Z Transforms

Lesson 20

6DT

Z Transforms – A Definition

- In a sense similar to the LT except it is associated with discrete time functions.
- Let's assume we have a continuous time function, $f(t)$, and let's create a discrete time function from it, $f(n)$ by sampling it at a rate T. We would then have:

$$f[n] \equiv f(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

- Taking the LT of *f(n)* we would have:

$$F(s) = \pounds[f[n]] = \int_{0}^{\infty} f(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) e^{-st} dt = \sum_{n=-\infty}^{\infty} \int_{0}^{\infty} f(t) \delta(t - nT) e^{-st} dt$$

$$= \sum_{n=-\infty}^{\infty} f(nT) e^{-snT}$$

Z Transforms – A Definition Continued

- Replacing e^{sT} with z we have the definition of the z-transform:

$$F(z) = \sum_{n=-\infty}^{\infty} f(nT) z^{-n}$$

- Note that this is a 2-sided summation. Since many signals are zero for $t<0$, we can also define a 1-sided z-transform as:

$$F(z) = \sum_{n=0}^{\infty} f(nT) z^{-n}$$

- Note that both versions $F(z)$ can be written in closed form when the series converges

Z-transform 2-sided Calculations - Examples

First, recall that the geometric series $\sum_{n=0}^{\infty} Ar^n$ converges to $\frac{A}{1 - r}$ provided that $|r| < 1$.

Example 1), let:

$f[n] = e^{aTn}$ for $n > 0$; 0 otherwise

$$F(z) = \sum_{-\infty}^{\infty} e^{aTn} z^{-n} = \sum_{0}^{\infty} e^{aTn} z^{-n} = \sum_{0}^{\infty} (\frac{e^{aT}}{z})^n$$

Using the result of a geometric series:

$$F(z) = \frac{1}{1 - \frac{e^{aT}}{z}} = \frac{z}{z - e^{aT}}$$

and will converge when $\left|\frac{e^{aT}}{z}\right| < 1$ or $|\mathrm{z}| > |e^{aT}|$

Geometric Series.

$\sum_{n=0}^{\infty} Ar^n \rightarrow \frac{A}{1-r}$ provided that $|r| < 1$.

Start with the partial sum $\sum_{n=0}^{k} Ar^n$ that is from 0 to k.

$$s_k = \sum_{n=0}^{k} Ar^n = A(1 + r + r^2 + \cdots + r^k)$$
$$rs_k = A(r + r^2 + \cdots + r^{k+1})$$
$$s_k - rs_k = A(1 + r + r^2 + \cdots + r^k) - A(r + r^2 + \cdots + r^{k+1}) = A(1 - r^{k+1})$$
$$s_k(1-r) == A(1 - r^{k+1})$$

$s_k = \frac{A(1-r^{k+1})}{1-r}$ This is called the partial sum of a geometric series. However, to converge as $k \rightarrow \infty$ the value of this partial sum can not approach infinity.

Therefore, from the denominator $r \neq 1$ and from the numerator r can not be greater than 1.

$\therefore \sum_{n=0}^{\infty} Ar^n \rightarrow \frac{A}{1-r}$ provided that $|r| < 1$

Examples Continued

Example 2), let:

$f[n] = -e^{aTn}$ for $n < 0$; 0 otherwise

$F(z) = -\sum_{-\infty}^{\infty} e^{aTn} z^{-n} = -\sum_{-\infty}^{-1} e^{aTn} z^{-n}$ let m = -n we have:

$= -\sum_{\infty}^{1} e^{-aTm} z^{m} = -\sum_{1}^{\infty} (e^{-aT} z)^{m} = -\sum_{0}^{\infty} (e^{-aT} z)^{p+1}$ where $\mathrm{m} = \mathrm{p} + 1$

$= -e^{-aT} z \sum_{0}^{\infty} (e^{-aT} z)^{p}$

Using the result of a geometric series:

$F(z) = -e^{-aT} z(\frac{1}{1 - e^{-aT} z}) = \frac{z}{z - e^{aT}}$

and will converge when $\left|e^{-aT} z\right| < 1$ or $\left|\mathrm{z}\right| < \left|e^{aT}\right|$

We see that we have to provide a region of convergence, in addition, to the function $F(z)$ to completely define the z-transform of a $f[n]$.

1-sided z-transforms

- Useful since most functions we use are zero for $t<0$.

$$F(z) = \sum_{n=0}^{\infty} f(nT) z^{-n}$$

1-sided z-transforms

- Some examples (see p. 396 Tables 6DT.1 & 6DT.2 for more):

(Unit sampling function) $\delta[n] = 1$ for $n = 0$, 0 otherwise; $F(z) = \sum_{0}^{0} 1 \times z^{-0} = 1$

$u[n] = 1$ for $n \geq 0$, 0 otherwise

$$F(z) = \sum_{0}^{\infty} 1 \times z^{-n} = \sum_{0}^{\infty} (\frac{1}{z})^n = \frac{1}{1 - \frac{1}{z}} = \frac{z}{z-1} \text{ converges for } |\mathrm{z}| > 1$$

$$nTu[n] \Rightarrow \sum_{0}^{\infty} nTz^{-n} = T\sum_{0}^{\infty} n(\frac{1}{z})^n = \frac{Tz^{-1}}{(1 - z^{-1})^2} = \frac{Tz}{(z-1)^2} \text{ converges for } |\mathrm{z}| > 1$$

$$n^2u[n] \Rightarrow \frac{T^2 z(z+1)}{(z-1)^3} \text{ converges for } |\mathrm{z}| > 1$$

Proof

$$nu[n] \Rightarrow \sum_{0}^{\infty} nTz^{-n} = T\sum_{0}^{\infty} n(\frac{1}{z})^{n}$$

$$= T\{0(\frac{1}{z})^{0} + 1(\frac{1}{z})^{1} + 2(\frac{1}{z})^{2} + 3(\frac{1}{z})^{3} + 4(\frac{1}{z})^{4} + \cdots\}$$ Step 1. Expand the sum

Proof

$$nu[n] \Rightarrow \sum_{0}^{\infty} nTz^{-n} = T\sum_{0}^{\infty} n(\frac{1}{z})^{n}$$

$$= T\{0(\frac{1}{z})^{0} + 1(\frac{1}{z})^{1} + 2(\frac{1}{z})^{2} + 3(\frac{1}{z})^{3} + 4(\frac{1}{z})^{4} + \cdots\}$$ Step 1. Expand the sum

$$= T\{0 + \frac{1}{z}[1 + 2(\frac{1}{z})^{1} + 3(\frac{1}{z})^{2} + 4(\frac{1}{z})^{3} + \cdots]\}$$ Step 2. Factor out $\frac{1}{z}$ from each term.

