
More on Complex Signals

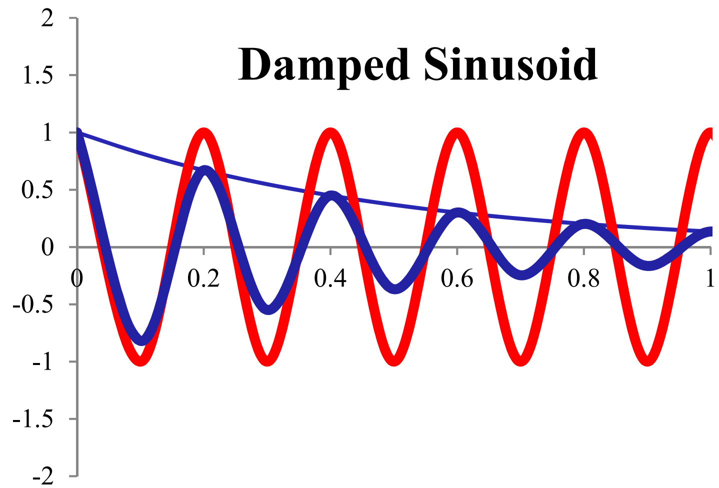
- Let's assume that $x(t)=Ae^{st}$ for all t, A is a real constant and s is complex and is given as $s = \alpha + j\omega$
- If $s = \alpha$ is real then $x(t)$ is a real exponential function $x(t)=Ae^{\alpha t}$
- If $s = j\omega$ is imaginary and using Euler's formula then $x(t)$ is a sinusoidal function $x(t)=Ae^{j\omega t} = A(\cos \omega t+j \sin \omega t)$ [Book uses, F: $\omega=2\pi F$]
- If s is complex then $x(t)$ is called a damped sinusoidal function for $s < 0$ and is of the form

$$x(t)=Ae^{st} = Ae^{\alpha t} (\cos \omega t + j \sin \omega t)$$

More on Complex Signals







Homework

1. Continuous and Discrete Signals Use Matlab to plot the signals; submit your code
   1. $f(t) = 1 - e^{-t}$ is a continuous signal. Draw its waveform.
   2. Draw the discrete version of $f(t)$ for $T=0.25$.
2. Periodic Signals
   1. Show that $tan\ t$ is periodic. What is its period?
   2. Is e^{-t} periodic? Why not?
   3. Is $e^{-t} sin(t)$ periodic? Describe?
3. Bounded Signals
   1. Prove that $f(t) = e^{-t}$ is bounded for $t > 0$.
   2. What about $f(t) = e^{-t}$ for all t.
4. Biosignals
   1. For a typical EEG, EKG, and EMG signal, is the signal periodic? If so, what is it's period.
   2. For a typical EEG, EKG, and EMG signal, is the signal bounded? If so, describe why.

Homework cont'd

5. Symmetry
   1. Is *cos t* even or odd? *sin t*? *tan t*?
   2. What about *cos t x sin t* ? *tan t x cos t* ?
   3. What is the symmetry of the product of:
      1. Two even functions
      2. Two odd Functions
      3. Even and Odd function
6. CT.1.2.1,CT.1.2,3
7. DT.1.2.1,DT.1.2.3

Complex Numbers and the Unit Impulse Function

Lesson #2
2CT.2,4,
3CT.2
Appendix A

What is the solution to?

1. $x^2+4x+3=0$

$$x_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4\times 3}}{2} = \frac{-4 \pm \sqrt{16-12}}{2}$$

$$= \frac{-4 \pm \sqrt{4}}{2} = \frac{-4 \pm 2}{2} = -1, -3$$

What is the solution to?

2. $x^2+4x+5=0$

$$x_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4 \times 5}}{2} = \frac{-4 \pm \sqrt{16 - 20}}{2}$$

$$= \frac{-4 \pm \sqrt{-4}}{2} \quad ?????$$

What is the Square Root of a Negative Number?

- We define the square root of a negative number as an imaginary number
- We define

$$\sqrt{-1} \Rightarrow j \text{ for engineers } (i \text{ for mathematicans})$$

- Then our solution becomes:

$$x_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4 \times 5}}{2} = \frac{-4 \pm \sqrt{16 - 20}}{2}$$

$$= \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm j\sqrt{4}}{2} = \frac{-4 \pm j2}{2} = -2 + j1, -2 - j1$$

The Complex Plane

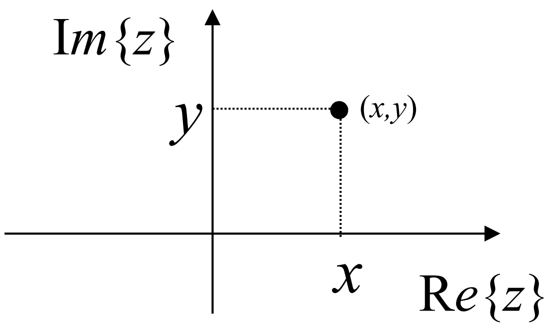
- $z = x + jy$ is a complex number where:

  $x = \mathrm{R}e\{z\}$ is the real part of z

  $y = \mathrm{I}m\{z\}$ is the imaginary part of z

- We can define the complex plane and we can define 2 representations for a complex number:

$$z = x + jy$$



Rectangular Form

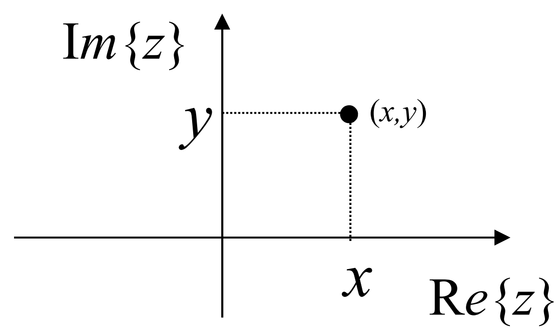
- Rectangular (or cartesian) form of a complex number is given as

$$z = x + jy$$

$x = \text{Re}\{z\}$ is the real part of z

$y = \text{Im}\{z\}$ is the imaginary part of z

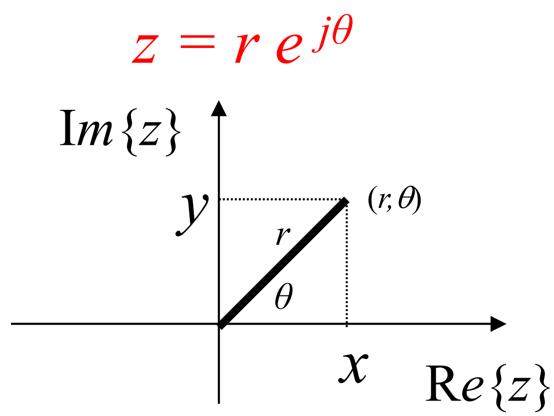
$$z = x + jy$$



Rectangular or Cartesian

Polar Form

- $z = r\,e^{j\theta} = r\angle\theta$ is a complex number where:
- r is the magnitude of z
- θ is the angle or argument of z (arg z)



Polar

Relationships between the Polar and Rectangular Forms

$$z = x + jy = r\, e^{j\theta}$$

- Relationship of Polar to the Rectangular Form:

$$x = \mathrm{R}e\{z\} = r \cos \theta$$

$$y = \mathrm{I}m\{z\} = r \sin \theta$$

- Relationship of Rectangular to Polar Form:

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \arctan(\frac{y}{x})$$

Relationships between the Polar and Rectangular Forms

$$z = x + jy = r\,e^{j\theta}$$

- Relationship of Polar to the Rectangular Form:

$$x = \mathrm{R}e\{z\} = r\cos\theta$$

$$y = \mathrm{I}m\{z\} = r\sin\theta$$

- Relationship of Rectangular to Polar Form:

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \arctan(\frac{y}{x})$$

Addition of 2 complex numbers

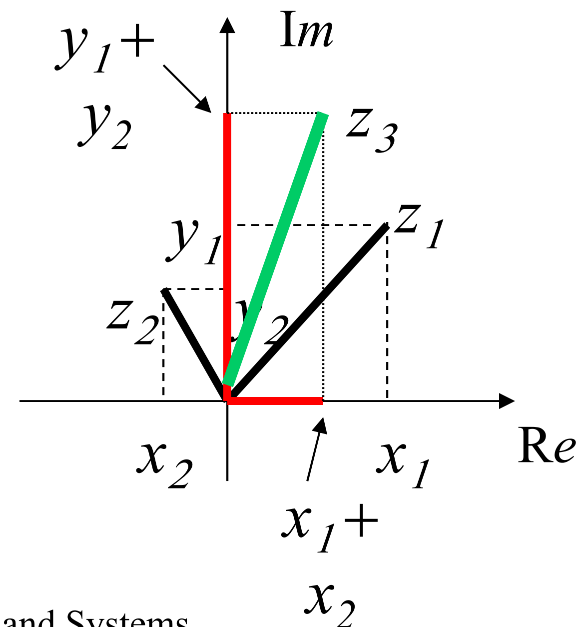
- When two complex numbers are added, it is best to use the rectangular form.
- The real part of the sum is the sum of the real parts and imaginary part of the sum is the sum of the imaginary parts.
- Example: $z_3 = z_1 + z_2$

$$z_1 = x_1 + jy_1; z_2 = x_2 + jy_2$$

$$z_3 = z_1 + z_2 = x_1 + jy_1 + x_2 + jy_2$$

$$= x_1 + x_2 + jy_1 + jy_2$$

$$= (x_1 + x_2) + j(y_1 + y_2)$$



Multiplication of 2 complex numbers

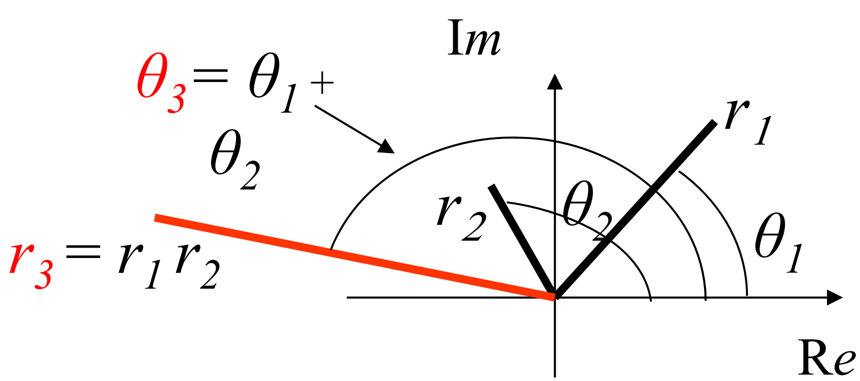
- When two complex numbers are multiplied, it is best to use the polar form:
- Example: $z_3 = z_1 \times z_2$

$$z_1 = r_1 e^{j(\theta_1)}; z_2 = r_2 e^{j(\theta_2)}$$

$$z_3 = z_1 \times z_2 = r_1 e^{j(\theta_1)} \times r_2 e^{j(\theta_2)}$$

$$= r_1 r_2 e^{j(\theta_1)} e^{j(\theta_2)} = r_1 r_2 e^{j(\theta_1 + \theta_2)}$$

- We multiply the magnitudes and add the phase angles



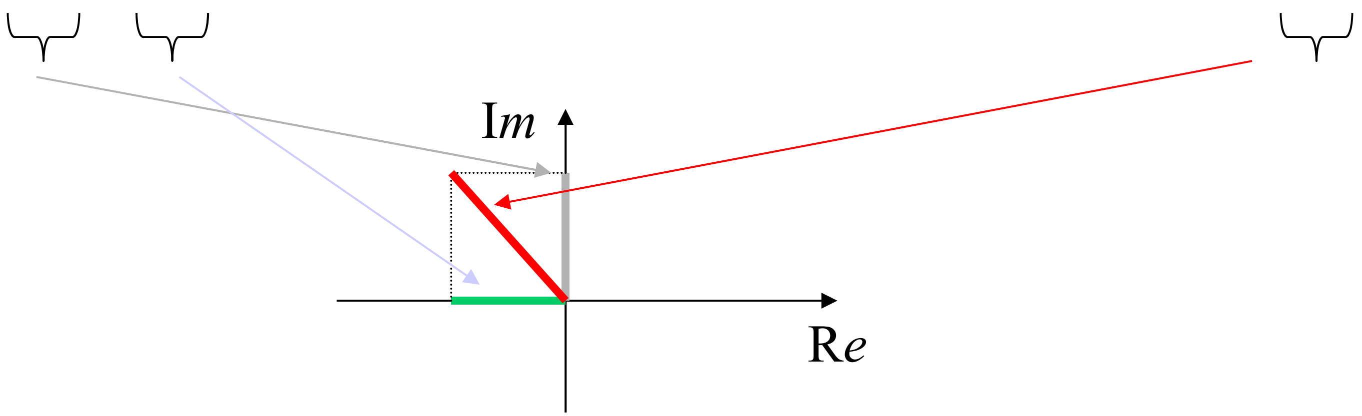
Some examples

$$5e^{j\frac{\pi}{2}} = 5\cos(\frac{\pi}{2}) + j5\sin(\frac{\pi}{2}) = 5\times 0 + j5\times 1 = j5$$

$$5e^{j\pi} = 5\cos(\pi) + j5\sin(\pi) = 5\times -1 + j5\times 0 = -5$$

$$5e^{-j\pi} = 5\cos(-\pi) + j5\sin(-\pi) = 5\cos(\pi) - j5\sin(\pi) = 5\times -1 + j5\times 0 = -5$$

$$5e^{j\frac{\pi}{2}} + 5e^{j\pi} = j5 - 5 = 5(-1 + j) = 5\times\sqrt{(-1)^2 + 1^2}\,e^{j\tan^{-1}(-1)} = 5\sqrt{2}e^{j\frac{3\pi}{4}}$$



Some examples

$$5e^{j\frac{\pi}{4}} + 5e^{j\frac{\pi}{2}}$$

$$5e^{j\frac{\pi}{4}} + 5e^{j\frac{\pi}{2}} = \underbrace{5\cos(\frac{\pi}{4})} + \underbrace{j5\sin(\frac{\pi}{4})} + \underbrace{5\cos(\frac{\pi}{2})} + \underbrace{j5\sin(\frac{\pi}{2})}$$

$$= \frac{5}{\sqrt{2}} + 0 + j\frac{5}{\sqrt{2}} + j5 \qquad = \frac{5}{\sqrt{2}} + j5 \times 1.707$$

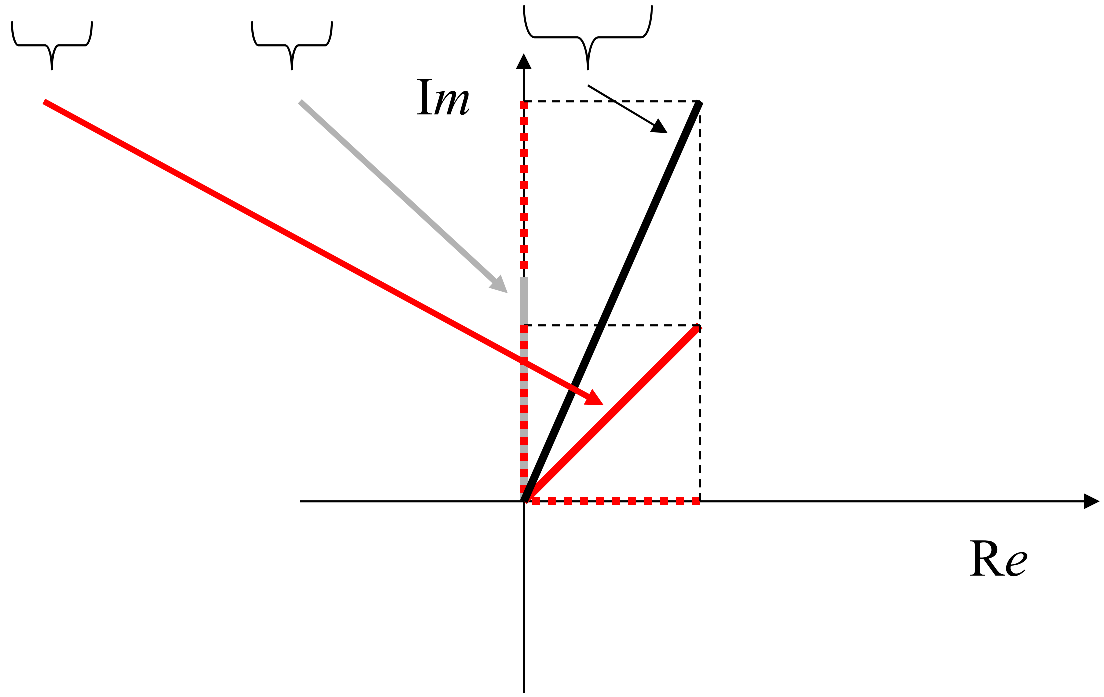
$$= 3.536 + j8.536$$

$$= \sqrt{3.536^2 + 8.536^2}\, e^{j\tan^{-1}(\frac{8.536}{3.536})} = \sqrt{12.5 + 72.86}\, e^{j\tan^{-1}(2.41)}$$

$$= \sqrt{84.367}\, e^{j\tan^{-1} 2.41} = 9.24 e^{j1.18}$$

Some examples

$$5e^{j\frac{\pi}{4}} + 5e^{j\frac{\pi}{2}} = 9.24e^{j1.18}$$



Complex Exponential Signals

- A complex exponential ***signal*** is define as:

$$z(t) = Ae^{j(\omega_o t + \theta)}$$

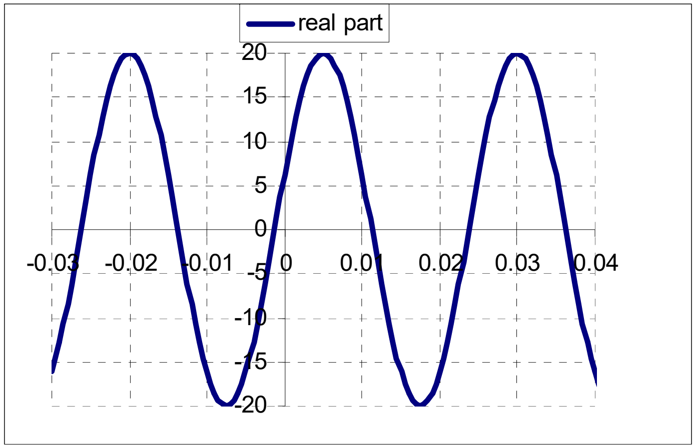
- Note that it is defined in polar form where
  - the magnitude of $z(t)$ is $|z(t)| = A$
  - the angle (or argument, arg $z(t)$) of $z(t) = (\omega_o t + \theta)$
    - Where ω_o is called the radian frequency and θ is the phase angle (phase shift)