Note that coefficents are not unity so that $\frac{1}{1-\frac{1}{z}}$ can be not be formed.

Proof

$$nu[n] \Rightarrow \sum_{0}^{\infty} nTz^{-n} = T\sum_{0}^{\infty} n(\frac{1}{z})^{n}$$

$= T\{0(\frac{1}{z})^0 + 1(\frac{1}{z})^1 + 2(\frac{1}{z})^2 + 3(\frac{1}{z})^3 + 4(\frac{1}{z})^4 + \cdots\}$ Step 1. Expand the sum

$= T\{0 + \frac{1}{z}[1 + 2(\frac{1}{z})^1 + 3(\frac{1}{z})^2 + 4(\frac{1}{z})^3 + \cdots]\}$ Step 2. Factor out $\frac{1}{z}$ from each term.

Note that coefficents are not unity so that $\frac{1}{1-\frac{1}{z}}$ can be not be formed.

$= T\{0 + \frac{1}{z}[1 + (\frac{1}{z})^1 + (\frac{1}{z})^2 + (\frac{1}{z})^3 + \cdots + (\frac{1}{z})^1 + 2(\frac{1}{z})^2 + 3(\frac{1}{z})^3 + \cdots]\}$ Step 3. Separate the sum into 2 sums: one is $\sum_{0}^{\infty} z^{-n}$ and the other is the remainder with coefficients not equal to unity.

$= T\{0 + \frac{1}{z}[1 + (\frac{1}{z})^1 + (\frac{1}{z})^2 + (\frac{1}{z})^3 + \cdots + \frac{1}{z}[1 + 2(\frac{1}{z})^1 + 3(\frac{1}{z})^2 + \cdots]]\}$ Step 4. Take the second sum and factor out $\frac{1}{z}$ from each term.

Proof

$$nu[n] \Rightarrow \sum_0^\infty nTz^{-n} = T\sum_0^\infty n(\frac{1}{z})^n$$

$= T\{0(\frac{1}{z})^0 + 1(\frac{1}{z})^1 + 2(\frac{1}{z})^2 + 3(\frac{1}{z})^3 + 4(\frac{1}{z})^4 + \cdots\}$ Step 1. Expand the sum

$= T\{0 + \frac{1}{z}[1 + 2(\frac{1}{z})^1 + 3(\frac{1}{z})^2 + 4(\frac{1}{z})^3 + \cdots]\}$ Step 2. Factor out $\frac{1}{z}$ from each term.

Note that coefficents are not unity so that $\frac{1}{1-\frac{1}{z}}$ can be not be formed.

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$= T\{0 + \frac{1}{z}[1 + (\frac{1}{z})^1 + (\frac{1}{z})^2 + (\frac{1}{z})^3 + \cdots + \frac{1}{z}[1 + 2(\frac{1}{z})^1 + 3(\frac{1}{z})^2 + \cdots]]\}$ Step 4. Take the second sum and factor out $\frac{1}{z}$ from each term.

$= T\{0 + \frac{1}{z}[1 + (\frac{1}{z})^1 + (\frac{1}{z})^2 + (\frac{1}{z})^3 + \cdots + \frac{1}{z}[1 + (\frac{1}{z})^1 + (\frac{1}{z})^2 + \cdots + \frac{1}{z}[1 + (\frac{1}{z})^1 + \cdots + (\frac{1}{z})^1 + \cdots]]]\}$ Step 5. Repeat Steps 3-4 for the entire sum. This can only be done in theory. Note we now have a infinite and nested set of sums of the form $\frac{1}{z}\sum_0^\infty z^{-n} \Rightarrow \frac{1}{z}[\frac{1}{1-\frac{1}{z}}]$.

Proof Continued

$= T\{\frac{1}{z}[\frac{1}{1-\frac{1}{z}} + \frac{1}{z}[\frac{1}{1-\frac{1}{z}} + \frac{1}{z}[\frac{1}{1-\frac{1}{z}} + \frac{1}{z}[\frac{1}{1-\frac{1}{z}} + \cdots]]]]\}$ Step 6. This is the resultant sum with $\frac{1}{z}\sum_{0}^{\infty} z^{-n} \Rightarrow \frac{1}{z}[\frac{1}{1-\frac{1}{z}}]$

Proof Continued

$= T\{\frac{1}{z}[\frac{1}{1-\frac{1}{z}} + \frac{1}{z}[\frac{1}{1-\frac{1}{z}} + \frac{1}{z}[\frac{1}{1-\frac{1}{z}} + \frac{1}{z}[\frac{1}{1-\frac{1}{z}} + \cdots]]]]\}$ Step 6. This is the resultant sum with $\frac{1}{z}\sum_{0}^{\infty} z^{-n} \Rightarrow \frac{1}{z}[\frac{1}{1-\frac{1}{z}}]$

$= T\{\frac{1}{z}(\frac{1}{1-\frac{1}{z}})[1+\frac{1}{z}[1+\frac{1}{z}[1+\frac{1}{z}[1+\cdots]]]]\}$ Step 7. Factor out the terms $\frac{1}{z}(\frac{1}{1-\frac{1}{z}})$ from the infinite nested sums.

What remains is an infinite nested sum of terms $[1+\frac{1}{z}[1+\frac{1}{z}[1+\frac{1}{z}[1+\cdots]]]]$

Proof Continued

$= T\{\frac{1}{z}[\frac{1}{1-\frac{1}{z}} + \frac{1}{z}[\frac{1}{1-\frac{1}{z}} + \frac{1}{z}[\frac{1}{1-\frac{1}{z}} + \frac{1}{z}[\frac{1}{1-\frac{1}{z}} + \cdots]]]]\}$ Step 6. This is the resultant sum with $\frac{1}{z}\sum_{0}^{\infty} z^{-n} \Rightarrow \frac{1}{z}[\frac{1}{1-\frac{1}{z}}]$

$= T\{\frac{1}{z}(\frac{1}{1-\frac{1}{z}})[1 + \frac{1}{z}[1 + \frac{1}{z}[1 + \frac{1}{z}[1 + \cdots]]]]\}$ Step 7. Factor out the terms $\frac{1}{z}(\frac{1}{1-\frac{1}{z}})$ from the infinite nested sums.