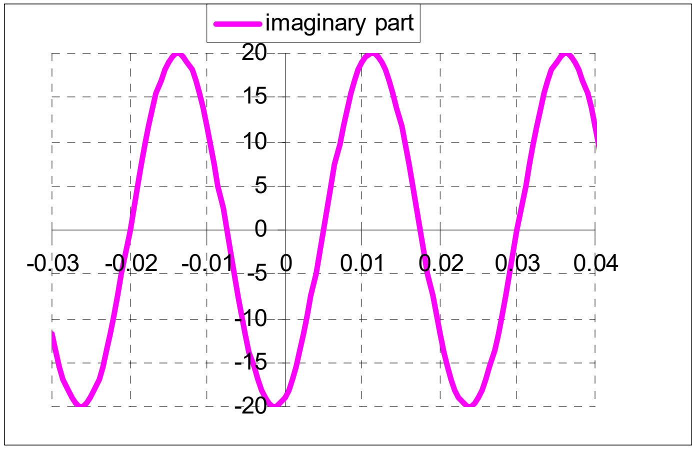
Plotting the waveform of a complex exponential signal

- For an complex signal, we plot the real part and the imaginary part separately.
- Example:

$$z(t) = 20e^{\,j(2\pi(40)t\text{-}0.4\pi)} = 20e^{\,j(80\pi t\text{-}0.4\pi)}$$

$$= 20\cos(80\pi t\text{-}0.4\pi) + j20\sin(80\pi t\text{-}0.4\pi)$$





Complex Exponential Function as a function of time

- Let's look at this $z(t) = 1e^{j2\pi(1)t} = e^{j2\pi t} = \cos 2\pi t + j \sin 2\pi t$

t=8/8 *seconds*

arg(*z*(*t*))=2π *x*8/8 = 2π ; *z*(*t*)= 1+ *j*0

t=2/8 *seconds*

arg(*z*(*t*))=2π *x*2/8= π /2; *z*(*t*)= 0 + *j*1

t=3/8 *seconds*

arg(*z*(*t*))=2π *x*3/8 = 3 π /4;

z(*t*)= -0.707+ *j*0.707

t=1/8 *seconds*

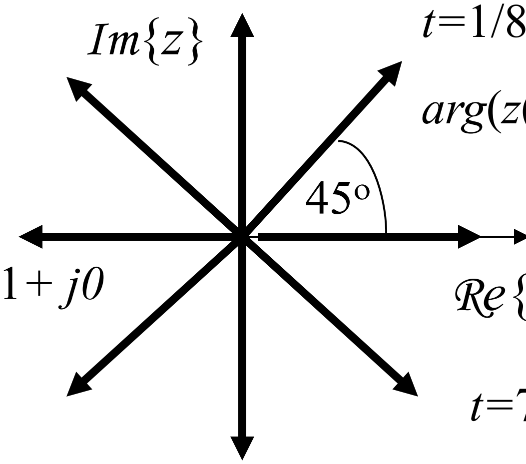
arg(*z*(*t*))=2π x1/8=π/4; *z*(*t*)=0.707+*j* 0.707

t=0 *seconds*

arg(*z*(*t*))=2π x0=0; *z*(*t*)=1+ *j*0

t=4/8 *seconds*

arg(*z*(*t*))=2π x4/8 = π; *z*(*t*)= -1+ *j*0



t=5/8 *seconds*

arg(*z*(*t*))=2π x5/8 = 5π /4;

z(*t*)= -0.707 - *j*0.707

t=7/8 *seconds*

arg(*z*(*t*))=2π x7/8= 7π /4;

z(*t*) = 0 .707- *j*0.707

t=6/8 *seconds*

arg(*z*(*t*))=2π x6/8 = 3π /2; *z*(*t*) = 0 - *j*

Phasor Representation of a Complex Exponential Signal

- Using the multiplication rule, we can rewrite the complex exponential signal as

$$z(t) = Ae^{j(\omega_o t+\phi)} = Ae^{j\omega_o t}e^{j\phi} = Ae^{j\phi}e^{j\omega_o t} = \mathbf{X}e^{j\omega_o t} = \mathbf{X}e^{j2\pi F_o t}$$

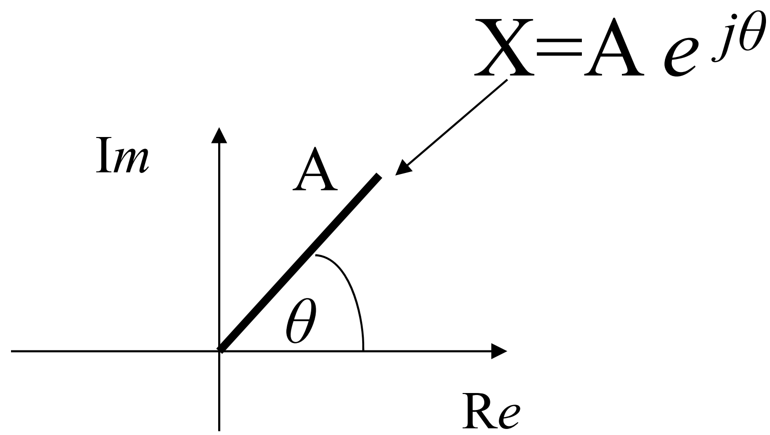
where **X** is a complex number equal to

$$\mathbf{X} = Ae^{j\phi}$$

- **X** is complex amplitude of the complex exponential signal and is also called a phasor

Graphing a phasor

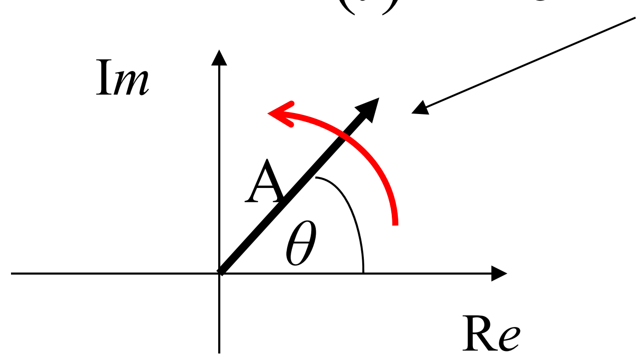
- **X**=A $e^{j\theta}$ can be graphed in the complex plane with magnitude A and angle θ:



Graphing a Complex Signal in terms of its phasors

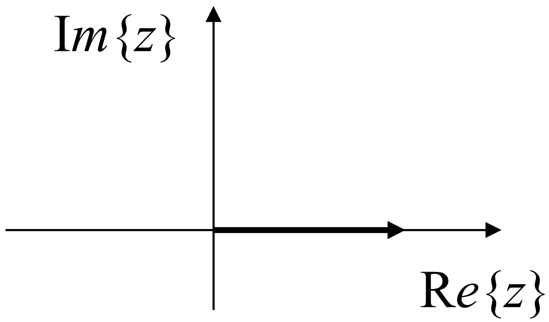
- Since a complex signal, $z(t)$, is a phasor multiplying a complex exponential signal $e^{j\omega_o t}$, then a complex signal can be viewed as a phasor rotating in time:

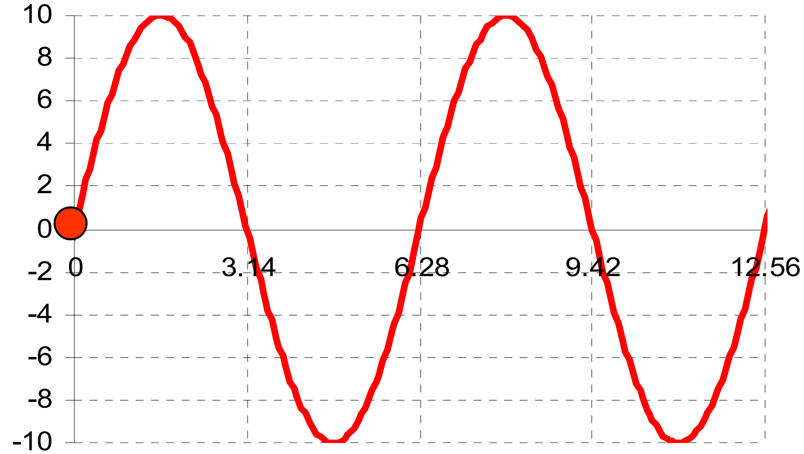
$$z(t) = Ae^{j(\omega_o t + \theta)} = \mathbf{X}e^{j\omega_o t}$$

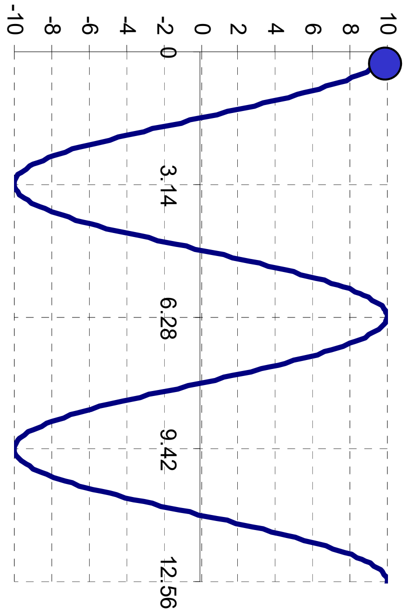


$$z(t) = Ae^{j(2\pi t)} = \mathbf{X}e^{j\omega_o t}$$

Rotating Phasor







Inverse Euler Formulas

- The inverse Euler formulas how the cosine and sine functions consist of complex exponentials

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Sinusoidal Signals

$$A\cos(\omega_o t+\theta) = A(\frac{e^{j(\omega_o t+\theta)}+e^{-j(\omega_o t+\theta)}}{2})$$
$$= A(\frac{e^{j\omega_o t}e^{j\theta}+e^{-j\omega_o t}e^{-j\theta}}{2})$$

$$A\cos(\omega_o t+\theta) = \frac{\mathbf{X}e^{j\omega_o t}+\mathbf{X}^{*}e^{-j\omega_o t}}{2}$$
$$= \frac{1}{2}z(t)+\frac{1}{2}z^{*}(t)$$
$$= \Re e\{z(t)\}$$

- Note that * means complex conjugate and $z(t)$ and $z^*(t)$ are called conjugate pairs
- This means that a cosine function consists of two complex exponential functions: one with positive frequency and one with negative frequency
- The amplitudes are complex conjugates

Complex Conjugate

- A conjugate of a complex number has the same real part but negative imaginary part of the complex number

$$x = a + jb$$

$$x^* = a - jb$$

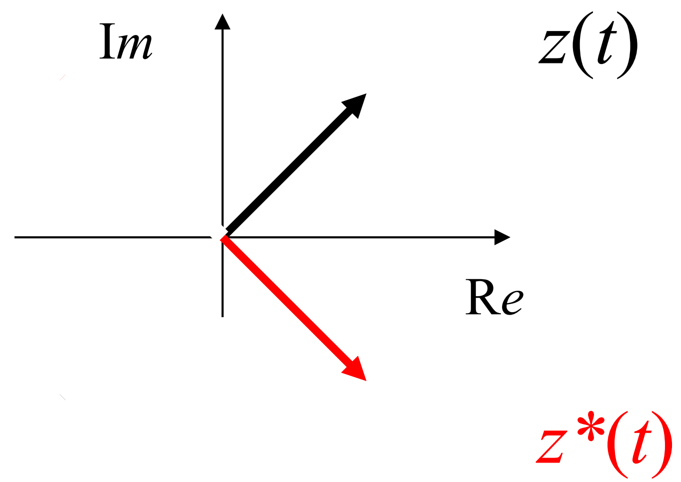
- Note the following important properties

$$x + x^* = 2\,\mathrm{Re}\{x\} = a + jb + (a - jb) = 2a$$

$$x - x^* = j2\,\mathrm{Im}\{x\} = a + jb - (a - jb) = j\,2b$$

Rotating Conjugate Pairs

Note that the imaginary part is zero since the imaginary parts of each cancel each other.



$$A\cos(\omega_o t+\theta)=\frac{1}{2}z(t)+\frac{1}{2}z^*(t)$$

$$=\frac{1}{2}[z(t)+z^*(t)]=\frac{1}{2}[\Re e\{z(t)\}+j\Im m\{z(t)\}+\Re e\{z^*(t)\}+j\Im m\{z^*(t)\}]$$

$$=\frac{1}{2}[2\Re e\{z(t)\}+j[\Im m\{z(t)\}+\Im m\{z^*(t)\}]]=\frac{1}{2}[2\Re e\{z(t)\}+j[\Im m\{z(t)\}-\Im m\{z(t)\}]]$$

$$=\Re e\{z(t)\}$$

Sinusoid Signal Addition

- Adding several sinusoid signals with the **same frequency** but with different amplitudes and phase angles to be in a form of a single sinusoidal signal

$$\sum_{1}^{N} A_k \cos(\omega_o t + \theta_k) = B\cos(\omega_o t + \phi)$$

- Proof of this uses identity 5:

$$A\cos(\omega_o t + \theta) = A\cos\theta\cos\omega_o t - A\sin\theta\sin\omega_o t$$

Proof of Sinusoid Signal Addition Algorithm

- The following shows how to calculate the sum of 2 phasors; it can be easily extended to more than 2 phasors:

$$A_1 \cos(\omega_o t + \theta_1) \quad + A_2 \cos(\omega_o t + \theta_2) \quad = B\cos(\omega_o t + \phi)$$

$$A_1 \cos\omega_o t \cos\theta_1 - A_1 \sin\omega_o t \sin\theta_1 + A_2 \cos\omega_o t \cos\theta_2 - A_2 \sin\omega_o t \sin\theta_2 = B\cos\omega_o t \cos\phi - B\sin\omega_o t \sin\phi$$

$$\underbrace{(A_1 \cos\theta_1 + A_2 \cos\theta_2)}\cos\omega_o t - (\underbrace{A_1 \sin\theta_1 + A_2 \sin\theta_2})\sin\omega_o t = \underbrace{B\cos\phi}\cos\omega_o t - \underbrace{B\sin\phi}\sin\omega_o t$$

Matching terms, we have:

$$B\cos\phi = A_1 \cos\theta_1 + A_2 \cos\theta_2$$

$$B\sin\phi = A_1 \sin\theta_1 + A_2 \sin\theta_2$$

Proof of Sinusoid Signal Addition Algorithm

$$B\cos\phi = A_1\cos\theta_1 + A_2\cos\theta_2$$

$$B\sin\phi = A_1\sin\theta_1 + A_2\sin\theta_2$$

Since

$$\sqrt{(B\cos\phi)^2 + (B\sin\phi)^2} = B$$

So

$$B = \sqrt{(A_1\cos\theta_1 + A_2\cos\theta_2)^2 + (A_1\sin\theta_1 + A_2\sin\theta_2)^2}$$

And

$$\frac{B\sin\phi}{B\cos\phi} = \tan\phi$$

$$\phi = \arctan\left(\frac{B\sin\phi}{B\cos\phi}\right)$$

$$= \arctan\left(\frac{A_1\sin\theta_1 + A_2\sin\theta_2}{A_1\cos\theta_1 + A_2\cos\theta_2}\right)$$

Example

$$A\cos(20\pi t+\phi)=1.7\cos(20\pi t+70\pi/180)+1.9\cos(20\pi t+200\pi/180)$$

$$A=\sqrt{(1.7\cos(70\pi/180)+1.9\cos(200\pi/180))^2+(1.7\sin(70\pi/180)+1.9\sin(200\pi/180))^2}$$

$$=1.532$$

$$\phi=\arctan(\frac{1.7\sin(70\pi/180)+1.9\sin(200\pi/180)}{1.7\cos(70\pi/180)+1.9\cos(200\pi/180)}=2.47rad\Rightarrow 141.79^\circ$$

$$=1.532\cos(20\pi t+2.47)$$

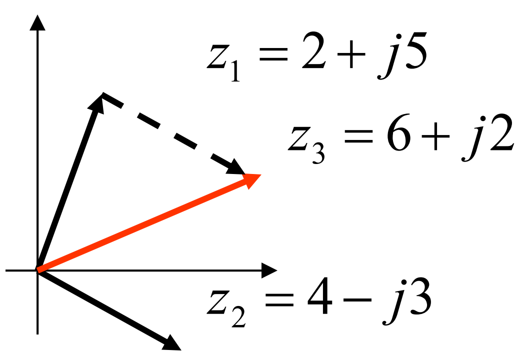
RECALL
Addition of Complex Numbers Rectangular form

- The real part of the sum is the sum of the real parts and imaginary part of the sum is the sum of the imaginary parts.

$$z_1 = x_1 + jy_1; z_2 = x_2 + jy_2$$

$$z_3 = z_1 + z_2 = x_1 + jy_1 + x_2 + jy_2$$

$$= (x_1 + x_2) + j(y_1 + y_2)$$



$$z_1 = 2 + j5; z_2 = 4 - j3$$

$$z_3 = z_1 + z_2 = 2 + j5 + 4 - j3$$

$$= (2+4) + j(5-3)$$

$$= 6 + j2$$

Phasor Addition

$$B\cos(\omega_o t+\phi)=\sum_1^N A_k\cos(\omega_o t+\theta_k)$$

$$\Re e\{Be^{j(\omega_o t+\phi)}\}=\sum_1^N \Re e\{A_k e^{j(\omega_o t+\theta_k)}\}$$

$$\Re e\{Be^{j\phi}e^{j\omega_o t}\}=\sum_1^N \Re e\{A_k e^{j\theta_k}e^{j\omega_o t}\}$$

$$\Re e\{Be^{j\phi}e^{j\omega_o t}\}=\Re e\{(\sum_1^N A_k e^{j\theta_k})e^{j\omega_o t}\}$$

Therefore,

$$Be^{j\phi}=\sum_1^N A_k e^{j\theta_k}$$

An Easier Method for Adding Sinusoids using Phasors

1. Represent the sinusoidal signals by complex exponential signals
2. From these exponential signals, take the each phasor in polar form and convert them to the Cartesian complex number form
3. Add the complex number to obtain a single complex number
4. Convert this complex number into its polar form
5. Using the phasor, reformate the sinusoidal signal by multiplying the phasor with $e^{j\omega_o t}$ and taking its real part

Example

$x_1(t) = 1.7\cos(20\pi t + 70\pi/180)$

Step 1) Formulate the phasor for signal 1.