What remains is an infinite nested sum of terms $[1 + \frac{1}{z}[1 + \frac{1}{z}[1 + \frac{1}{z}[1 + \cdots]]]]$

$= T\{\frac{1}{z}(\frac{1}{1-\frac{1}{z}})[1 + \frac{1}{z} + (\frac{1}{z})^2 + (\frac{1}{z})^2 + \cdots]\}$ Step 8. Multiply these term out to revel a sum of the form $\sum_{0}^{\infty} z^{-n}$

which can be replaced by $[\frac{1}{1-\frac{1}{z}}]$.

$= T\{\frac{1}{z}(\frac{1}{1-\frac{1}{z}})(\frac{1}{1-\frac{1}{z}})\} = T\{z^{-1}(\frac{1}{1-z^{-1}})(\frac{1}{1-z^{-1}})\}$ Step 9. Multiply top and both of each term by z.

Proof Continued

$= T\{\frac{1}{z}[\frac{1}{1-\frac{1}{z}} + \frac{1}{z}[\frac{1}{1-\frac{1}{z}} + \frac{1}{z}[\frac{1}{1-\frac{1}{z}} + \frac{1}{z}[\frac{1}{1-\frac{1}{z}} + \cdots]]]]\}$ Step 6. This is the resultant sum with $\frac{1}{z}\sum_{0}^{\infty} z^{-n} \Rightarrow \frac{1}{z}[\frac{1}{1-\frac{1}{z}}]$

$= T\{\frac{1}{z}(\frac{1}{1-\frac{1}{z}})[1+\frac{1}{z}[1+\frac{1}{z}[1+\frac{1}{z}[1+\cdots]]]]\}$ Step 7. Factor out the terms $\frac{1}{z}(\frac{1}{1-\frac{1}{z}})$ from the infinite nested sums.

What remains is an infinite nested sum of terms $[1+\frac{1}{z}[1+\frac{1}{z}[1+\frac{1}{z}[1+\cdots]]]]$

$= T\{\frac{1}{z}(\frac{1}{1-\frac{1}{z}})[1+\frac{1}{z}+(\frac{1}{z})^2+(\frac{1}{z})^2+\cdots]\}$ Step 8. Multiply these term out to revel a sum of the form $\sum_{0}^{\infty} z^{-n}$

which can be replaced by $[\frac{1}{1-\frac{1}{z}}]$.

$= T\{\frac{1}{z}(\frac{1}{1-\frac{1}{z}})(\frac{1}{1-\frac{1}{z}})\} = T\{z^{-1}(\frac{1}{1-z^{-1}})(\frac{1}{1-z^{-1}})\}$ Step 9. Multiply top and both of each term by z.

$= T\{\frac{1}{z}(\frac{z}{z-1})(\frac{z}{z-1})\} = \frac{Tz}{(z-1)^2}$ converges for $|z| > 1$

1-sided z-transforms

- Some examples (see p. 396 Tables 6DT.1 & 6DT.2 for more):

$$e^{aTn}u[nT] \Rightarrow \frac{z}{z-e^{aT}} \text{ converges for } |\mathrm{z}| > e^{aT}$$

$$a^{Tn}u[nT] = \sum_{n=0}^{\infty} a^{nT} z^{-n} = \sum_{n=0}^{\infty} (\frac{a^T}{z})^n = \frac{1}{1-\frac{a^T}{z}}$$

$$= \frac{z}{z-a^T} \text{ converges for } |\mathrm{z}| > a^T$$

Properties of z-transforms

$$Z\{f_1[nT] + f_2[nT]\} = F_1(z) + F_2(z)$$

$$Z\{af[nT]\} = aF(z)$$

$$Z\{f[nT - T]\} = z^{-1}F(z)$$

$$Z\{f[nT - mT]\} = z^{-m}F(z)$$

$$Z\{nTf[nT]\} = -Tz\frac{dF(z)}{dz}$$

If $f[nT] = \sum_{k=0}^{n} f_1[nT - kT]f_2[kT]$, then $F(z) = F_1(z)F_2(z)$

Proof of Delay Property

Given:

$$Z\{f[nT-T]\} = z^{-1}F(z)$$

$$Z\{f[nT]\} = F(z)$$

$$Z\{f[nT-T]\} = \sum_{n=0}^{\infty} f(nT-T)z^{-n} = \sum_{n=0}^{\infty} f([n-1]T)z^{-n}$$

Let $n - 1 = m$

$$\sum_{n=0}^{\infty} f([n-1]T)z^{-n} = \sum_{m=-1}^{\infty} f(mT)z^{-m-1} = \sum_{m=0}^{\infty} f(mT)z^{-m}z^{-1}$$

Since $f(-T) = 0$

$$= z^{-1}\sum_{m=0}^{\infty} f(mT)z^{-m} = z^{-1}F(z)$$

Proof of (nT)f[nT] Property

$$Z\{nTf[nT]\} = \sum_{n=0}^{\infty} nTf[nT]z^{-n} = -Tz\frac{dF(z)}{dz}$$

$$F(z) = \sum_{n=0}^{\infty} f[nT]z^{-n}$$

$$\frac{dF(z)}{dz} = \frac{d\sum_{n=0}^{\infty} f[nT]z^{-n}}{dz} = \sum_{n=0}^{\infty} \text{-}nf[nT]z^{-n-1} = \sum_{n=0}^{\infty} \text{-}nz^{-1}f[nT]z^{-n}$$

$$\therefore -Tz\frac{dF(z)}{dz} = -Tz\sum_{n=0}^{\infty} \text{-}nz^{-1}f[nT]z^{-n} = \sum_{n=0}^{\infty} Tznz^{-1}f[nT]z^{-n}$$

$$= \sum_{n=0}^{\infty} nTf[nT]z^{-n} = Z\{nTf[nT]\}$$

Another Proof of (nT)f[nT] Property

Proof of $Z\{nTf[nT]\} = -Tz\frac{dF(z)}{dz}$

$$Z\{nTf[nT]\} = \sum_{n=0}^{\infty} nTf(nT)z^{-n} = T\sum_{n=0}^{\infty}\frac{nf(nT)}{z^n}$$

$$= T(0f(nT)z^{-0} + 1f(nT)z^{-1} + 2f(nT)z^{-2} + \cdots) = T(1f(nT)z^{-1} + 2f(nT)z^{-2} + \cdots)$$

$$Z\{f[nT]\} = \sum_{n=0}^{\infty} f(nT)z^{-n} = F(z);$$

$$-T\frac{dF(z)}{dz} = -T\frac{d}{dz}\sum_{n=0}^{\infty} f(nT)z^{-n}$$

$$= -Tz(\frac{d}{dz}f(nT)z^{-0} + \frac{d}{dz}f(nT)z^{-1} + \frac{d}{dz}f(nT)z^{-2}\cdots)$$

$$= -Tz(0 + -1f(nT)z^{-2} + -2f(nT)z^{-3} + \cdots) = T(1f(nT)z^{-1} + 2f(nT)z^{-2} + \cdots)$$