$X_1 = A_1 e^{j\phi_1} = 1.7e^{j70\pi/180}$

Step 2) Convert its phasor to rectangular form

$X_1 = 1.7\cos 70\pi/180 + j1.7\sin 70\pi/180$

$= 0.5814 + j1.597$

$x_2(t) = 1.9\cos(20\pi t + 200\pi/180)$

Step 1) Formulate the phasor for signal 2.

$X_2 = A_2 e^{j\phi_1} = 1.9e^{j200\pi/180}$

Step 2) Convert its phasor to rectangular form

$X_2 = 1.9\cos 200\pi/180 + j1.9\sin 200\pi/180$

$= -1.785 - j0.6498$

Step 3) Add the 2 phasors in rectangular form

$X_3 = X_1 + X_2$

$= 0.5814 + j1.597 + (-1.785 - j0.6498)$

$= -1.204 + j0.948$

Step 4) Convert the resultant phasor to polar form

$X_3 = -1.204 + j0.948 = 1.532e^{j2.475}$

Step 5) Formulate the resultant signal from the phasor

$x_3(t) = 1.532\cos(20\pi t + 2.475)$

NOTE!!!!

- The reason why we prefer the complex exponential representation of the real cosine signal:

$$x(t) = \Re e\{z(t)\} = \Re e\{Ae^{j(\omega_o t+\theta)}\}$$

$$= A\cos(\omega_o t + \theta)$$

- In solving equations and making other calculations, it easier to use the complex exponential form and then take the Real Part.

Phasors

- Note that the real sinusoidal function

  $f(t) = A\cos(\omega t + \phi)$

  can be represented by a complex function

  $f(t) = A\cos(\omega t + \phi) = Re[Ae^{j(\omega t + \phi)}]$

- Let's represent this function by a phasor which is its magnitude and phase angle:

$$f(t) = A\cos(\omega t + \phi) = \mathrm{Re}[Ae^{j(\omega t + \phi)}] = \mathrm{Re}[Ae^{j\phi}e^{j\omega t}] \Rightarrow A\angle\phi$$

- Therefore, we can use phasors to represent complex functions which makes it easy to solve and calculate system solutions

Complex Numbers

- Constants:

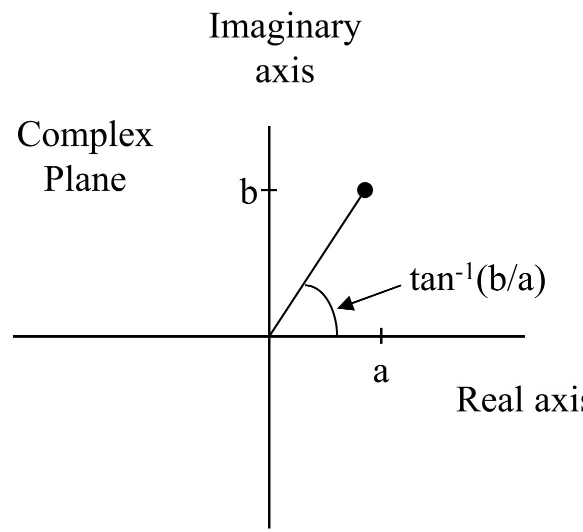
$s = a + jb$ Rectangular Form

a is called the Real part of s

b is called the Imaginary part of s

$$= \sqrt{a^2 + b^2}\, e^{j \tan^{-1}(b/a)}$$

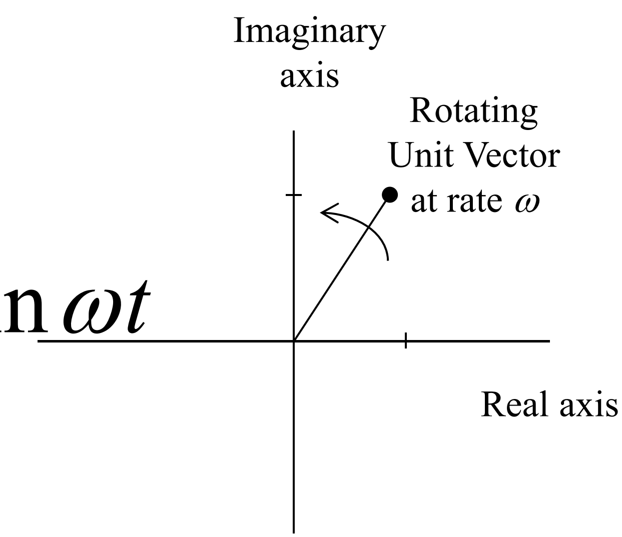
$$= \sqrt{a^2 + b^2} \angle \tan^{-1}(b/a)$$ Polar Form



- Functions:

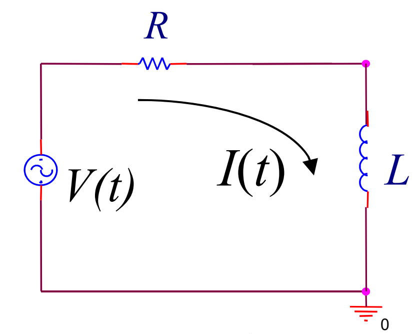
$$Example: e^{j\omega t} = \cos\omega t + j\sin\omega t$$

(recall: $e^{j\theta} = \cos\theta + j\sin\theta$)



Example Using ODE with Trigonometry

- Let's calculate the current $I(t)$ assuming $V(t) = A\cos\omega t$



$$RI(t) + L\frac{dI(t)}{dt} = V(t) = A\cos\omega t$$

Use Trigonometric functions

$$\text{Let } I(t) = I\cos(\omega t + \theta); \quad \frac{dI(t)}{dt} = -I\omega\sin(\omega t + \theta)$$

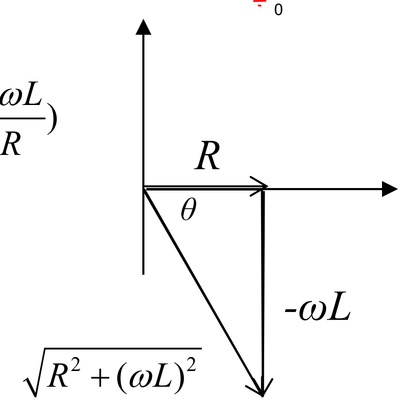
$$RI\cos(\omega t + \theta) - I\omega L\sin(\omega t + \theta) = A\cos\omega t$$

To solve for I and θ, use the identities:

$$\cos(A+B) = \cos A\cos B - \sin A\sin B; \quad \sin(A+B) = \sin A\cos B + \cos A\sin B$$

$$RI[\cos\omega t\cos\theta - \sin\omega t\sin\theta] - I\omega L[\sin\omega t\cos\theta + \cos\omega t\sin\theta] = A\cos\omega t$$

$$RI[-\sin\theta] - I\omega L\cos\theta = 0 \Rightarrow R[\sin\theta] = -\omega L\cos\theta \Rightarrow \frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{-\omega L}{R} \Rightarrow \theta = \tan^{-1}\left(\frac{-\omega L}{R}\right)$$



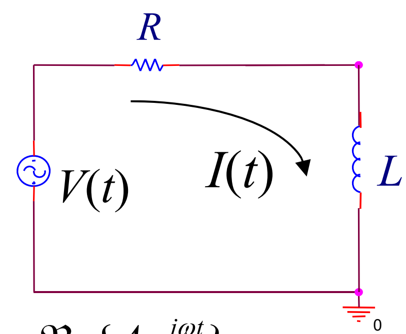
$$RI\cos\theta - I\omega L\sin\theta = A \Rightarrow I = \frac{A}{R\cos\theta - \omega L\sin\theta} = \frac{A}{R\dfrac{R}{\sqrt{R^2 + (\omega L)^2}} - \omega L\left(\dfrac{-\omega L}{\sqrt{R^2 + (\omega L)^2}}\right)}$$

$$I = \frac{A}{\dfrac{R^2 + (\omega L)^2}{\sqrt{R^2 + (\omega L)^2}}} = \frac{A}{\sqrt{R^2 + (\omega L)^2}}$$

$$I(t) = \frac{A}{\sqrt{R^2 + (\omega L)^2}}\cos\left(\omega t + \tan^{-1}\left(\frac{-\omega L}{R}\right)\right) \quad MESSY!!!!$$

Example Using ODE with Complex Exponentials

- Let's calculate the current *I*(*t*) assuming *V(t)= A cos ωt*



$$RI(t) + L\frac{dI(t)}{dt} = V(t) = A\cos\omega t$$

Use complex exponent functions

Let $I(t) = I\cos(\omega t + \theta) = \Re e\{Ie^{j\theta}e^{j\omega t}\}$; Let $V(t) = A\cos(\omega t) = \Re e\{Ae^{j\omega t}\}$;

$$\frac{dI(t)}{dt} = j\omega Ie^{j\theta}e^{j\omega t}$$

$$RIe^{j\theta}e^{j\omega t} + j\omega LIe^{j\theta}e^{j\omega t} = Ae^{j\omega t}$$

$$RIe^{j\theta} + j\omega LIe^{j\theta} = A$$

$$Ie^{j\theta} = \frac{A}{R + j\omega L} = \frac{A}{\sqrt{R^2 + (\omega L)^2}}e^{-j\tan^{-1}\frac{\omega L}{R}}$$

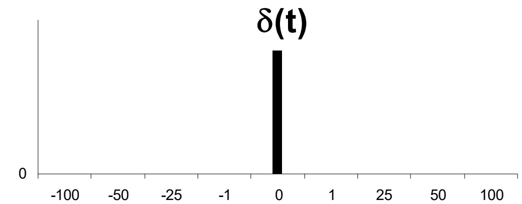
$$I(t) = \Re e\{\frac{A}{\sqrt{R^2 + (\omega L)^2}}e^{-j\tan^{-1}\frac{\omega L}{R}}e^{j\omega t}\} = \frac{A}{\sqrt{R^2 + (\omega L)^2}}\cos(\omega t - \tan^{-1}\frac{\omega L}{R})$$

A Special Function – Unit Impulse Function

- The unit impulse function, $\delta(t)$, also known as the Dirac delta function, is defined as

$\delta(t) = 0$ for $t \neq 0$;

$= \text{undefined for } t = 0$



and has the following special property:

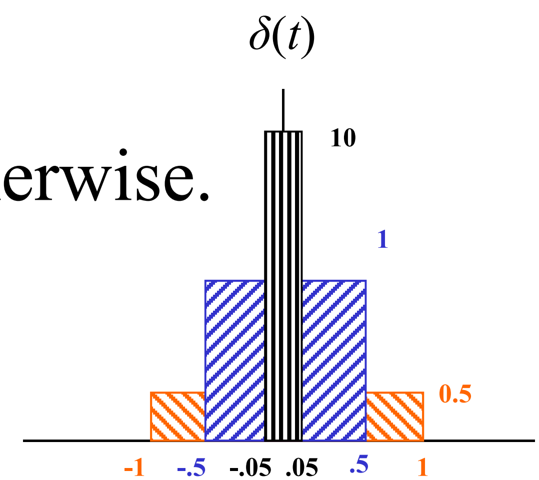
$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau)dt = f(\tau)$$

$$\therefore \int_{-\infty}^{\infty} \delta(t)dt = 1$$

Unit Impulse Function Continued

- A consequence of the delta function is that it can be approximated by a narrow pulse as the width of the pulse approaches zero while the area under the curve = 1

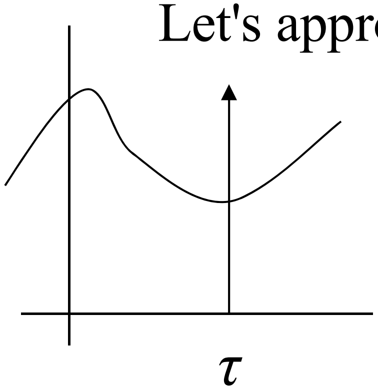
$$\lim_{\varepsilon \to 0} \delta(t) \approx 1/\varepsilon \text{ for } -\varepsilon/2 < t < \varepsilon/2; = 0 \text{ otherwise.}$$



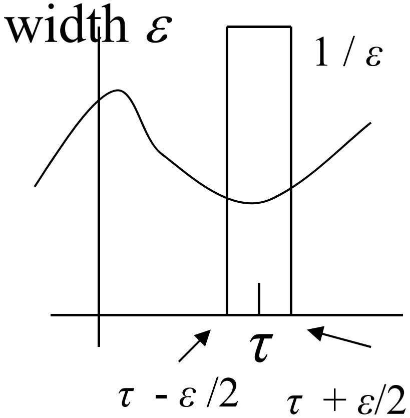
Unit Impulse Function Continued

$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau)\,dt$$

Let's approximate $\delta(t-\tau)$ with a pulse of height $\dfrac{1}{\varepsilon}$ and width ε



$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau)\,dt \approx \int_{\tau-\varepsilon/2}^{\tau+\varepsilon/2} f(t)\frac{1}{\varepsilon}\,dt$$



If we take the limit of this integral as $\varepsilon \to 0$,

the approximation integral approaches the original integral

$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau)\,dt = \lim_{\varepsilon\to 0}\int_{\tau-\varepsilon/2}^{\tau+\varepsilon/2} f(t)\frac{1}{\varepsilon}\,dt \to \lim_{\varepsilon\to 0} f(\tau)\frac{1}{\varepsilon}\varepsilon = f(\tau),$$

since as $\varepsilon \to 0$, the integral is zero except at $t = \tau$

Unit Impulse Sequence

- The unit impulse only has a value at n=0. The notation used to represent the unit impulse is called the (Kronecker) delta function:

$$\delta[n] = 1 \text{ for } n=0, 0 \text{ elsewhere}$$

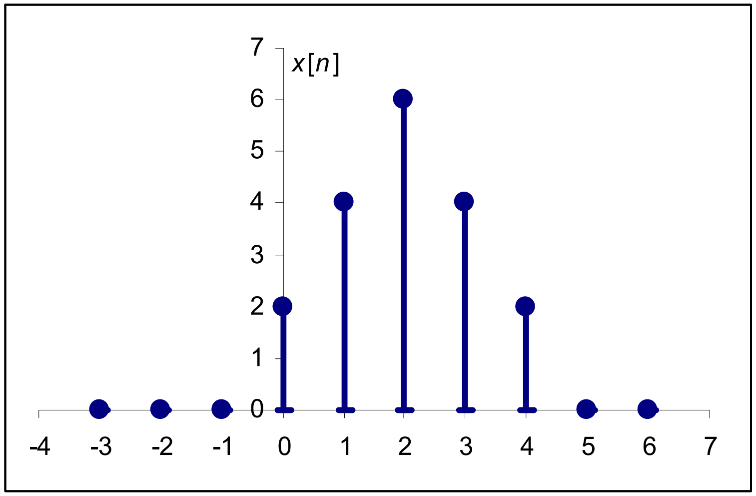
- Therefore, shifted impulses are:

$$\delta[n-2] = 1 \text{ for } n=2, 0 \text{ elsewhere}$$

$$\delta[n-k] = 1 \text{ for } n=k, 0 \text{ elsewhere}$$

Application of the Unit Impulse

- One may use the unit impulse to represent our first sequence as:



$$x[n] = 2\delta[n] + 4\delta[n-1] + 6\delta[n-2] + 4\delta[n-3] + 2\delta[n-4]$$

n	-1	0	1	2	3	4	5
2δ[n]	0	2	0	0	0	0	0
$4\delta[n\text{-}1]$	0	0	4	0	0	0	0
$6\delta[n\text{-}2]$	0	0	0	6	0	0	0
$4\delta[n\text{-}3]$	0	0	0	0	4	0	0
$2\delta[n\text{-}4]$	0	0	0	0	0	2	0
0	0	0	0	0	0	0	0
$x[n]$	0	2	4	6	4	2	0

Unit Impulse Representation of a Sequence

- In fact, any sequence can be represented as sum of unit impulse functions.

$$x[n] = \sum_k x[k]\delta[n-k]$$

$$= \cdots + x[-1]\delta[n+1] + x[0]\delta[n]$$

$$+ x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots$$

Unit Impulse Function for Discrete SignalsType equation here.

$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau)\,dt$$

Let's approximate $\delta(t-\tau)$ with a pulse of height $\frac{1}{\varepsilon}$ and width ε

$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau)\,dt \approx \int_{\tau-\varepsilon/2}^{\tau+\varepsilon/2} f(t)\frac{1}{\varepsilon}\,dt$$

If we take the limit of this integral as $\varepsilon \to 0$,

the approximation integral approaches the original integral

$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau)\,dt = \lim_{\varepsilon\to 0}\int_{\tau-\varepsilon/2}^{\tau+\varepsilon/2} f(t)\frac{1}{\varepsilon}\,dt \to \lim_{\varepsilon\to 0} f(\tau)\frac{1}{\varepsilon}\varepsilon = f(\tau),$$

since as $\varepsilon \to 0,$ the integral is zero except at $t = \tau$

Uses of Delta Function

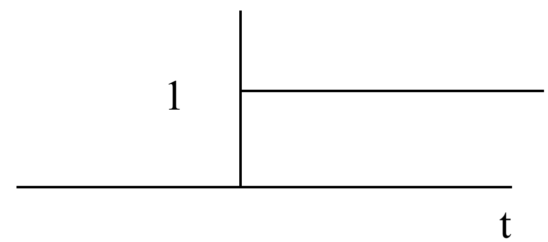
- Modeling of electrical, mechanical, physical phenomenon:
  - point charge,
  - impulsive force,
  - point mass
  - point light

Another Special Function – Unit Step Function

- The unit step function, $u(t)$ is defined as:

$u(t) = 1$ for $t \geq 0$;

$= 0$ for $t < 0$.



and is related to the delta function as follows:

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

Integration of the Delta Function

- $\delta(t) \longrightarrow u(t)$
- $u(t) \longrightarrow tu(t)$ *1st order*
- $tu(t) \longrightarrow \frac{t^2}{2!}u(t)$ *2nd order*

$\cdot$

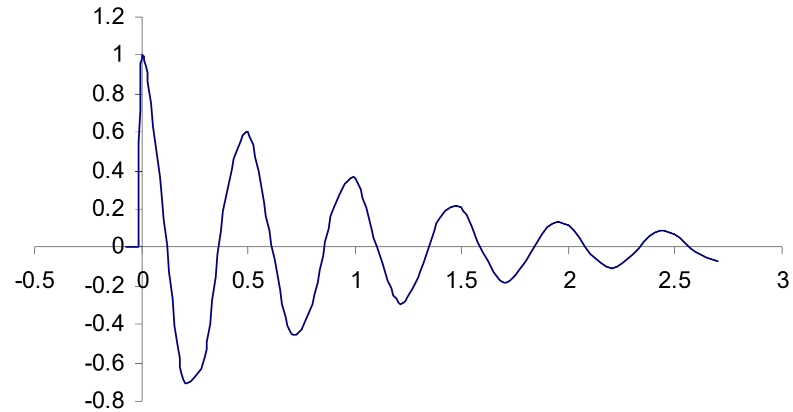
$\cdot$

$\cdot$

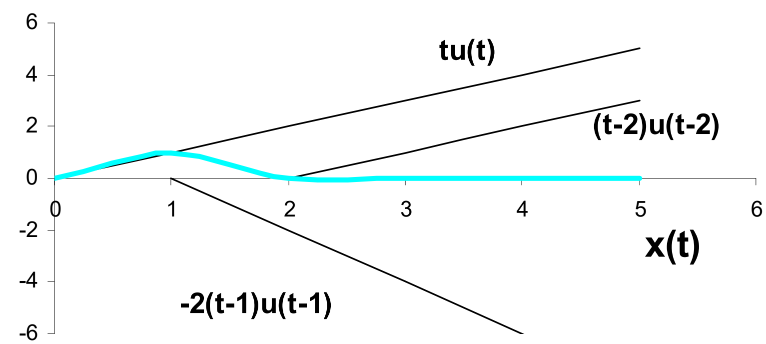
- $\longrightarrow \frac{t^n}{n!}u(t)$ n^{th} order

Signal Representations using the Unit Step Function

- $x(t) = e^{-\sigma t} \cos(\omega t) u(t)$



- $x(t) = t\,u(t) - 2\,(t-1)u(t-1) + (t-2)\,u(t-2)$



Homework

- Complex numbers
  - Convert 1+j1 to its magnitude/angle representation (phasor)
  - Convert 1/(1+j1) to a phasor
  - Draw $e^{j\omega t}$ and $e^{j(\omega t+\alpha)}$ in the complex plane
  - For the series R-L circuit in class, calculate the voltage across the inductor.
  - Appendix A.4, A.7
- Unit Impulse and Unit Step Functions
  - Using unit step functions, construct a single pulse of magnitude 10 starting at t=5 and ending at t=10.
  - Repeat problem 1) with 2 pulses where the second is of magnitude 5 starting at t=15 and ending at t=25.
  - Is the unit step function a bounded function?
  - Is the unit impulse function a bounded function?
  - 2CT.2.4a,b

Systems

Lecture #3

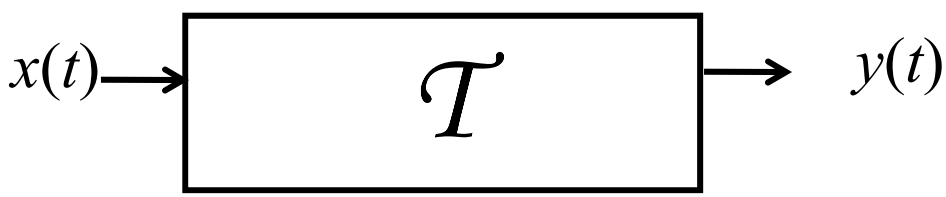
1.3

Representation of a System

- How do represent a system mathematically?
  - Since a system transforms a signal into another we write an equation:

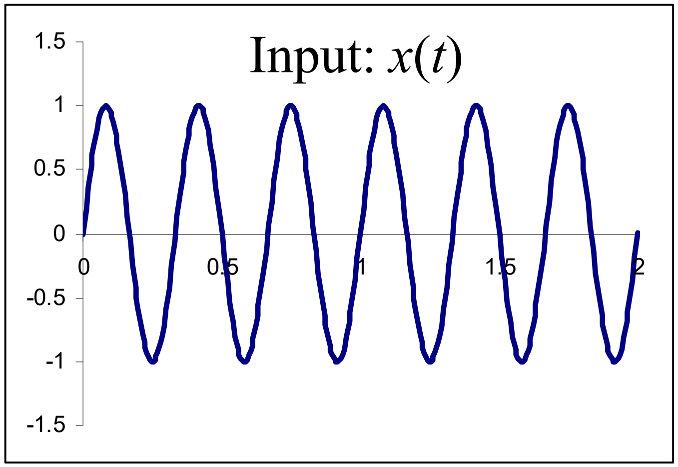
$$y(t)=\mathcal{T}\{x(t)\}$$

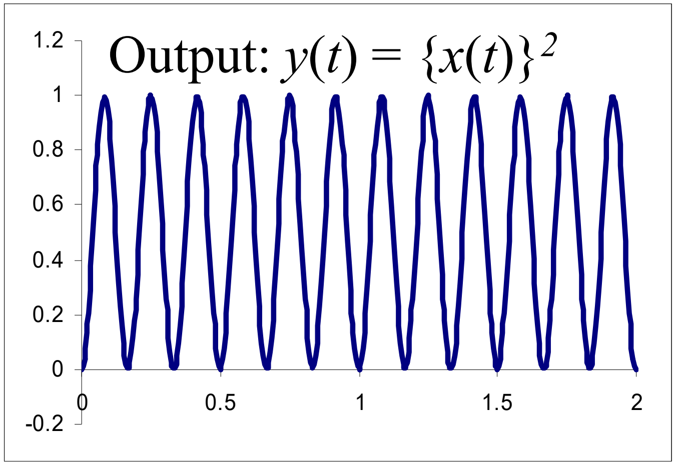
  - where $\mathcal{T}$ is an operator to symbolize a system,
  - $x(t)$ is the signal that goes into the system: input signal *(*or source)
  - And $y(t)$ is transformed signal or output signal (or solution of the equation)
- We can also represent it by a flow diagram



Example of a Continuous-Time System

- A squarer system: $y(t)=\{x(t)\}^2$
  - The output equals the square of the input.
  - This is the result of putting the sine wave into the squarer





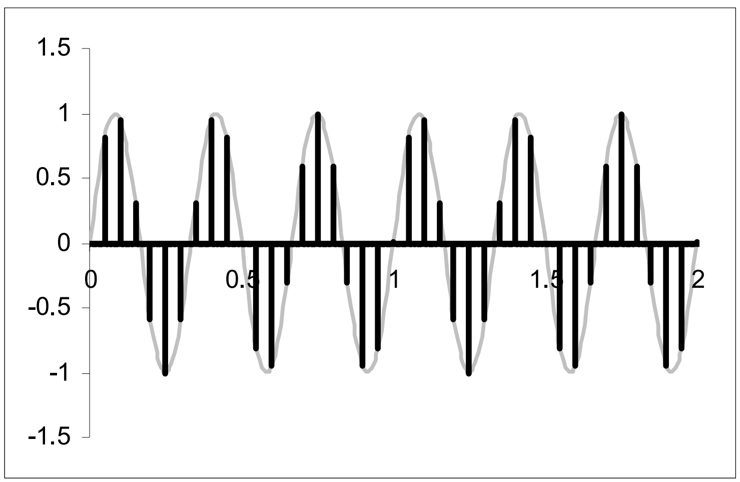
- This is an example of a continuous-time system
- We might be able to build this using an electronic circuit

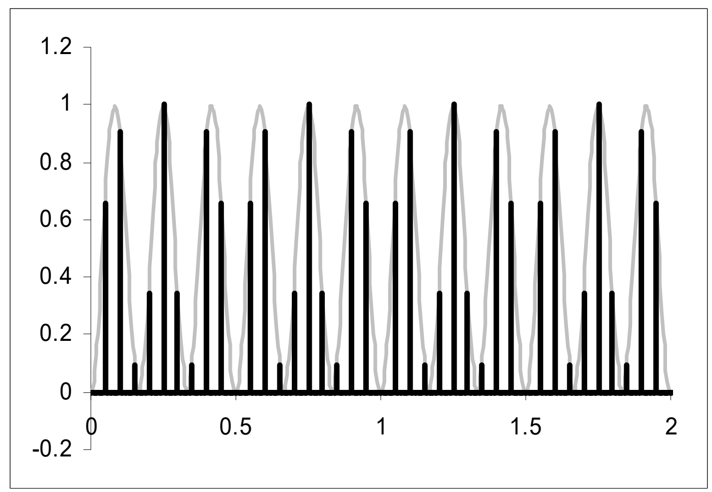
Discrete-Time Systems

- If we put a discrete-time signal into a system the output may be a discrete-time signal
- This is called a Discrete-time system.

$$y[n]=\mathcal{T}\,\{x[n]\}$$

- Using our squarer example: $y[n]=\{x[n]\}^2$





Mixed Systems

- Continuous-to-Discrete systems

$$y[n] = \mathcal{T}\{x(t)\}$$

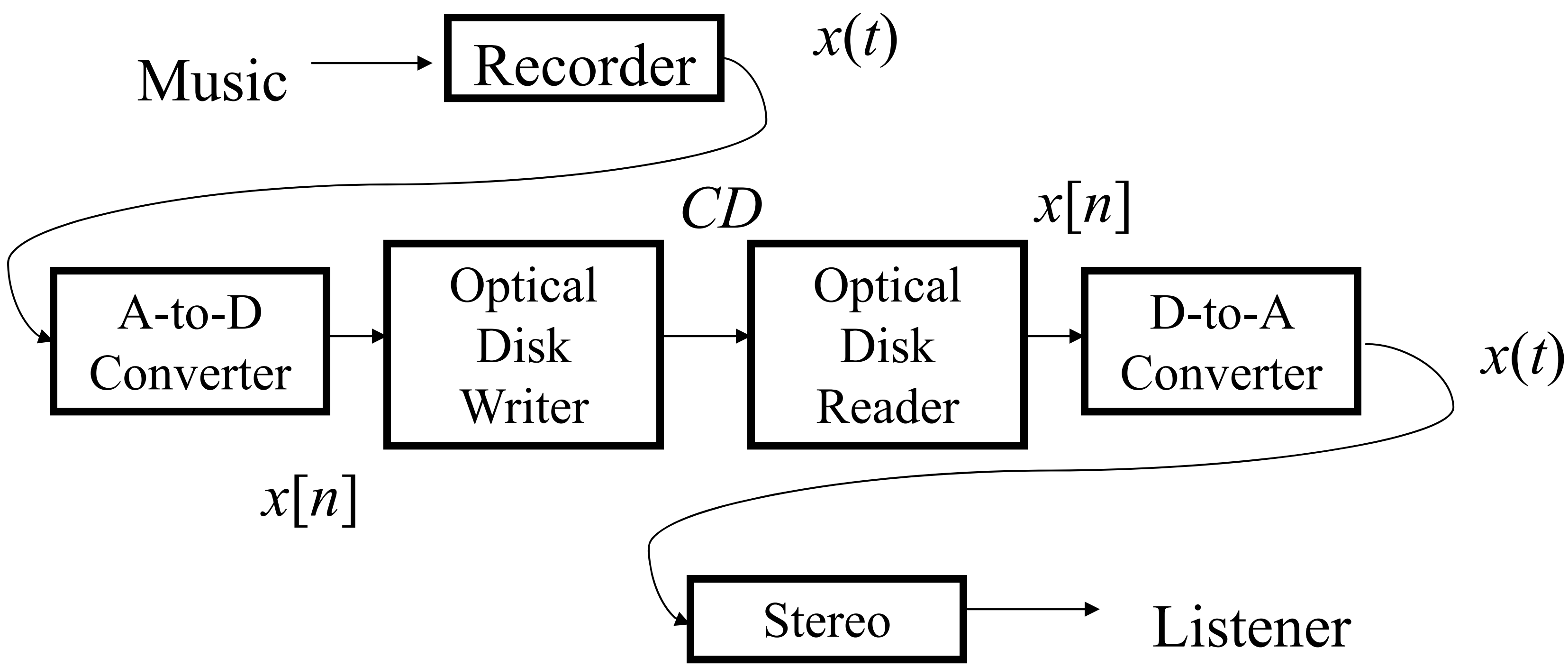
  - Example: a sampler: $y[n]=x(nT_s)$
    - This is also called a A-to-D converter
- Discrete-to-Continuous systems

$$y(t) = \mathcal{T}\{x[n]\}$$

  - Example: An D-to-A converter
    - The opposite of a sampler
    - Takes the samples a recreates the Continuous Signal

An Example

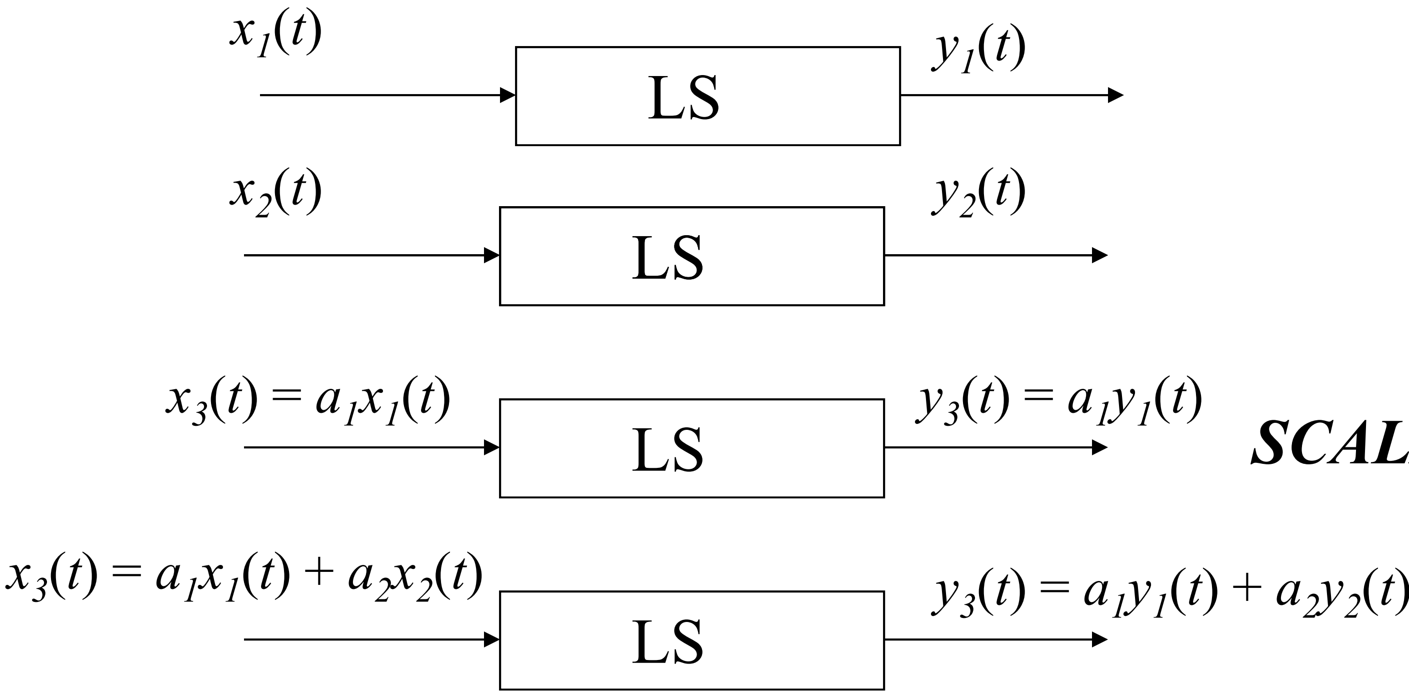
- Example: A music CD



Some Basic Properties of Linear Systems

- If a system is Linear, or better yet Linear and Time Invariant (LTI), it is easier to analyze and understand than systems that are non-linear and/or vary with time.
- All LTI systems must be
  - Linear and support superposition
  - Causal
  - Time Invariant