Properties of z-transforms

If $Z\{nTf[nT]\} = -Tz\frac{dF(z)}{dz}$; then

$$Z\{(nT)^2 f[nT]\} = Z\{nTg[nT]\} = -Tz\frac{dG(z)}{dz}$$

where $g[nT] = nTf[nT]$ but $G(z) = -Tz\frac{dF(z)}{dz}$;

$$\frac{dG(z)}{dz} = -T\{\frac{dF(z)}{dz} + z\frac{d^2F(z)}{dz^2}\}$$

And

$$Z\{(nT)^2 f[nT]\} = -Tz\frac{dG(z)}{dz} = -Tz(-T\{\frac{dF(z)}{dz} + z\frac{d^2F(z)}{dz^2}\})$$

$$= T^2\{z\frac{dF(z)}{dz} + z^2\frac{d^2F(z)}{dz^2}\}$$

$Z\{(nT)^M f[nT]\} = -Tz\frac{dG(z)}{dz}$; where $G(z) = Z\{(nT)^{M-1} f[nT]\}$

Inverting the z-transform

- Several Methods:
  - Taylor series expansion:

Since $F(z) = f(0) + f(T)z^{-1} + f(2T)z^{-2} + \cdots + f(nT)z^{-n} + \cdots$

We can define a new function $\phi(y)$ where $y = z^{-1}$

$\varphi(y) = f(0) + f(T)y^{1} + f(2T)y^{2} + \cdots + f(nT)y^{n} + \cdots$

and note that this is just a Taylor series expansion and therefore, calculate the coefficients of this series [which are the values of $f(n)$] by $f(nT) = \frac{1}{n!}\frac{d^{n}\phi}{dy^{n}}|_{y=0}$

Long division of the closed form of $F(z)$ should also yield the same results.

Inverting the z-transform Continued

– Residue Method which relates the zT, $F(z)$, *to* the LT, $F(s)$:

It can be shown that

$$F(z) = \sum \text{residues of } \frac{F(s)}{1\text{-}e^{sT} z^{-1}} \text{ at the poles of } F(s)$$

The residues are calculated for a pole (s = a) of order n as:

$$= \frac{1}{(n-1)!} \frac{d^{n-1}}{ds^{n-1}} [(s-a)^n \frac{F(s)}{1\text{-}e^{sT} z^{-1}}] |_{s=a}$$

Inverting the z-transform Continued

Partial Fraction Expansion of $\frac{F(z)}{z}$ (since most discrete functions we deal with are of the form $f[n]u[n]$ and, therefore, due to $u[n]$, $F(z)$ will have a z in its numerator;

e.g., $Z\{f[n]u[n]\} = F(z) = \frac{z \times \prod_i (z - z_i)^{p_i}}{\prod_k (z - z_k)^{p_k}}$ and will have terms of the form $= \frac{zA_k}{(z - z_k)^{p_k}}$ which can be inverted.)

$$F(z) = \frac{2z}{(z-2)(z-1)^2}$$

$$\frac{F(z)}{z} = \frac{2}{(z-2)(z-1)^2} = \frac{K_1}{z-2} + \frac{M_0}{(z-1)^2} + \frac{M_1}{z-1}$$

$$K_1 = \frac{2}{(z-1)^2}\Big|_{z=2} = \frac{2}{(2-1)^2} = 2$$

$$M_0 = \frac{2}{(z-2)}\Big|_{z=1} = \frac{2}{(1-2)} = -2$$

$$M_1 = \frac{d}{dz}\left[\frac{2}{(z-2)}\right]\Big|_{z=1} = -\frac{2}{(z-2)^2}\Big|_{z=1} = -\frac{2}{(1-2)^2} = -2$$

Inverting the z-transform Continued

$$\frac{F(z)}{z} = \frac{2}{z-2} + \frac{-2}{(z-1)^2} + \frac{-2}{z-1}$$

$$F(z) = \frac{2z}{z-2} - \frac{2z}{(z-1)^2} - \frac{2z}{z-1}$$

See slide 78

$$a^{Tn}u[nT] = \sum_{n=0}^{\infty} a^{nT} z^{-n} = \sum_{n=0}^{\infty} (\frac{a^T}{z})^n = \frac{1}{1 - \frac{a^T}{z}} = \frac{z}{z - a^T} \text{ converges for } |z| > a^T$$

See slide 69

$$nTu[n] \Rightarrow \sum_{0}^{\infty} nTz^{-n} = T\sum_{0}^{\infty} n(\frac{1}{z})^n = \frac{Tz^{-1}}{(1-z^{-1})^2} = \frac{Tz}{(z-1)^2} \text{ converges for } |z| > 1$$

$$f(n) = 2[2^n - n - 1]\ u[n] \text{ for } T = 1$$

Another Inversion Example

$$F(z) = \frac{z(z^2 - 2z - 1)}{(z^2 + 1)^2} = \frac{z(z^2 - 2z - 1)}{(z - j)^2 (z + j)^2}$$

$$\frac{F(z)}{z} = \frac{(z^2 - 2z - 1)}{(z - j)^2 (z + j)^2}$$

$$= \frac{A_0}{(z - j)^2} + \frac{A_1}{z - j} + \frac{A_0 *}{(z + j)^2} + \frac{A_1 *}{z + j}$$

$$A_0 = \frac{(z^2 - 2z - 1)}{(z + j)^2}\Big|_{z=j} = \frac{(j^2 - 2j - 1)}{(j + j)^2}$$