Linearity for Continuous Signals



SCALAR

SUPERPOSITION

Shorthand

$$x_k(t) \rightarrow y_k(t)$$

$$\sum_k a_k x_k(t) \rightarrow \sum_k a_k y_k(t)$$

Same for Discrete Signals

$$x_k[n] \rightarrow y_k[n]$$

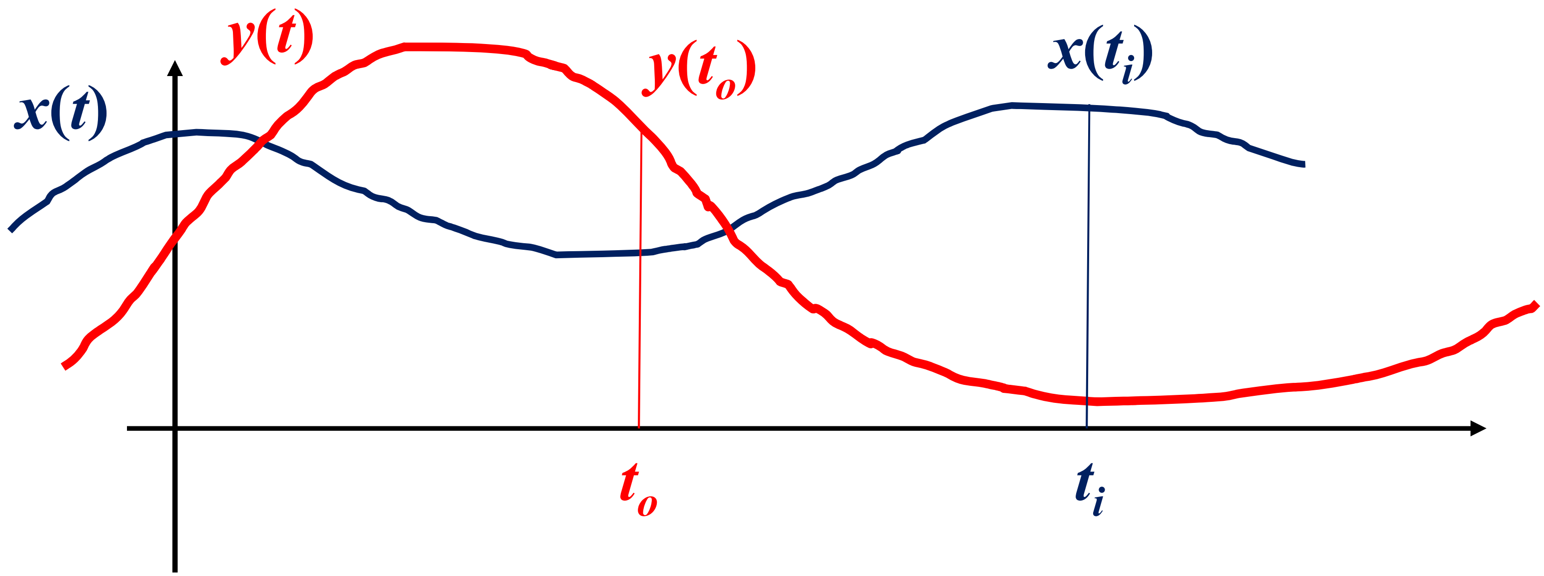
$$\sum_k a_k x_k[n] \rightarrow \sum_k a_k y_k[n]$$

Causality

- A system is causal if the output at any time depends **only** on the input values up to that time
- $y(t_o)$ does not depend on $x(t_i)$ that occur at times after t_o, $t_i > t_o$.
- True for all real time physical systems
- Not true for system-processed recorded signals or spatial varying signal
  - Such systems can look ahead or left, right, up & down
  - E.g., a Morphing System

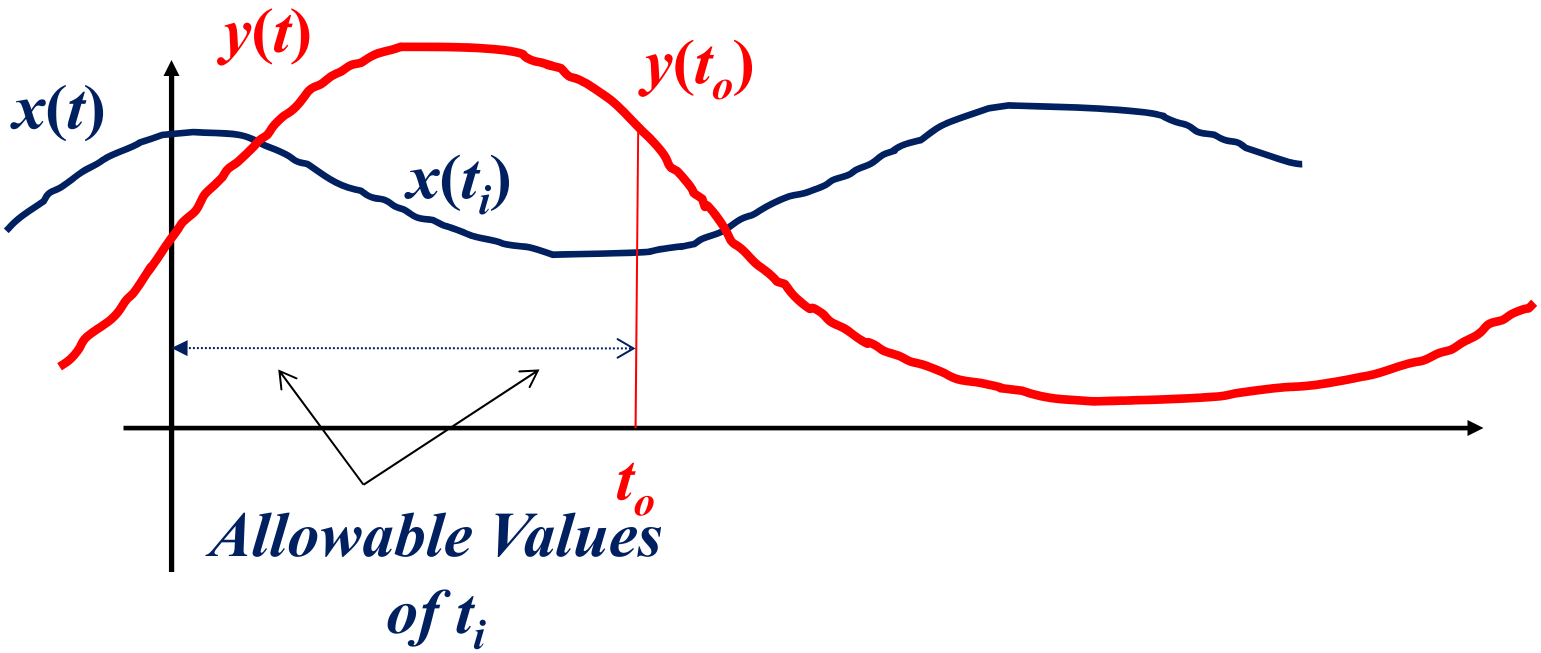
Causality

- Not Causal



Causality

- Causal



Time Invariance

Continuous Signals

$$x_k(t) \longrightarrow y_k(t)$$

Delay $x(t)$ by t_0 yields same response only later

$$x_k(t-t_0) \longrightarrow y_k(t-t_0)$$

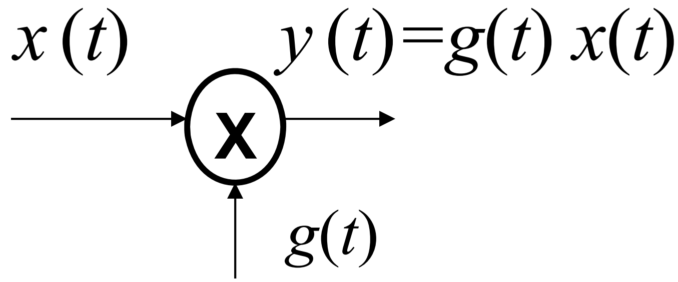
Discrete Signals

$$x_k[n] \longrightarrow y_k[n]$$

$$x_k[n-n_0] \longrightarrow y_k[n-n_0]$$

A Non-LTI System

A multiplier which is a function of time



Check Superposition:

$x_1(t)$ yields $y_1(t)=g(t)\,x_1(t)$

$x_2(t)$ yields $y_2(t)=g(t)\,x_2(t)$

let $x_3(t)= a_1 x_1(t)+a_2 x_2(t)$ then

$y_3(t)=g(t)\,x_3(t) = g(t)\,[a_1 x_1(t)+a_2 x_2(t)]$

$= a_1 y_1(t)+a_2 y_2(t)$

OK

Check Time Invariance:

$x_1(t)=x(t)$ yields $y(t)=g(t)\,x(t)$

$x_2(t) = x(t-\tau)$ yields $y_2(t)=g(t)\,x_2(t)$

$=g(t)x(t-\tau)$

But to be TI

$x_2(t)= x(t-\tau)$ yields $y_2(t) = y(t-\tau)$

$= g(t-\tau)\,x(t-\tau)$

Not OK

Another Non-LTI System

A system with an additive constant

$$y(t) = x(t) + K$$

Check Superposition:

For Superposition to hold, we need to have:

let $x(t) = a_1 x_1(t) + a_2 x_2(t)$ then $y(t) = a_1 x_1(t) + a_2 x_2(t) + K$

But for this system:

$y(t) = y_1(t) + y_2(t) = a_1 x_1(t) + K + a_2 x_2(t) + K$

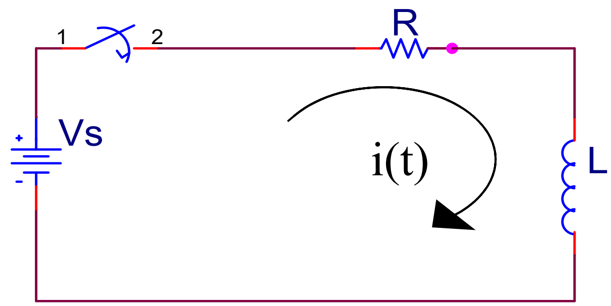
Not OK

How Does One Describe LTI Systems

- For Continuous Systems – By Using Ordinary Differential Equations (ODE)

- For Discrete Systems – By Using Difference Equations

1^{st} Order Linear ODE: Simple Electrical Circuit



R = Resistance
L = Inductance
Vs = Voltage

$$Vs = i(t)R + L\frac{di(t)}{dt}$$

$$\frac{di}{dt} + \frac{R}{L}i = \frac{Vs}{L}$$

1^{st} Order Linear ODE

Solve for i(t) assuming: $i(t) = K_1 e^{-At} + K_2$ with the initial condition that i(0)=0. The 2 terms are need due to the following: Since the source Vs is a constant (battery), we assume that the output must a component which is a constant, K_2. Since the differential equation is requires that the output and its derivative be proportional to each other, we assume that the output must have a component which is proportional to an exponential function, $K_1 e^{-At}$.

1^{st} Order Linear ODE: Simple Electrical Circuit

$$Vs = i(t)R + L\frac{di(t)}{dt}$$

$$\frac{di}{dt} + \frac{R}{L}i = \frac{Vs}{L}$$

Substituting $i(t) = K_1 e^{-At} + K_2$ in the equation, we get

Note that the first derivative, equals

$$\frac{di}{dt} = -AK_1 e^{-At} + 0$$

$$-AK_1 e^{-At} + 0 + \frac{R}{L}(K_1 e^{-At} + K_2) = \frac{Vs}{L}$$

Resorting we have

$$-AK_1 e^{-At} + \frac{R}{L}K_1 e^{-At} + \frac{R}{L}K_2 = \frac{Vs}{L}$$

This implies

$$-AK_1 e^{-At} + \frac{R}{L}K_1 e^{-At} = 0$$

$$\frac{R}{L}K_2 = \frac{Vs}{L}$$

$$-AK_1 e^{-At} + \frac{R}{L}K_1 e^{-At} = 0$$

$$-A + \frac{R}{L} = 0; A = \frac{R}{L}$$

$$\frac{R}{L}K_2 = \frac{Vs}{L}; K_2 = \frac{Vs}{R}$$

Therefore,

$$i(t) = K_1 e^{-\frac{R}{L}t} + \frac{Vs}{R}$$

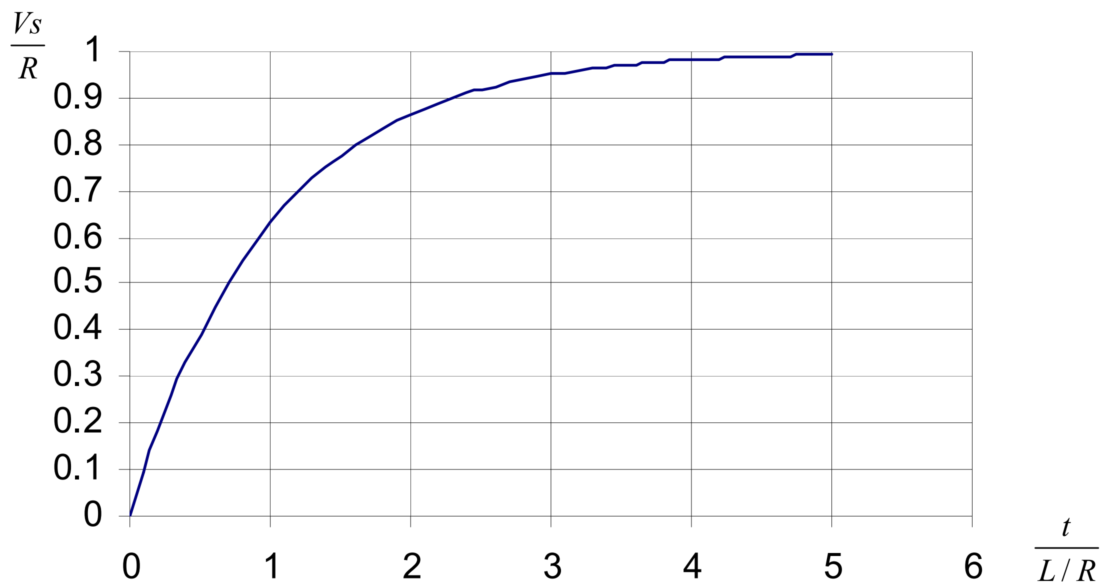
But the initial condition states that $i(0) = 0$

$$i(0) = K_1 e^{-\frac{R}{L}0} + \frac{Vs}{R} = K_1 + \frac{Vs}{R} = 0$$

$$K_1 = -\frac{Vs}{R}$$

$$i(t) = \frac{Vs}{R}(1 - e^{-\frac{R}{L}t})$$

1st Order Linear ODE: Simple Electrical Circuit



$$i(t)=\frac{Vs}{R}(1-e^{-\frac{R}{L}t})=\frac{Vs}{R}(1-e^{-\frac{t}{L/R}})$$

$\frac{L}{R}$ is called the time constant and we see that within 3 time constants 95% of its final value is reached.

Another 1st Order LODE : Drug Concentration in Blood Being Removed by the Liver

$$\dot{D} + K_L D = \frac{R_D}{V_C}$$

Where K_L = drug loss rate

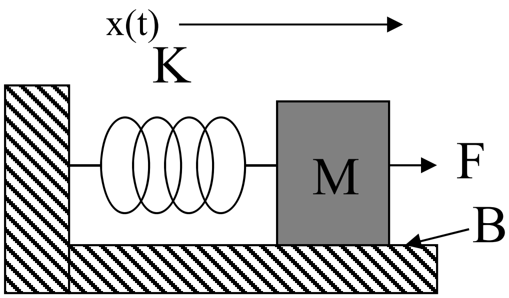
Vc = Volume of circulatory system in liters

R_D is the rate of drug input (μg/min)

In a similar way as in the RL circuit, we can solve this for

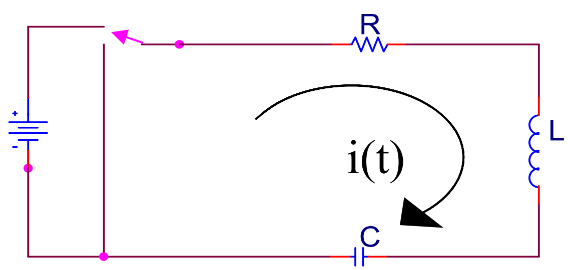
$$D(t) = \frac{R_D}{V_C K_L}(1 - e^{-K_L t})$$

2nd Order LODE



$$M\ddot{x}+B\dot{x}+Kx=F(t)$$

M = Mass
B = Friction
K = Spring constant



$$L\ddot{i}+R\dot{i}+\frac{1}{C}i=0$$

R = Resistance
L = Inductance
C = Capacitance

Homework

- Linear Systems
  - Is $y(t)=x(t)^2$ a linear system? Prove your point.
  - Is $y(t)=t^2$ a linear system? Prove your point.
  - CT.1.3.1
- ODE
  - Solve and plot the solution to the equation: $dx/dt + 6\ x = 0$; $x(0) = 5$; use Matlab to obtain the plot
  - Solve and plot the solution to the equation : $dx/dt + 6\ x = 6$; $x(0) = 0$; use Matlab to obtain the plot

LTI ODE Continued

Lecture #4

Introduction of the p Operator

Let's start with this 2nd order differential equation

to represent some system.

$$a\frac{d^2y(t)}{dt^2}+b\frac{dy(t)}{dt}+cy(t)=e\frac{dx(t)}{dt}+fx(t)$$

where $x(t)$ is the input or source,

$y(t)$ is the solution or the response of the differential equation,

and $a,\ b,\ c,\ d,\ f$ are the coefficeints.

Let's define the p operators:

$$p\cong\frac{d}{dt},p^2\cong\frac{d^2}{dt^2},\ldots,p^n\cong\frac{d^n}{dt^n}$$

and rewrite the differential equation as

$$ap^2y(t)+bpy(t)+cy(t)=epx(t)+fx(t)$$

Introduction of the p Operator

And now for some mathematical blasphemy!!!

$$ap^2 y(t) + bpy(t) + cy(t) = epx(t) + fx(t)$$

$$[ap^2 + bp + c]y(t) = [ep + f]x(t)$$

$$y(t) = \frac{ep + f}{ap^2 + bp + c} x(t)$$

This is **<u>not</u>** a solution for $y(t)$ but a way of reducing an equation into a simpler form.

Introduction of the p Operator

Note that $[ap^2 + bp + c]$ and $[ep + f]$ are polynomials in p and we can rewrite these polynomials as $[ap^2 + bp + c] = A(p)$ and $[ep + f] = B(p)$ and we get

$$A(p)y(t) = B(p)x(t)$$

$$y(t) = \frac{B(p)}{A(p)}x(t)$$

This is **<u>not</u>** a solution for y(t) but a way of reducing an equation into a simpler form.

Components of the Solution of LODE

$$A(p)y(t) = B(p)x(t)$$

Let's define $y(t)$ which is

the total response of the system

$$y(t) = y_s(t) + y_{sf}(t)$$

where $y_s(t)$ is called the response due to the source

and is the solution to this

$$A(p)y_s(t) = B(p)x(t)$$

and $y_{sf}(t)$ is called the source free response or the transient response.

and is the soltuion to this equation

$$A(p)y_{sf}(t) = 0$$

We can now use suposition, solve two simpler equations,

and add them up to get the total solution.