$$= \frac{(-1 - 2j - 1)}{(2j)^2} = \frac{2(-1 - j)}{-4}$$

$$= \frac{1}{2}(1 + j)$$

$$A_0 * = \frac{1}{2}(1 - j)$$

$$A_1 = \frac{d}{dz}\left[\frac{(z^2 - 2z - 1)}{(z + j)^2}\right]\Big|_{z=j}$$

$$= \left[\frac{(2z - 2)(z + j)}{(z + j)^3} - \frac{2(z^2 - 2z - 1)}{(z + j)^3}\right]\Big|_{z=j}$$

$$= \left[\frac{(2j - 2)(j + j)}{(j + j)^3} - \frac{2(j^2 - 2j - 1)}{(j + j)^3}\right]$$

$$= \left[\frac{(4(-1) - 4j) - 2(-1) + 4j + 2)}{(j + j)^3}\right] = 0$$

Likewise for $A_1 *$

$$A_1 * = 0$$

Another Inversion Example Continued

$$F(z) = \frac{1+j}{2}\frac{z}{(z-j)^2} + \frac{1-j}{2}\frac{z}{(z+j)^2} = \frac{1}{2}\sqrt{2}[e^{j\pi/4}\frac{z}{(z-e^{j\pi/2})^2} + e^{-j\pi/4}\frac{z}{(z-e^{-j\pi/2})^2}]$$

This looks like the term $\frac{z}{(z-e^{j\pi/2})^2}$ is related to $\frac{z}{(z-e^{j\pi/2})}$

$$Z\{e^{j\frac{\pi}{2}n}u[n]\} = \frac{z}{(z-e^{j\pi/2})};\ \text{Recall } Z\{nf[n]\} = -z\frac{dF(z)}{dz}; T=1$$

$$\frac{dF(z)}{dz} = \frac{d\frac{z}{(z-e^{j\pi/2})}}{dz} = \frac{1}{(z-e^{j\pi/2})} - \frac{z}{(z-e^{j\pi/2})^2} = \frac{z-e^{j\pi/2}-z}{(z-e^{j\pi/2})^2} = \frac{-e^{j\pi/2}}{(z-e^{j\pi/2})^2}$$

$$-z\frac{dF(z)}{dz} = -z\frac{d\frac{z}{(z-e^{j\pi/2})}}{dz} = \frac{ze^{j\pi/2}}{(z-e^{j\pi/2})^2} = Z\{ne^{j\frac{\pi}{2}n}u[n]\}$$

Likewise

$$Z\{ne^{-j\frac{\pi}{2}n}u[n]\} = \frac{ze^{-j\pi/2}}{(z-e^{-j\pi/2})^2}$$

Another Inversion Example Continued

Continuing

$$F(z)=\frac{1}{2}\sqrt{2}[e^{j\pi/4}e^{-j\pi/2}\,\frac{ze^{j\pi/2}}{(z-e^{j\pi/2})^2}+e^{-j\pi/4}e^{j\pi/2}\,\frac{ze^{-j\pi/2}}{(z-e^{-j\pi/2})^2}]$$

$$f[n]=\frac{1}{2}\sqrt{2}[e^{-j\pi/4}ne^{j\pi n/2}+e^{j\pi/4}ne^{-j\pi n/2}]u[n]$$

$$=\frac{1}{2}\sqrt{2}[ne^{j(\pi n/2-\pi/4)}+ne^{-j(\pi n/2-\pi/4)}]u[n]$$

$$=\sqrt{2}[n\{\frac{e^{j(\pi n/2-\pi/4)}+e^{-j(\pi n/2-\pi/4)}}{2}\}]u[n]$$

$$=\sqrt{2}[n\cos(\pi n/2-\pi/4)]u[n]$$

Difference Equations

- It can be shown that for a function which equals zero for $t<0$, any initial conditions will not affect the following formulation:

If we have a system given by this equation:

$$a_m y[n+mT] + a_{m-1} y[n+(m-1)T] + \cdots + a_0 y[n] = b_p x[n+pT] + b_{p-1} x[n+(p-1)T] + \cdots + b_0 x[n]$$

Then applying the z-transform to this system will yield:

$$(a_m z^m + a_{m-1} z^{m-1} + \cdots + a_0) Y(z) = (b_p z^p + b_{p-1} z^{p-1} + \cdots + b_0) X(z)$$

Or

$$\frac{Y(z)}{X(z)} = \frac{b_p z^p + b_{p-1} z^{p-1} + \cdots + b_0}{a_m z^m + a_{m-1} z^{m-1} + \cdots + a_0}$$

And we have the system function (which is the response due to the unit sampling function, δ(n)), *H(z)* as:

$$H(z) = \frac{Y(z)}{X(z)}$$

Example

Assume we have

$$y[k+2]-3y[k+1]+2y[k]=2v[k+1]-2v[k]$$

where $v[k]=k$ for $k \ge 0$, 0 otherwise. Then,

$$V(z)=\frac{z}{(z-1)^2}$$

$$(z^2-3z+2)Y(z)=2(z-1)V(z)$$

$$\frac{Y(z)}{V(z)}=H(z)=\frac{2(z-1)}{(z^2-3z+2)}=\frac{2(z-1)}{(z-1)(z-2)}=\frac{2}{(z-2)}$$

And

$$Y(z)=\frac{2}{(z-2)}V(z)=\frac{2}{(z-2)}\frac{z}{(z-1)^2}=\frac{2z}{(z-2)(z-1)^2}$$

From a preceding example, we have

$$f[n]=2[2^n-n-1]u[n]$$

Example

But $H(z) = \dfrac{2}{(z-2)}$ which is not in any table,

but if: $h[n] = Z^{-1}\{H(z)\}$

Then, $h[n+1] = Z^{-1}\{zH(z)\}$

$= Z^{-1}\{\dfrac{2z}{(z-2)}\} = 2 \times 2^n$

$h[n+1] = 2^{n+1}$ for $n \geq 0$

$n+1 \Rightarrow m$

Therefore, $h[m] = 2^m$ for $m \geq 1$

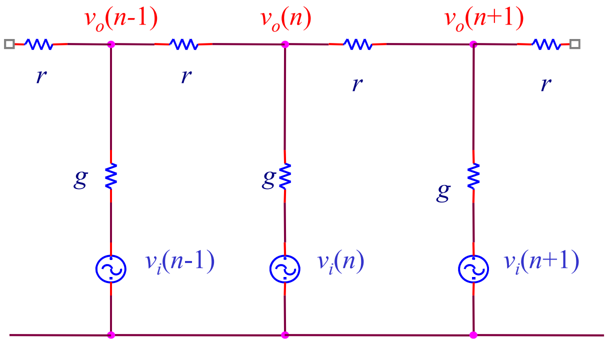
OR

$H(z) = \dfrac{2}{(z-2)} = z^{-1}\dfrac{2z}{(z-2)}$; but $Z^{-1}\{\dfrac{2z}{(z-2)}\} = 2 \times 2^n = 2^{n+1}u[n]$

Alternatively, using $Z\{f[n-1]\} = z^{-1}F(z)$

And $Z^{-1}\{\dfrac{2}{(z-2)}\} = Z^{-1}\{z^{-1}\dfrac{2z}{(z-2)}\} = Z^{-1}\{z^{-1}F(z)\} = f[n-1] = 2^n u[n-1]$

Another Example



Let's calculate the nodal voltage of this iterative network:

By Kirchoff's Current Law at node n, we have:

$$\frac{v_o[n-1]-v_0[n]}{r} = \frac{v_o[n]-v_0[n+1]}{r} + \frac{v_o[n]-v_i[n]}{1/g}$$

$rgv_i[n] = -v_0[n+1] + (2+rg)v_o[n] - v_o[n-1]$ and let $\alpha = \text{rg}$

$(\text{-}z + (2+\alpha) - z^{-1})V_o(z) = \alpha V_1(z)$

$$H(z) = \frac{-\alpha z}{z^2-(2+\alpha)z+1};$$

$$p_{1,2} = (1+\frac{\alpha}{2}) \pm \sqrt{(1+\frac{\alpha}{2})^2 - 1}$$

since $\text{c} = 1$, then $p_1 \times p_2 = 1$ or $p_1 = p$ and $p_2 = 1/p$

Therefore,

$$H(z) = \frac{-\alpha z}{z^2-(2+\alpha)z+1} = \frac{-\alpha z}{(z-p)(z-1/p)}$$

$$\frac{H(z)}{z} = \frac{-\alpha}{(z-p)(z-1/p)} = \frac{K_1}{(z-p)} + \frac{K_2}{(z-1/p)}$$

$$K_1 = \frac{-\alpha}{z-1/p}\Big|_{z=p} = \frac{-\alpha}{p-1/p} = \frac{\alpha p}{1-p^2}$$

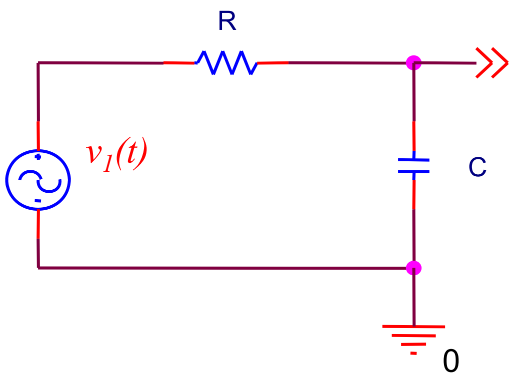
$$K_2 = \frac{-\alpha}{z-p}\Big|_{z=1/p} = \frac{-\alpha}{1/p-p} = \frac{-\alpha p}{1-p^2}$$

$$H(z) = \frac{\alpha p}{1-p^2}\left(\frac{z}{z-p} - \frac{z}{z-1/p}\right) \text{ where } p < |z| < 1/p$$

$$h[n] = \frac{\alpha p}{1-p^2}[p^n - p^{-n}]u[n]$$

By the way this looks like our first exercise when $f(t) = e^{at}$ for $t > 0$; 0 otherwise - (the p pole part) and $f(t) = -e^{at}$ for $t < 0$; 0 otherwise - the $1/p$ pole part. So the left hand side of $h[n]$ can be associated with a $n > 0$ 1-sided ZT and the right hand side of $h[n]$ with a $n < 0$ 1-sided ZT.

One More Example

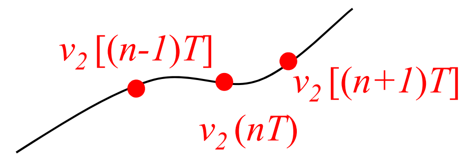


How do we solve this on a digit computer?

$$\frac{dv_2(t)}{dt} = \frac{1}{RC}[v_1(t) - v_2(t)]$$

We use an iterative techique but first we must approximate the derivative.

First we define a discrete version of $v_2(t) = v_2(nT)$ and $v_1(t) = v_1(nT)$



$$\frac{dv_2(t)}{dt} \approx \frac{v_2[(n+1)T] - v_2[nT]}{T}$$ known as the forward Euler Algorithm

$$\frac{v_2[(n+1)T] - v_2[nT]}{T} = -\frac{1}{RC}v_2[nT] + \frac{1}{RC}v_1[nT]$$

$$v_2[(n+1)T] - (1 - \frac{T}{RC})v_2[nT] = \frac{T}{RC}v_1[nT] = \frac{T}{RC}u(n)$$

$$[z - (1-\alpha)]V_2(z) = \alpha V_1(z) = \frac{\alpha z}{z-1} \text{ where } \alpha = \mathrm{T/RC}$$