Components of the Solution of LODE Source Response

- $y_s(t)$ - Source Response has the same form as the source and is a solution of:

$$A(p)y_s(t)=B(p)x(t)$$

- If $x(t)$ is a constant then $y_s(t)$ is a constant, if $x(t)$ is a polynomial then $y_s(t)$ is a polynomial, if $x(t)$ is a sinusoid then $y_s(t)$ is a sinusoid, etc.

Components of the Solution of LODE Source-Free Response

- $y_{sf}(t)$ - Source-free Response is also called the Natural Mode Response, Transient Response, Homogeneous, or Characteristic Response and is a solution of

$$A(p)y_{sf}(t)=0 \text{ Homogeneous Equation}$$
$$A(p)=0 \text{ Characteristic Equation}$$

- Functions which satisfy the Homogeneous Equation are called eigenfunctions (e.g., Ke^{at})
- The values of p which are solutions to the Characteristic Equation are called eigenvalues.

Components of the Solution of LODE Source-Free Response Stability

- The source-free response is independent of the source and always appears. It is a function of the system under examination.
- It determines the stability of the system.
- For a stable system,
  1. it is expected that the source-free response is also know as the transient response since it is expected that this component will effectively "end".
  2. When the source-free response ends, the response due to the source or steady state remains.
- For an unstable system, the source-free response may never end.

Components of the Solution of LODE: Source Response

Let us look back at the RL circuit 1st Order ODE: $Vs = i(t)R + L\frac{di(t)}{dt}$

Source Response: $i_s(t)$

Since the source is a constant, Vs, then source response is a constant

$$i_s(t) = K$$

$$\frac{di_s(t)}{dt} + \frac{R}{L}i_s(t) = \frac{Vs}{L}$$

Substituting $i_s(t)$ into the differential equation, we have

$$K = \frac{Vs}{R}$$

$$i_s(t) = \frac{Vs}{R}$$

Components of the Solution of LODE: Source Free Response: Using the Homogenous Equation

Source Free Response: $i_{sf}(t)$

The Homogenous equation: $L\dfrac{di(t)}{dt}+i(t)R=0$

Solutions of the Homogenous equation (the eigenfunctions for a 1st order ODE) are

$$i_{sf}(t)=Ae^{at}$$

Substituting $i_{sf}(t)$ into the homogenous equation, we have

$$\frac{di_{sf}(t)}{dt}=aAe^{at}$$

$$aAe^{at}+\frac{R}{L}Ae^{at}=0$$

$$a=-\frac{R}{L}$$

$$\therefore i_{sf}(t)=Ke^{-\frac{R}{L}t}$$

Components of the Solution of LODE: Source Free Response: Using the Characteristic Equation

Source Free Response: $i_{sf}(t)$

Note that the solution of the Characteristic equation (the eigenvalue) is

$$(p+\frac{R}{L})=0$$

$$p=-\frac{R}{L}$$

and, therefore, same solution: $i_{sf}(t)=Ae^{-\frac{R}{L}t}$

Components of the Solution of LODE: Total Solution

Then the total response is:

$$i(t) = i_s(t) + i_{sf}(t)$$

$$= \frac{Vs}{R} + Ae^{-\frac{R}{L}t}$$

The constant *A* can be found from initial conditions of *i*(*t*)

$$i(0) = \frac{Vs}{R} + Ae^{-\frac{R}{L}0} = \frac{Vs}{R} + A$$

$$A = i(0) - \frac{Vs}{R}$$

$$i(t) = \frac{Vs}{R} + (i(0) - \frac{Vs}{R})e^{-\frac{R}{L}t}$$

Components of the Solution of LODE

In our problem $i(0) = 0$:

$$i(t) = \frac{Vs}{R} + (i(0) - \frac{Vs}{R})e^{-\frac{R}{L}t}$$

$$i(t) = \frac{Vs}{R} - \frac{Vs}{R}e^{-\frac{R}{L}t}$$

$$= \frac{Vs}{R}(1 - e^{-\frac{R}{L}t})$$

Free Response of a 2nd ODE Solutions of the Characteristic Equation

$$ap^2 + bp + c = 0$$

Let p_1,p_2 be the roots of the characteristic equation, then

$$p_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The 4 Free Response Cases of a 2nd ODE Solutions of the Homogeneous Equation

Case	Roots		Solution	Type
$b^2 - 4ac > 0$	$p_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Real, Unequal, Negative	$y_{sf}(t) = C_1 e^{p_1 t} + C_2 e^{p_2 t}$ where C_1 and C_2 are real	Overdamped Stable
$b^2 - 4ac = 0$	$p_{1,2} = p = \frac{-b}{2a}$	Real, Equal, Negative	$y_{sf}(t) = (C_1 t + C_2) e^{pt}$ where C_1 and C_2 are real	Critically damped Stable
$b^2 - 4ac < 0$	$\boldsymbol{p}_{1,2} = \frac{-b \pm j\sqrt{4ac - b^2}}{2a} = \alpha \pm j\omega$	Complex conjugates, Unequal	$y_{sf}(t) = e^{-\alpha t}(C_1 e^{j\omega t} + C_2 e^{-j\omega t})$ where $p_{1,2} = -\alpha \pm j\omega$ $C_2 = C_1{*}; C_1 = Ce^{j\theta}$ are complex conjugates $y_{sf}(t) = e^{-\alpha t} C(e^{j(\omega t + \theta)} + e^{-j(\omega t + \theta)})$ $= e^{-\alpha t} 2C \cos(\omega t + \theta)$	Underdamped Stable
$b^2 - 4ac < 0$ & $b = 0$	$\boldsymbol{p}_{1,2} = \pm j \frac{\sqrt{4ac}}{2a} = \pm j\omega$	Imaginary	$y_{sf}(t) = C_1 e^{j\omega t} + C_2 e^{-j\omega t}$ $= 2C \cos(\omega t + \theta)$ where $p_{1,2} = \pm j\omega$ & $C_2 = C_1{*}; C_1 = Ce^{j\theta}$	Undamped or Oscillatory Unstable

Solutions to the Source Free Response of 2nd Order ODE

Overdamped $y_{sf}(t) = C_1 e^{p_1 t} + C_2 e^{p_2 t}$; note that p_1 and p_2 are negative

Critically Damped $y_{sf}(t) = (C_1 t + C_2) e^{pt}$; note that p is negative

Underdamped $y_{sf}(t) = C_1 e^{p_1 t} + C_2 e^{p_2 t} = e^{-\alpha t}(C_1 e^{j\omega t} + C_2 e^{-j\omega t})$ where $p_{1,2} = -\alpha \pm j\omega$

but we can show that

$C_2 = C_1^*$; and assuming

$C_1 = a + jb = Ce^{j\theta}$; then $C_2 = a - jb = Ce^{-j\theta}$

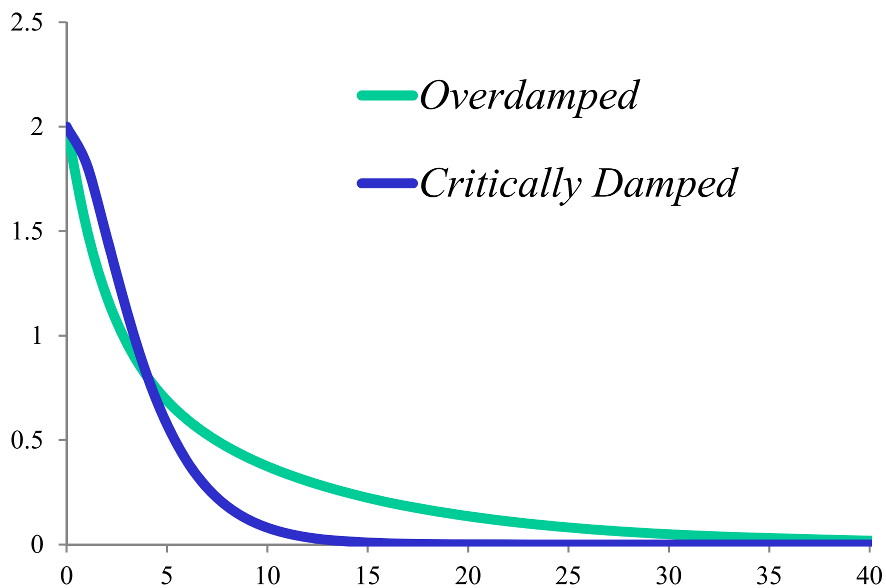
$$y_{sf}(t) = e^{-\alpha t}(Ce^{j\theta}e^{j\omega t} + Ce^{-j\theta}e^{-j\omega t}) = e^{-\alpha t}(Ce^{j(\omega t+\theta)} + Ce^{-j(\omega t+\theta)})$$

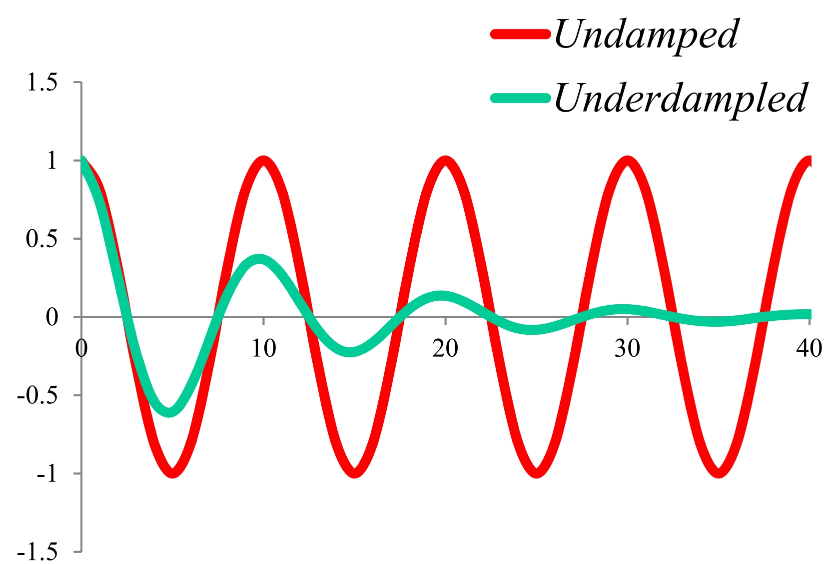
$$= e^{-\alpha t}C(e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}) = e^{-\alpha t}2C\cos(\omega t + \theta)$$

Undamped

$$y_{sf}(t) = C_1 e^{j\omega t} + C_2 e^{-j\omega t} = 2C\cos(\omega t + \theta);\text{where } p_{1,2} = \pm j\omega;\ C_2 = C_1^*; C_1 = Ce^{j\theta}$$

Source Free Responses of 2nd Order ODE





Proof of the Complex Conjugate Constants Underdamped Case

$$y(t) = e^{-\alpha t}(C_1 e^{j\omega t} + C_2 e^{-j\omega t}) = C_1 e^{s_1 t} + C_2 e^{s_1^* t}; \text{where } s_1 = -\alpha + j\omega$$

$$y(0) = C_1 + C_2$$

$$\dot{y}(t) = s_1 C_1 e^{s_1 t} + s_1^* C_2 e^{s_1^* t}$$

$$\dot{y}(0) = s_1 C_1 + s_1^* C_2$$

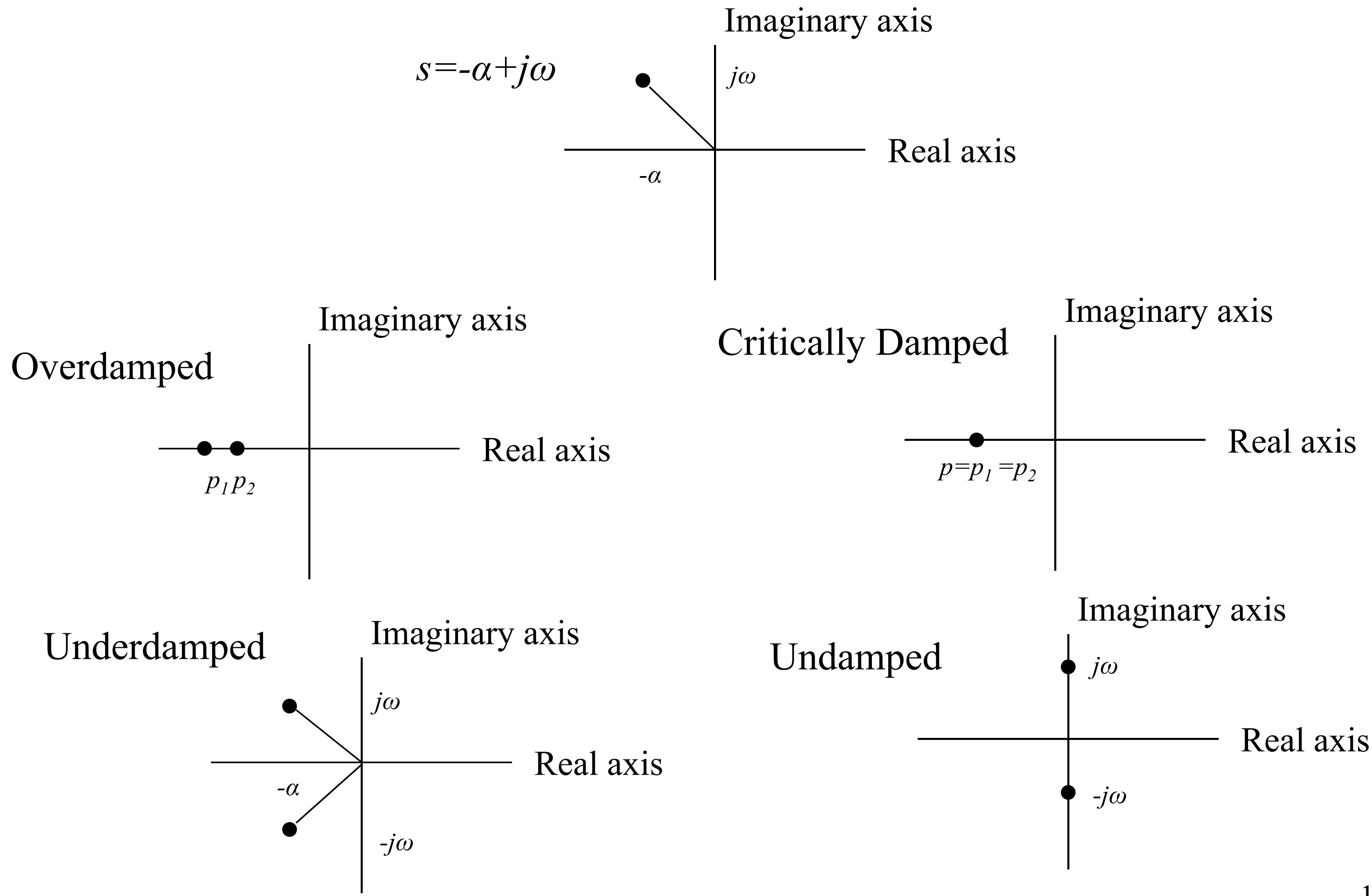
$$C_1 = y(0) - C_2$$

$$\dot{y}(0) = s_1[y(0) - C_2] + s_1^* C_2$$

$$C_2 = \frac{\dot{y}(0) - s_1 y(0)}{s_1^* - s_1} = \frac{\dot{y}(0) - (-\alpha + j\omega)y(0)}{-j2\omega} = \frac{y(0)}{2} + j\frac{\dot{y}(0) + \alpha y(0)}{2\omega}$$

$$C_1 = y(0) - \frac{\dot{y}(0) - s_1 y(0)}{s_1^* - s_1} = \frac{s_1^* y(0) - \dot{y}(0)}{s_1^* - s_1} = \frac{(-\alpha - j\omega)y(0) - \dot{y}(0)}{-j2\omega} = \frac{y(0)}{2} - j\frac{\dot{y}(0) + \alpha y(0)}{2\omega}$$

Complex Plane



Complete Response of 2nd Order ODE
An Example

$\ddot{y} + 4\dot{y} + 3y = 4e^{-2t}; y(0) = 2; \dot{y}(0) = 4$ System with initial conditions

$p^2 + 4p + 3 = 0$ Characteristic Equation

$p_{1,2} = -3, -1$ Eigenvalues - Overdamped

$y_{sf}(t) = A_1 e^{-3t} + A_2 e^{-t}$ Eigenfunctions and source free response

$y_s(t) = A_3 e^{-2t}$ Source free response

$y(t) = A_1 e^{-3t} + A_2 e^{-t} + A_3 e^{-2t}$ Total response

$$\dot{y}(t) = -3A_1 e^{-3t} - A_2 e^{-t} - 2A_3 e^{-2t}$$

$$\ddot{y}(t) = 9A_1 e^{-3t} + A_2 e^{-t} + 4A_3 e^{-2t}$$

Complete Response of 2nd Order ODE Solution

Substitute the solution, the first and second derivatives into the system

$$(9A_1e^{-3t} + A_2e^{-t} + 4A_3e^{-2t}) + 4(-3A_1e^{-3t} - A_2e^{-t} - 2A_3e^{-2t}) + 3(A_1e^{-3t} + A_2e^{-t} + A_3e^{-2t}) = 4e^{-2t}$$

Resort the equation into like terms

$$(9 - 12 + 3)A_1e^{-3t} + (1 - 4 + 3)A_2e^{-t} + (4 - 8 + 3)A_3e^{-2t} = 4e^{-2t}$$

Notice that the coefficents of the eigenfunctions are zero since they must satisfy the Homogeneous Equation.

What is left is the source term, the response due to the sources:

$$(0)A_1e^{-3t} + (0)A_2e^{-t} - A_3e^{-2t} = 4e^{-2t}$$

From this A_3 is determined:

$$-A_3e^{-2t} = 4e^{-2t} \Rightarrow A_3 = -4$$

Using the initial conditions of the response and the first derivative, A_1 and A_2 are determined.

$$y(0) = A_1 + A_2 - 4 = 2 \Rightarrow A_1 + A_2 = 6$$

$$\dot{y}(0) = -3A_1 - A_2 + 8 = 4 \Rightarrow -3A_1 - A_2 = -4$$

$$-2A_1 = 2 \Rightarrow A_1 = -1; A_2 = 7$$

The total solution is:

$$\therefore y(t) = -e^{-3t} + 7e^{-t} + -4_3e^{-2t}$$

Natural Frequency and Damping Ratio

$ap^2 + bp + c = 0$

$p^2 + \frac{b}{a}p + \frac{c}{a} = 0$

$p_{1,2} = -\frac{b}{2a} \pm \sqrt{(\frac{b}{2a})^2 - \frac{c}{a}}$

If $b = 0$, then $p_{1,2} = \pm j\sqrt{\frac{c}{a}} = \pm j\omega_o$

Define $\frac{c}{a} = \omega_o{}^2$

Define $\frac{b}{2a} = \xi\omega_o \Rightarrow \xi = \frac{b}{2a\omega_o} = \frac{b}{2a\sqrt{\frac{c}{a}}} = \frac{b}{2\sqrt{ac}}$

We call ω_o the undamped natural frequency and ξ the damping ratio

$p^2 + 2\xi\omega_o p + \omega_o{}^2 = 0 \Rightarrow p_{1,2} = -\frac{b}{2a} \pm \sqrt{(\frac{b}{2a})^2 - \frac{c}{a}} \Rightarrow -\xi\omega_o \pm \sqrt{(\xi\omega_o)^2 - \omega_o{}^2}$

$p_{1,2} = -\xi\omega_o \pm \sqrt{(\xi\omega_o)^2 - \omega_o{}^2} = -\xi\omega_o \pm \omega_o\sqrt{\xi^2 - 1}$

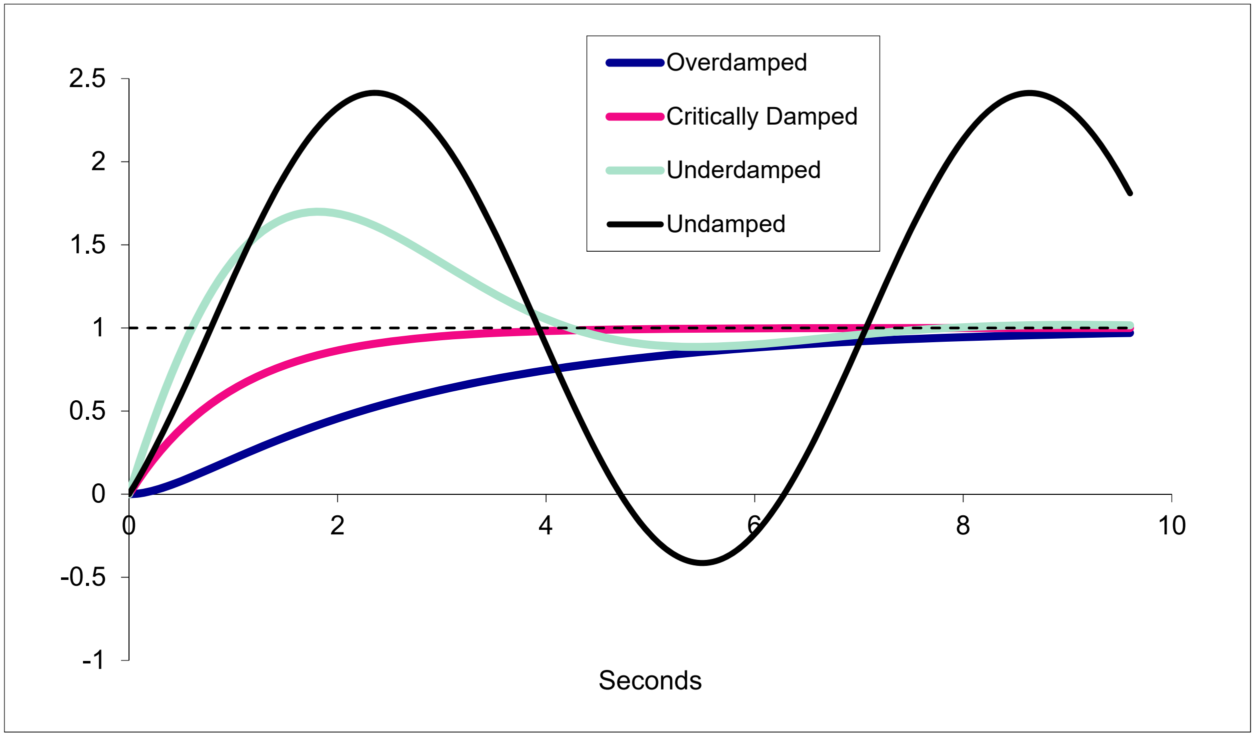
For $\xi > 1$, overdamped, $p_{1,2} = -\xi\omega_o \pm \omega_o\sqrt{\xi^2 - 1} = \omega_o(-\xi \pm \sqrt{\xi^2 - 1})$

For $\xi = 1$, critically damped, $p_{1,2} = -\omega_o$

For $\xi < 1$, underdamped, $p_{1,2} = -\xi\omega_o \pm j\omega_o\sqrt{1 - \xi^2}$

For $\xi = 0$, undamped, $p_{1,2} = \pm j\omega_o$

Natural Frequency and Damping Ratio Unit Step Response



Unit Step Response Overdamped

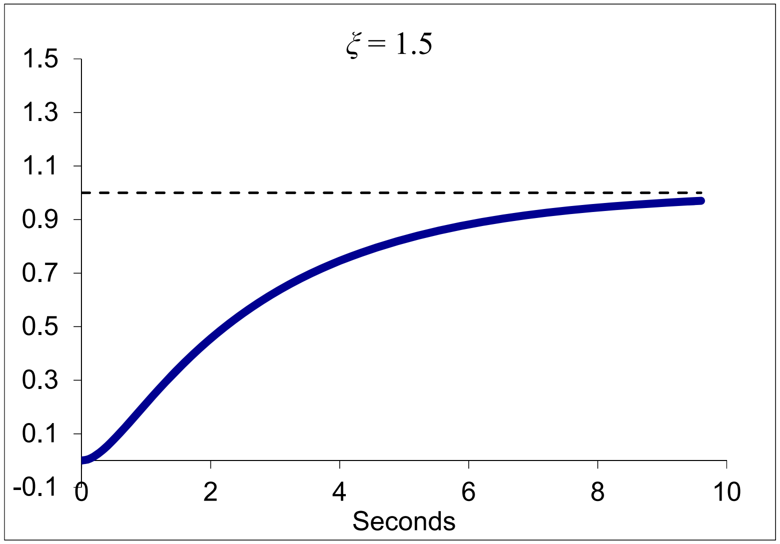
$$p^2 y(t) + \frac{b}{a} p y(t) + \frac{c}{a} y(t) = \frac{1}{a} x(t)$$

$$p^2 y(t) + 2\xi\omega_o p y(t) + \omega_o^{\,2} y(t) = Kx(t) = 1$$

Source response is $y_s(t) = A = \dfrac{1}{\omega_o^{\,2}}$

For $\xi > 1$, overdamped, $p_{1,2} = (-\xi \pm \sqrt{\xi^2 - 1})\omega_o$

Source free response is $y_{sf}(t) = C_1 e^{(-\xi - \sqrt{\xi^2-1})\omega_o t} + C_2 e^{(-\xi + \sqrt{\xi^2-1})\omega_o t}$



$$y(t) = \frac{1}{\omega_o^{\,2}} + C_1 e^{(-\xi - \sqrt{\xi^2-1})\omega_o t} + C_2 e^{(-\xi + \sqrt{\xi^2-1})\omega_o t}$$

$$y(0) = 0 = \frac{1}{\omega_o^{\,2}} + C_1 + C_2; C_1 = -(\frac{1}{\omega_o^{\,2}} + C_2)$$

$$\dot{y}(0) = 0 + (-\xi - \sqrt{\xi^2 - 1})\omega_o C_1 + (-\xi + \sqrt{\xi^2 - 1})\omega_o C_2$$

$$\dot{y}(0) = -(-\xi - \sqrt{\xi^2 - 1})\omega_o[(\frac{1}{\omega_o^{\,2}} + C_2)] + (-\xi + \sqrt{\xi^2 - 1})\omega_o C_2 = (\xi + \sqrt{\xi^2 - 1})\omega_o[(\frac{1}{\omega_o^{\,2}} + C_2)] + (-\xi + \sqrt{\xi^2 - 1})\omega_o C_2$$

$$= -(-\xi - \sqrt{\xi^2 - 1})\omega_o \frac{1}{\omega_o^{\,2}} + (\xi + \sqrt{\xi^2 - 1})\omega_o C_2 + (-\xi + \sqrt{\xi^2 - 1})\omega_o C_2 = (\xi + \sqrt{\xi^2 - 1})\frac{1}{\omega_o} + (2\sqrt{\xi^2 - 1})\omega_o C_2$$

$$\dot{y}(0) = -(-\xi - \sqrt{\xi^2 - 1})\frac{1}{\omega_o} + (2\sqrt{\xi^2 - 1})\omega_o C_2 \Rightarrow C_2 = \frac{\dot{y}(0) + (-\xi - \sqrt{\xi^2 - 1})\frac{1}{\omega_o}}{(2\sqrt{\xi^2 - 1})\omega_o}$$

$$C_2 = \frac{\dot{y}(0)\omega_o + (-\xi - \sqrt{\xi^2 - 1})}{2\omega_o^{\,2}\sqrt{\xi^2 - 1}}; C_1 = (\frac{\xi - \sqrt{\xi^2 - 1} - \dot{y}(0)\omega_o}{2\omega_o^{\,2}\sqrt{\xi^2 - 1}})$$

$$y(t) = \frac{1}{\omega_o^{\,2}} + \frac{-\dot{y}(0)\omega_o + \xi - \sqrt{\xi^2 - 1}}{2\omega_o^{\,2}\sqrt{\xi^2 - 1}} e^{(-\xi - \sqrt{\xi^2-1})\omega_o t} + \frac{\dot{y}(0)\omega_o - \xi - \sqrt{\xi^2 - 1}}{2\omega_o^{\,2}\sqrt{\xi^2 - 1}} e^{(-\xi + \sqrt{\xi^2-1})\omega_o t}$$

Unit Step Response
Critically Damped

$$p^2 y(t) + \frac{b}{a} p y(t) + \frac{c}{a} y(t) = \frac{1}{a} x(t)$$

$$p^2 y(t) + 2\xi\omega_o p y(t) + \omega_o^2 y(t) = Kx(t) = 1$$

Source response is $y_s(t) = A = \frac{1}{\omega_o^2}$

For $\xi = 1$, critically damped, $p_{1,2} = -\omega_o$

Source free response is $y_{sf}(t) = (C_1 t + C_2)e^{-\omega_o t}$

$$y(t) = \frac{1}{\omega_o^2} + (C_1 t + C_2)e^{-\omega_o t}$$

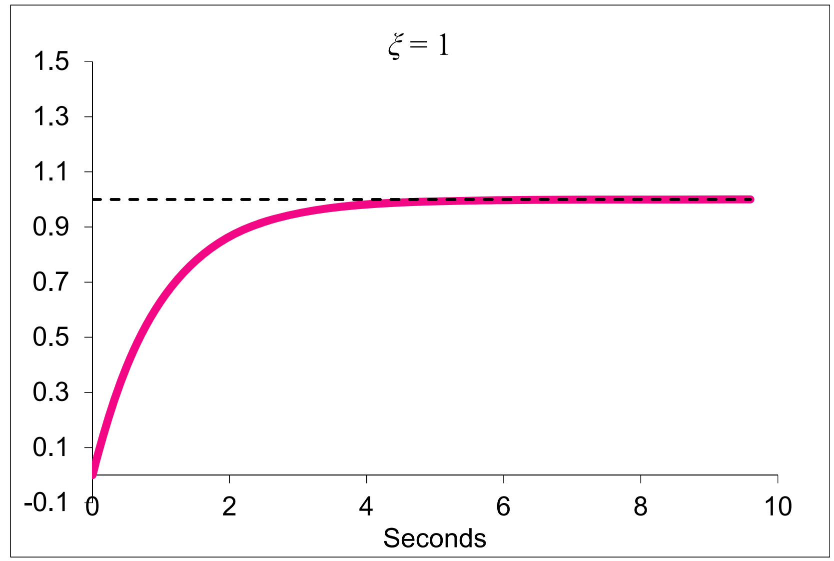
$$y(0) = 0 = \frac{1}{\omega_o^2} + C_2$$

$$C_2 = -\frac{1}{\omega_o^2}$$

$$\dot{y}(0) = 0 + C_1 - \omega_o C_2$$

$$C_1 = \dot{y}(0) + \omega_o C_2 = \dot{y}(0) - \frac{1}{\omega_o}$$

$$y(t) = \frac{1}{\omega_o^2} + ([\dot{y}(0) - \frac{1}{\omega_o}]t - \frac{1}{\omega_o^2})e^{-\omega_o t}$$



Unit Step Response Underdamped

$$p^2 y(t) + \frac{b}{a} p y(t) + \frac{c}{a} y(t) = \frac{1}{a} x(t)$$

$$p^2 y(t) + 2\xi\omega_o p y(t) + \omega_o^2 y(t) = Kx(t) = 1$$

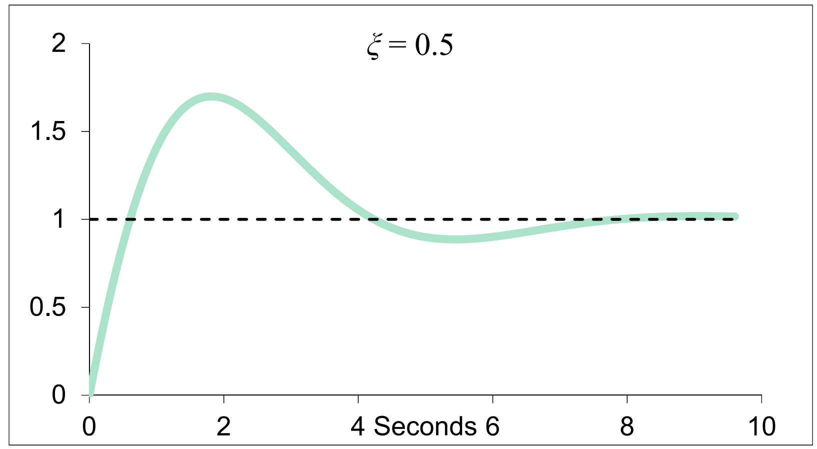
Source response is $y_s(t) = A = \frac{1}{\omega_o^2}$

For $\xi < 1$, underdamped, $p_{1,2} = -\xi\omega_o \pm j\omega_o\sqrt{1-\xi^2}$

Source free response is $y_{sf}(t) = e^{-\xi\omega_o t} C\cos(\omega_o\sqrt{1-\xi^2}\,t + \theta)$

$$y(t) = \frac{1}{\omega_o^2} + e^{-\xi\omega_o t} C\cos(\omega_o\sqrt{1-\xi^2}\,t + \theta)$$

$$y(0) = 0 = \frac{1}{\omega_o^2} + C\cos(\theta); C\cos(\theta) = -\frac{1}{\omega_o^2}$$



$$\dot{y}(0) = 0 + -\xi\omega_o C\cos(\theta) - \omega_o\sqrt{1-\xi^2}\,C\sin(\theta) = \xi\omega_o\frac{1}{\omega_o^2} - \omega_o\sqrt{1-\xi^2}\,C\sin(\theta); C\sin(\theta) = -\frac{\dot{y}(0) - \frac{\xi}{\omega_o}}{\omega_o\sqrt{1-\xi^2}}$$

$$\theta = \tan^{-1}\left(\frac{\dot{y}(0)\omega_o - \xi}{\sqrt{1-\xi^2}}\right); C = -\frac{1}{\omega_o^2\cos\left(\tan^{-1}\left(\frac{\dot{y}(0)\omega_o - \xi}{\sqrt{1-\xi^2}}\right)\right)}$$

$$y(t) = \frac{1}{\omega_o^2} - e^{-\xi\omega_o t}\frac{1}{\omega_o^2\cos\left(\tan^{-1}\left(\frac{\dot{y}(0)\omega_o - \xi}{\sqrt{1-\xi^2}}\right)\right)}\cos\left(\omega_o\sqrt{1-\xi^2}\,t + \tan^{-1}\left(\frac{\dot{y}(0)\omega_o - \xi}{\sqrt{1-\xi^2}}\right)\right)$$

Unit Step Response Undamped

$$p^2 y(t) + \frac{b}{a} p y(t) + \frac{c}{a} y(t) = \frac{1}{a} x(t)$$

$$p^2 y(t) + 2\xi\omega_o p y(t) + \omega_o^2 y(t) = Kx(t) = 1$$

Source response is $y_s(t) = A = \dfrac{1}{\omega_o^2}$

For $\xi = 0$, undamped, $p_{1,2} = \pm j\omega_o$

Source free response is $y_{sf}(t) = C\cos(\omega_o t + \theta)$

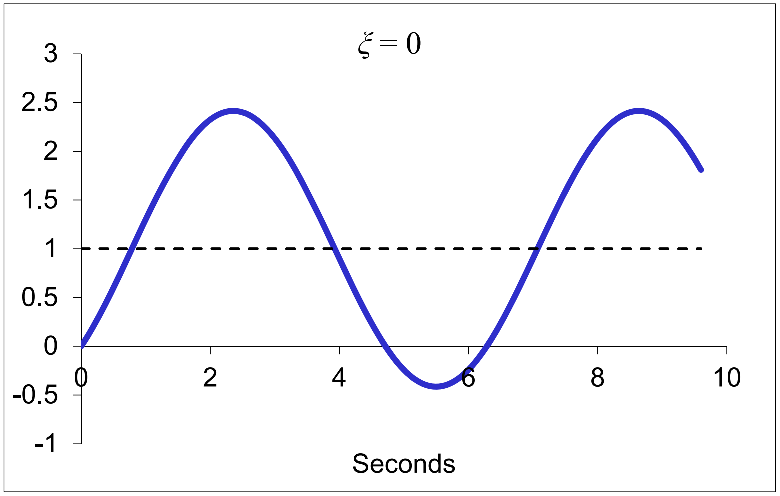
$$y(t) = \frac{1}{\omega_o^2} + C\cos(\omega_o t + \theta)$$

$$y(0) = 0 = \frac{1}{\omega_o^2} + C\cos(\theta); C\cos(\theta) = -\frac{1}{\omega_o^2}$$

$$\dot{y}(0) = 0 - \omega_o C\sin(\theta); C\sin(\theta) = -\frac{\dot{y}(0)}{\omega_o}$$

$$\theta = \tan^{-1}(\dot{y}(0)\omega_o); C = -\frac{1}{\omega_o^2 \cos(\tan^{-1}(\dot{y}(0)\omega_o))}$$

$$y(t) = \frac{1}{\omega_o^2} - \frac{1}{\omega_o^2 \cos(\tan^{-1}(\dot{y}(0)\omega_o))}\cos(\omega_o t + \tan^{-1}(\dot{y}(0)\omega_o))$$



Poles and Zeroes

- Source Response:

$$A(p)y(t) = B(p)x(t)$$
$$y(t) = \frac{B(p)}{A(p)}x(t)$$
$$y(t) = H(p)x(t)$$
$$H(p) = \frac{B(p)}{A(p)}$$

*$H(p)$ is known as the **system function** or **network response***

- We can think of the solutions of $B(p) = 0$ as the zeroes of the system
- We can think of the solutions of $A(p) = 0$ as the poles of the system

Poles and Zeros Continued

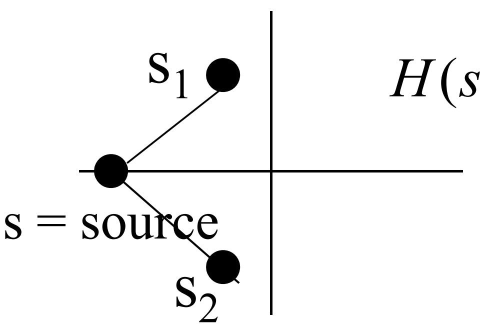
- If we assume that the source *x(t)* has the form $x(t)=e^{st}$ where s is a complex number, we can plot the poles and zeroes in the complex plane to graphically see the response to a particular source function. Note that *H(s)* is a complex number with magnitude and angle.