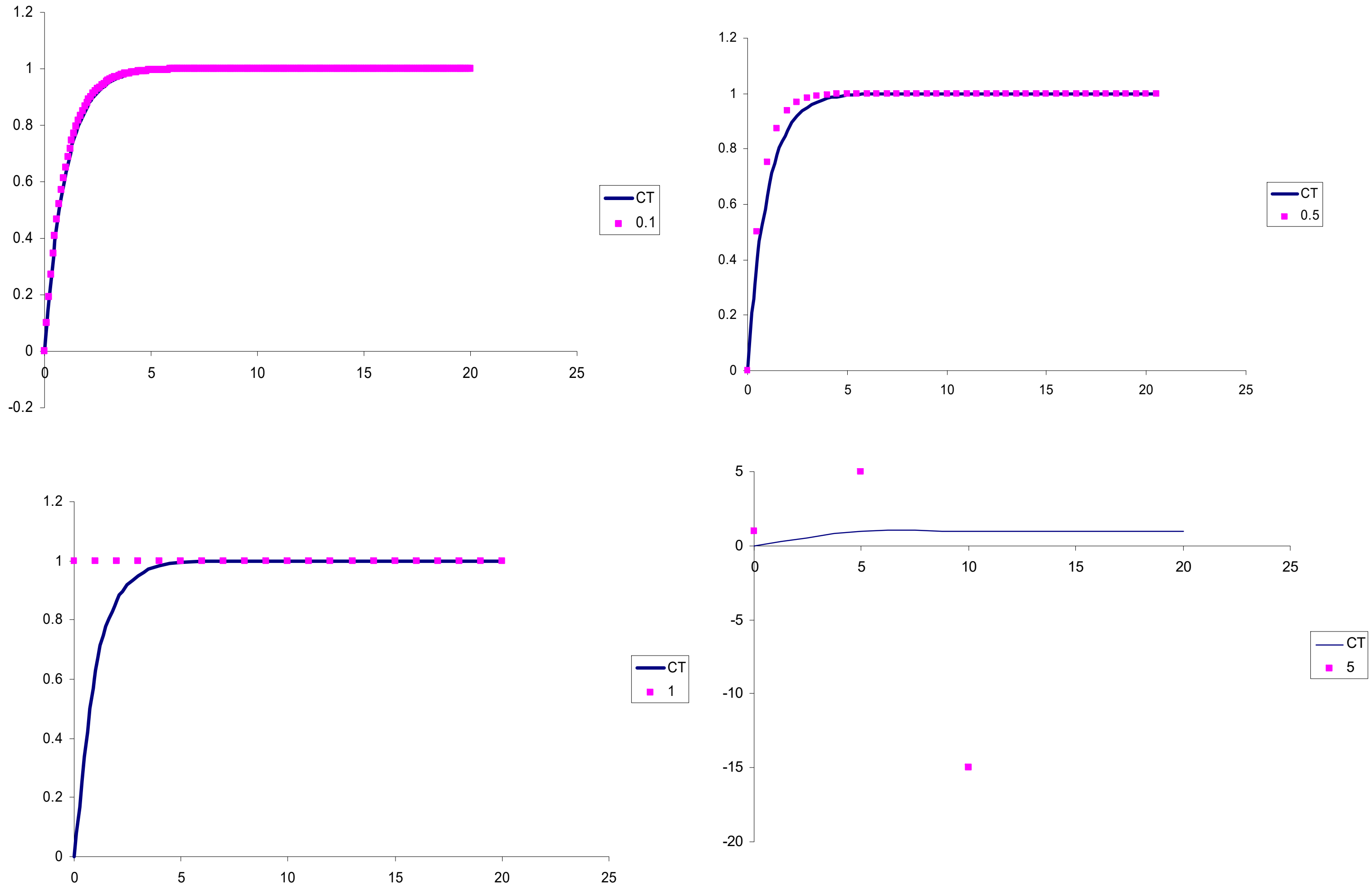
$$\frac{V_2(z)}{z} = \frac{\alpha}{(z-1)[z-(1-\alpha)]} = \frac{K_1}{z-1} + \frac{K_2}{z-(1-\alpha)}$$

$$K_1 = \frac{\alpha}{[1-(1-\alpha)]} = 1, K_2 = \frac{\alpha}{[(1-\alpha)-1]} = -1$$

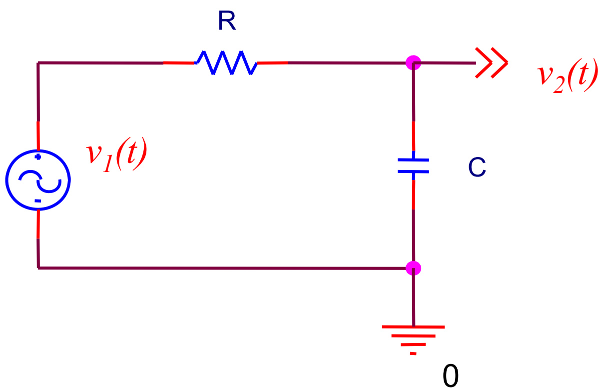
$$V_2(z) = \frac{z}{z-1} - \frac{z}{z-(1-\alpha)}$$

$$v_2[n] = [1-(1-\alpha)^n]u[n]$$

Forward Euler Algorithm Continued Estimate of the output voltage vs sampling time



Continued

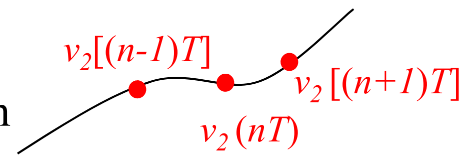


How do we solve this on a digit computer?

$$\frac{dv_2(t)}{dt} = \frac{1}{RC}[v_1(t) - v_2(t)]$$

We use an iterative techique but first we must approximate the derivative.

First we define a discrete version of $v_2(t) = v_2(nT)$ and $v_1(t) = v_1(nT)$



$$\frac{dv_2(t)}{dt} \approx \frac{v_2[nT] - v_2[(n-1)T]}{T}$$ known as the backward Euler Algorithm

$$\frac{v_2[nT] - v_2[(n-1)T]}{T} = -\frac{1}{RC}v_2[nT] + \frac{1}{RC}v_1[nT]$$

$$-v_2[(n-1)T] + (1+\frac{T}{RC})v_2[nT] = \frac{T}{RC}v_1[nT] = \frac{T}{RC}u(n)$$

$$[-z^{-1} + (1+\alpha)]V_2(z) = \alpha V_1(z) = \frac{\alpha z}{z-1} \text{ where } \alpha = \text{T/RC}$$

$$\frac{V_2(z)}{z} = \frac{\alpha}{(z-1)[(1+\alpha) - z^{-1}]} = \frac{\frac{\alpha}{1+\alpha}z}{(z-1)[z - \frac{1}{1+\alpha}]}$$

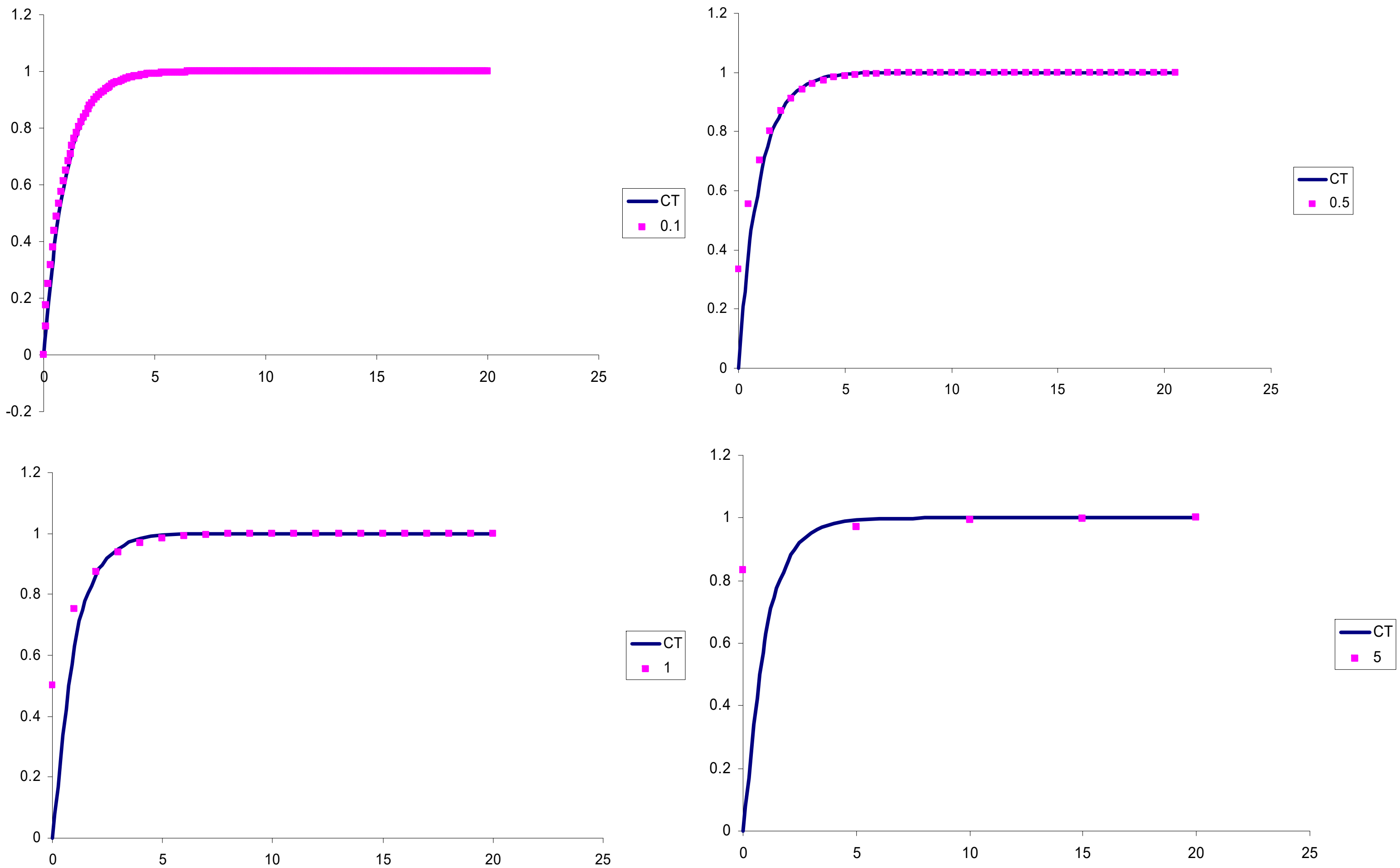
$$\frac{V_2(z)}{z} = \frac{K_1}{z-1} + \frac{K_2}{z - \frac{1}{1+\alpha}}$$

$$K_1 = \frac{\frac{\alpha}{1+\alpha}(1)}{[1 - \frac{1}{1+\alpha}]} = 1, K_2 = \frac{\frac{\alpha}{1+\alpha}(\frac{1}{1+\alpha})}{[(\frac{1}{1+\alpha}) - 1]} = -\frac{1}{1+\alpha}$$

$$V_2(z) = \frac{z}{z-1} - \frac{\frac{1}{1+\alpha}z}{z - (1-\alpha)}$$

$$v_2[n] = [1 - (\frac{1}{1+\alpha})^{n+1}]u[n]$$

Backward Euler Algorithm Continued
Estimate of the output voltage vs sampling time



Time Domain to Frequency Domain Transformations

Time Domain		Transformation Type	Frequency Domain
Time Signal Type	Quadratic Content		
Periodic & Continuous $x(t)$	Finite	Fourier Series FS $a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j\frac{2\pi k}{T}t}\,dt$	Discrete Spectrum a_k
Non-Periodic & Continuous $x(t)$	Finite	Continuous Time Fourier Transform CTFT $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$	Continuous Spectral Density $X(j\omega)$
Discrete $x[n]$	Finite	Discrete Time Fourier Transform DTFT $X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n}$	Continuous Spectral Density $X(e^{j\hat{\omega}})$
Discrete $x[n]$	Finite	Discrete Fourier Transform DFT $X(k) = \sum_{n=0}^{L-1} x[n]e^{-j\frac{2\pi}{N}kn}$	Discrete Spectral Density $X(k)$
Non-Periodic, Continuous & Zero for $t < 0$ $x(t)$	Not Necessarily Finite	Laplace Transform $X(s) = \int_{0}^{\infty} x(t)e^{-st}\,dt$	Complex Spectrum $X(s)$
Non-Periodic, Discrete & Zero for $n < 0$ $x[n]$	Not Necessarily Finite	1-sided Z Transform $X(z) = \sum_{0}^{\infty} x[n]z^{-n}$	Complex Spectrum $X(z)$

Homework

- Problems: 2.16 ,2.17

2.16

Consider the LTI system and find the output $y[n]$ for $x[n] = u[n]$:

$$y[n] - \frac{1}{3} y[n-1] = x[n], n \geq 0$$

2.17

Find $x[n]$.

$$X(z) = \frac{3 - \frac{5}{6} z^{-1}}{(1 - \frac{1}{4} z^{-1})(1 - \frac{1}{3} z^{-1})}$$

Homework

- Problems: 2-19, 2-25

  2.19

  Find the z - transform for

  $$x[n] = (\frac{1}{2})^n u[n] + (\frac{1}{3})^n u[n]$$

  2.25

  Find the difference equation for

  $$H(z) = \frac{1}{(1 - \frac{1}{3}z^{-1})(1 - \frac{1}{6}z^{-1})}$$

  Find the output for an input of $u[n]$, a unit step.