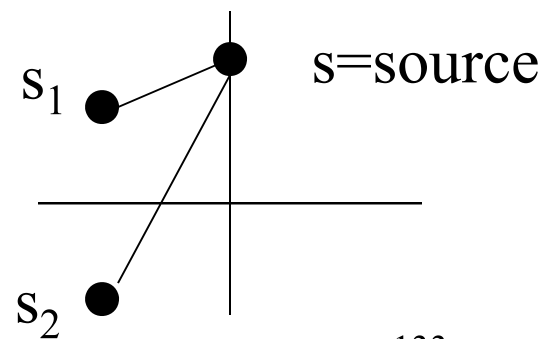
$$y(t) = H(p)x(t) = H(p)e^{st}$$
$$= H(s)e^{st}$$

since $pe^{st} = \frac{d}{dt}e^{st} = se^{st}$

For example, if *H(p)* have no zeroes and 2 poles, then

$$H(s) = \frac{A}{(s-s_1)(s-s_2)}$$





Homework

- 2^{st} Order ODEs
    1. Using Matlab, plot the response for following systems. Identity what type of system each is. Submit your code:

$$a)\ddot{x} + 10\dot{x} + 4x = 0; x(0) = 5; \dot{x}(0) = 0$$

$$b)\ddot{x} + 4\dot{x} + 4x = 0; x(0) = 5; \dot{x}(0) = 0$$

$$c)\ddot{x} + 1\dot{x} + 4x = 0; x(0) = 5; \dot{x}(0) = 0$$

$$d)\ddot{x} + 4x = 0; x(0) = 5; \dot{x}(0) = 0$$

    2. 2.2 An LTI system is described by the second-order ODE:

$$\ddot{y} + 7\dot{y} + 10y = x(t)$$

        a. Use the p operator notation to find the roots of the characteristic equation.
        b. Assume y(0)=0, dy(0)/dt=9 and x(t)=0, find y(t).
        c. Now Let x(t) = 10 and same initial conditions, find y(t).

Homework

3. A quadratic low-pass filter is described by the second-order ODE:

$$\ddot{y} + (2\xi\omega_n)\dot{y} + \omega_n^{\;2} y = x(t)$$

a. The characteristic equation for this ODE in terms of the p operator has complex conjugate roots. Find an algebraic expression for the position of the roots.

b. Let $x(t)$=1. Find the steady-state output.

c. Let $x(t)$=0, $y(0)$=0 and $dy(0)/dt$=10, Find and sketch $y(t)$ for ξ =.5 and ω_n=1

Homework

4. An alternate way of writing the ODE for an underdamped system is

$$\ddot{y} + 2a\dot{y} + (b^2 + a^2)y = x(t)$$

Let a=.5 and b=.86603. Repeat a, b, and c in problem 2.

Homework

5. BioSignals
   - An heart signal is sampled at the rate 250 s/s and is passed to EKG which has an input consisting of a low pass filter. The filter is a resistor and capacitor in series where the output of the filter is taken across the capacitor. What should be the value of the Capacitor if the Resistor is 1k ohms and the time constant of the filter so that the transient response is completed within 1/10 of the sample time? What is the cutoff frequency of this filter?

Homework

6. Respiration may be modeled with the following second order equation, where $y(t)$ is the respiration signal and $x(t)$ is the additional load above the resting respiration on the body:

$$\ddot{y} + \omega_n^2 y = x(t)$$

a. Calculate the resting respiration rate. Assume $y(0)=0$ and $dy(0)/dt=\omega_n$.

b. Calculate the respiration rate when $x(t)=\cos(\omega_n t)$. Assume the same initial conditions as part a.

c. Use Matlab to graph the signals for both parts. Assume $\omega_n=2\pi$.

Natural Frequency and Damping Ratio

$$p^2 y(t) + \frac{b}{a} p y(t) + \frac{c}{a} y(t) = \frac{1}{a} x(t)$$

$$p^2 y(t) + 2\xi\omega_o p y(t) + \omega_o^2 y(t) = Kx(t) = 1$$

Source response is $y_s(t) = A = \frac{1}{\omega_o^2}$

For $\xi > 1$, overdamped, $p_{1,2} = (-\xi \pm \sqrt{\xi^2 - 1})\omega_o$

Source free response is $y_{sf}(t) = C_1 e^{(-\xi-\sqrt{\xi^2-1})\omega_o t} + C_2 e^{(-\xi+\sqrt{\xi^2-1})\omega_o t}$

$$y(t) = \frac{1}{\omega_o^2} + C_1 e^{(-\xi-\sqrt{\xi^2-1})\omega_o t} + C_2 e^{(-\xi+\sqrt{\xi^2-1})\omega_o t}$$

$$y(0) = 0 = \frac{1}{\omega_o^2} + C_1 + C_2; C_1 = -(\frac{1}{\omega_o^2} + C_2)$$

$$\dot{y}(0) = 0 + (-\xi - \sqrt{\xi^2 - 1})\omega_o C_1 + (-\xi + \sqrt{\xi^2 - 1})\omega_o C_2$$

$$\dot{y}(0) = -(-\xi - \sqrt{\xi^2 - 1})\omega_o [(\frac{1}{\omega_o^2} + C_2)] + (-\xi + \sqrt{\xi^2 - 1})\omega_o C_2$$

$$\dot{y}(0) = -(-\xi - \sqrt{\xi^2 - 1})\omega_o (\frac{1}{\omega_o^2}) - (-\xi - \sqrt{\xi^2 - 1})\omega_o C_2 + (-\xi + \sqrt{\xi^2 - 1})\omega_o C_2$$

$$\dot{y}(0) = -(-\xi - \sqrt{\xi^2 - 1})\omega_o (\frac{1}{\omega_o^2}) + [(-\xi + \sqrt{\xi^2 - 1})\omega_o - (-\xi - \sqrt{\xi^2 - 1})\omega_o] C_2$$

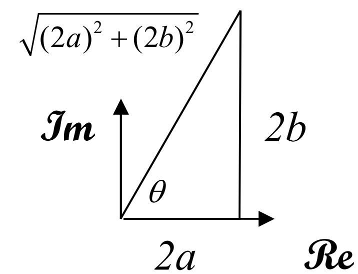
$$\dot{y}(0) + (-\xi - \sqrt{\xi^2 - 1})\omega_o (\frac{1}{\omega_o^2}) = 2\omega_o \sqrt{\xi^2 - 1} C_2$$

$$C_2 = \frac{\dot{y}(0)\omega_o + (-\xi - \sqrt{\xi^2 - 1})}{2\omega_o^2 \sqrt{\xi^2 - 1}}; C_1 = (\frac{\xi - \sqrt{\xi^2 - 1} - \dot{y}(0)\omega_o}{2\omega_o^2 \sqrt{\xi^2 - 1}})$$

$$y(t) = \frac{1}{\omega_o^2} + \frac{-\dot{y}(0)\omega_o + \xi - \sqrt{\xi^2 - 1}}{2\omega_o^2 \sqrt{\xi^2 - 1}} e^{(-\xi-\sqrt{\xi^2-1})\omega_o t} + \frac{\dot{y}(0)\omega_o - \xi - \sqrt{\xi^2 - 1}}{2\omega_o^2 \sqrt{\xi^2 - 1}} e^{(-\xi+\sqrt{\xi^2-1})\omega_o t}$$

Proof of the Underdamped Case

- With C_1 and C_2 are complex conjugates, the proof is concluded:



$$y(t) = e^{-\alpha t}(C_1 e^{j\omega t} + C_2 e^{-j\omega t})$$

$$= e^{-\alpha t}[(C_1 + C_2)\cos\omega t + j(C_1 - C_2)\sin\omega t]$$

If $C_1 = a + jb = Ce^{j\theta}$, then

$$C_1 + C_2 = C_1 + C_1^* = 2a$$

$$C_1 - C_2 = C_1 - C_1^* = j2b$$

$$y(t) = e^{-\alpha t}[(2a)\cos\omega t + j(j2b)\sin\omega t] = e^{-\alpha t}[(2a)\cos\omega t - (2b)\sin\omega t]$$

Define $\theta = \tan^{-1}(\frac{2b}{2a}) = \tan^{-1}(\frac{|C_1 - C_2|}{C_1 + C_2})$

$$y(t) = e^{-\alpha t}[\sqrt{(2a)^2 + (2b)^2}(\cos\theta\cos\omega t - \sin\theta\sin\omega t)] = e^{-\alpha t}[\sqrt{(2a)^2 + (2b)^2}\cos(\omega t + \theta)]$$

$$= e^{-\alpha t}[2\sqrt{(a)^2 + (b)^2}\cos(\omega t + \theta)] = e^{-\alpha t}[2C\cos(\omega t + \theta)]$$

Another version of the Underdamped Case

- With C_1 and C_2 are complex conjugates:

$$y(t) = e^{-\alpha t}(C_1 e^{j\omega t} + C_2 e^{-j\omega t}) = e^{-\alpha t}(C_1 e^{j\omega t} + C_1^* e^{-j\omega t})$$

$$= e^{-\alpha t}(C_1 e^{j\omega t} + [C_1 e^{j\omega t}]^*)$$

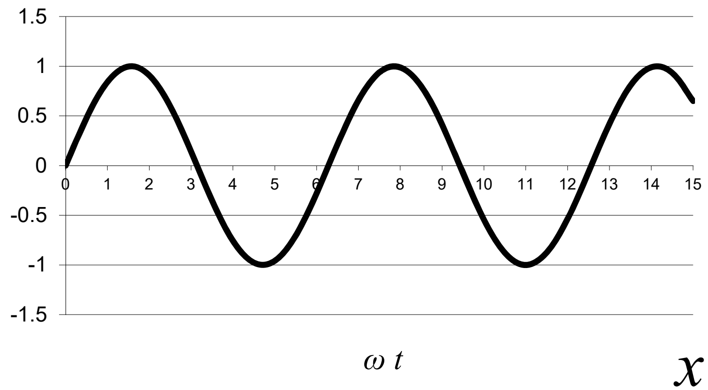
$$= e^{-\alpha t}(2\,\mathrm{Re}\{C_1 e^{j\omega t}\})$$

$$\text{Assume } \mathrm{C}_1 = a + jb = Ce^{j\theta} =;\text{where } C = \sqrt{a^2 + b^2};\ \theta = \tan^{-1}\frac{b}{a}$$

$$y(t) = e^{-\alpha t}(2\,\mathrm{Re}\{C_1 e^{j\omega t}\}) = e^{-\alpha t}(2\,\mathrm{Re}\{Ce^{j\theta}e^{j\omega t}\}) = e^{-\alpha t}(2\,\mathrm{Re}\{Ce^{j\omega t+\theta}\})$$

$$= e^{-\alpha t}\, 2C\cos(\omega t + \theta)$$

Continuous and Discrete Time Signals

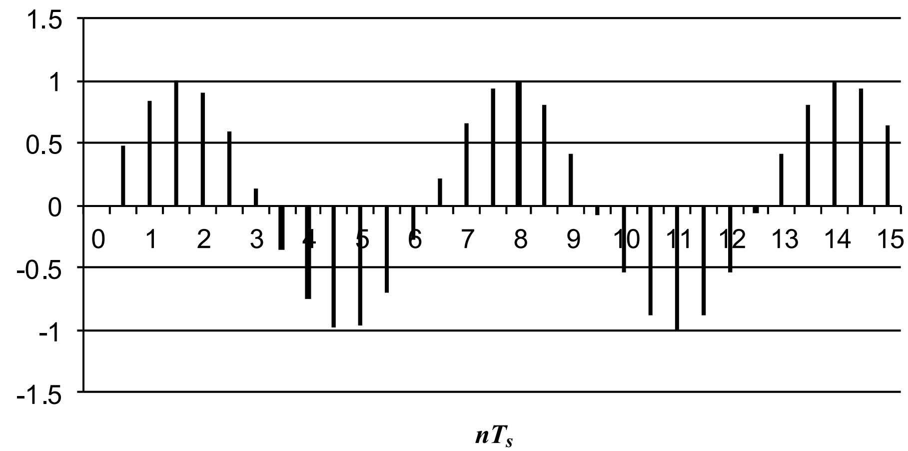


Continuous Signal

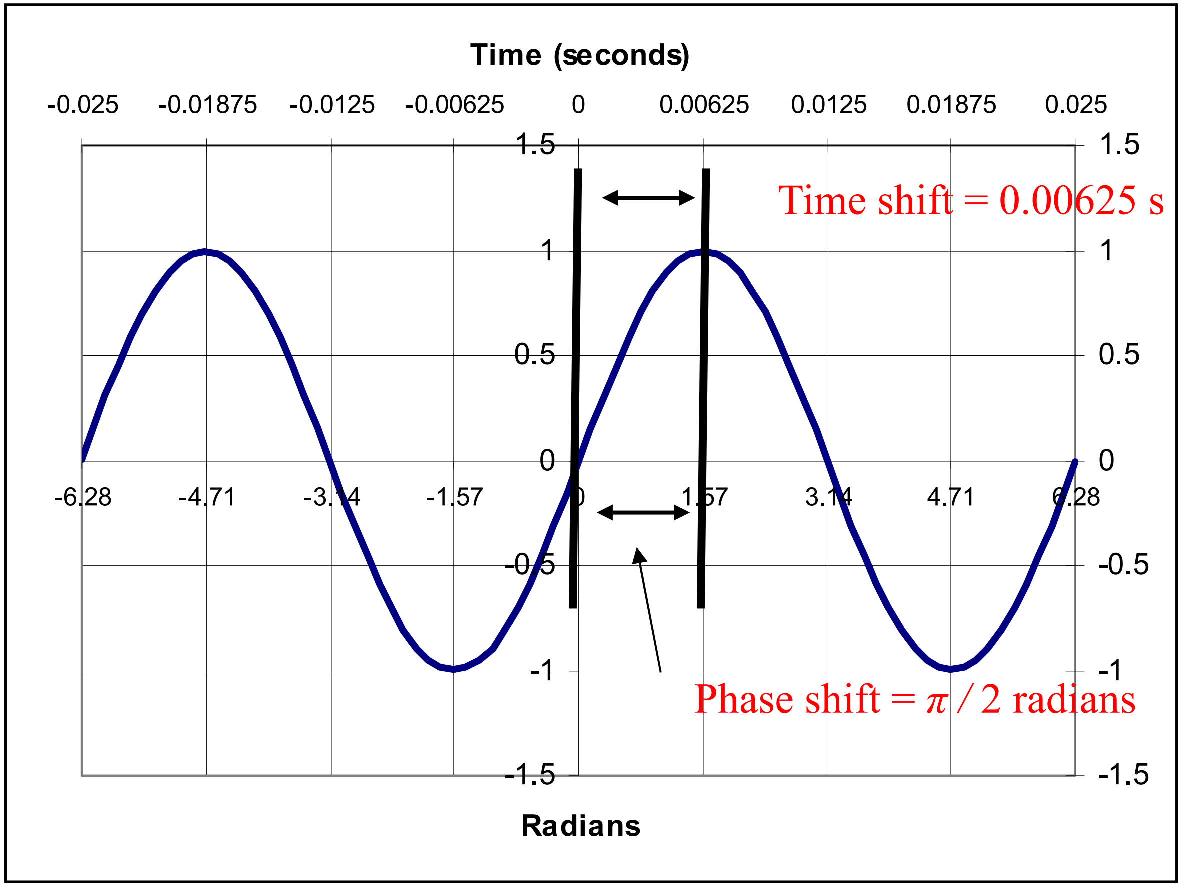
$$x(t) = \sin(\omega t)$$

Discrete Signal

$$x(nT) = x[n] = \sin(\omega n T_s) \text{ for } N_1 < n < N_2$$



Phase shift and Time Shift



$$x(t) = \cos(2\pi 40t - \frac{\pi}{2})$$

$$f = 40Hz;$$

$$T = \frac{1}{40} = 0.025 \text{ sec}$$

phase shift:

$$\theta = -\frac{\pi}{2}$$

time shift:

$$t_s = -\frac{-\frac{\pi}{2}}{2\pi 40} = \frac{1}{160} = 0.00625 \text{ sec}$$

$$x(t) = \cos(2\pi 40(t - 0.00625))$$