# ***Systems***

Lecture #3

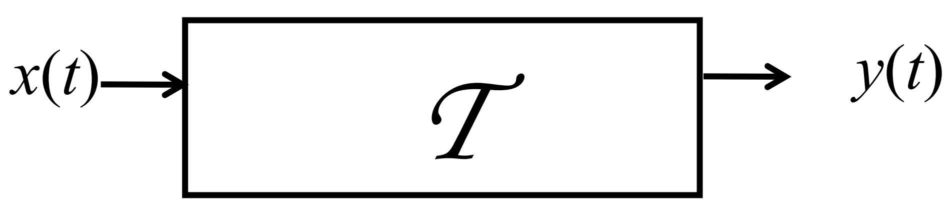
1.3

# *Representation of a System*

- How do represent a system mathematically?
  - Since a system transforms a signal into another we write an equation:

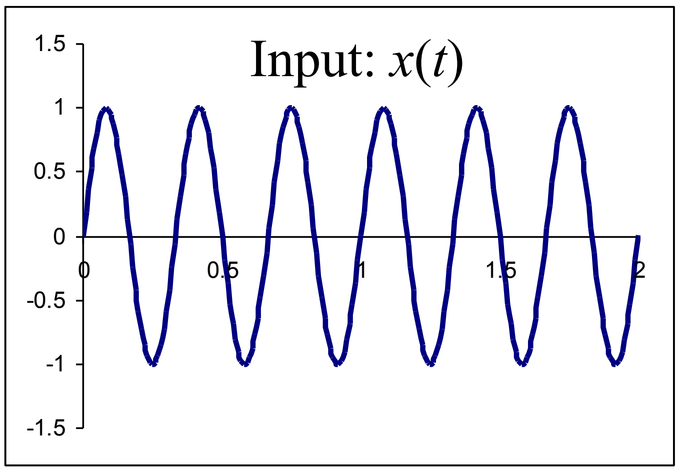
$$y(t) = \mathcal{T}\{x(t)\}$$

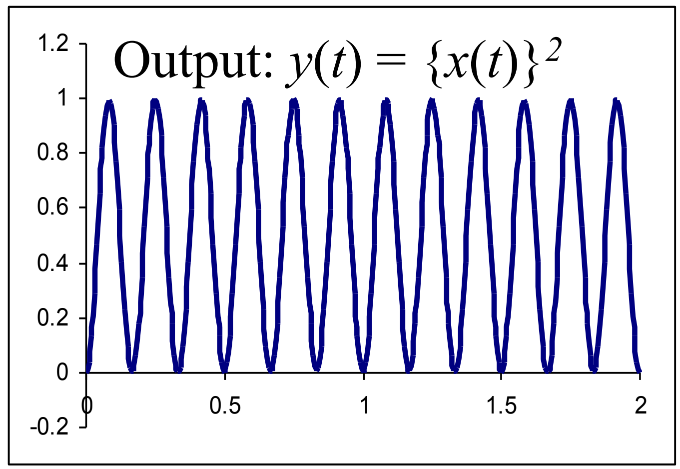
  - where $\mathcal{T}$ is an operator to symbolize a system,
  - $x(t)$ is the signal that goes into the system: input signal *(*or source)
  - And $y(t)$ is transformed signal or output signal (or solution of the equation)
- We can also represent it by a flow diagram



# *Example of a Continuous-Time System*

- A squarer system: $y(t)=\{x(t)\}^2$
  - The output equals the square of the input.
  - This is the result of putting the sine wave into the squarer



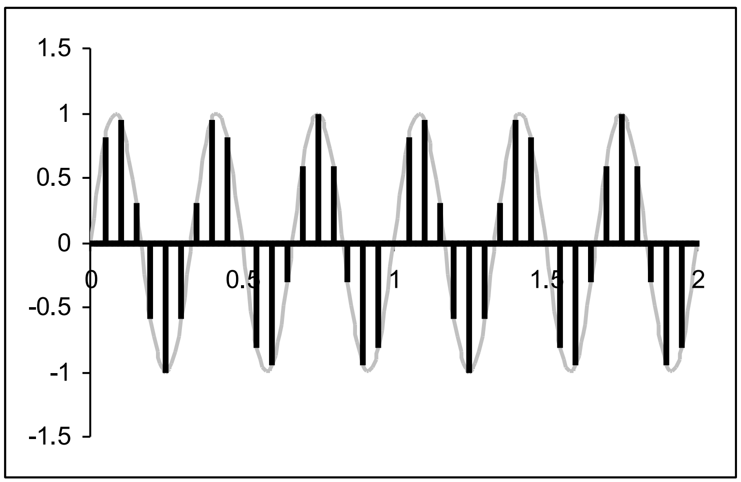

- This is an example of a continuous-time system
- We might be able to build this using an electronic circuit

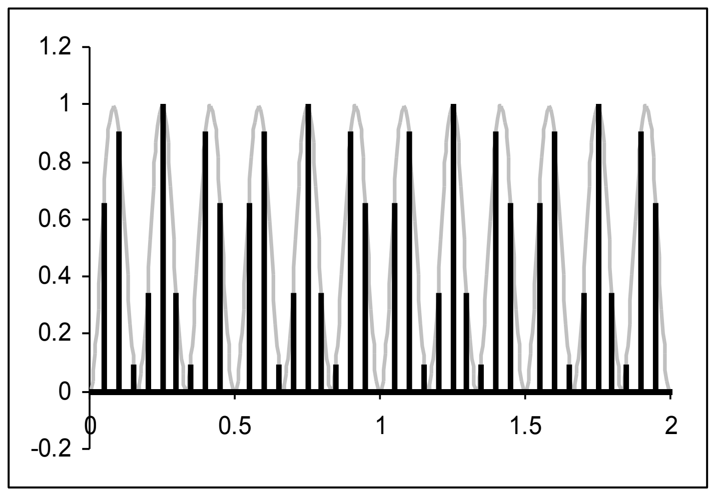
# Discrete-Time Systems

- If we put a discrete-time signal into a system the output may be a discrete-time signal
- This is called a Discrete-time system.

$$y[n] = \mathcal{T}\{x[n]\}$$

- Using our squarer example: $y[n] = \{x[n]\}^2$



# *Mixed Systems*

- Continuous-to-Discrete systems

$$y[n] = \mathcal{T}\{x(t)\}$$

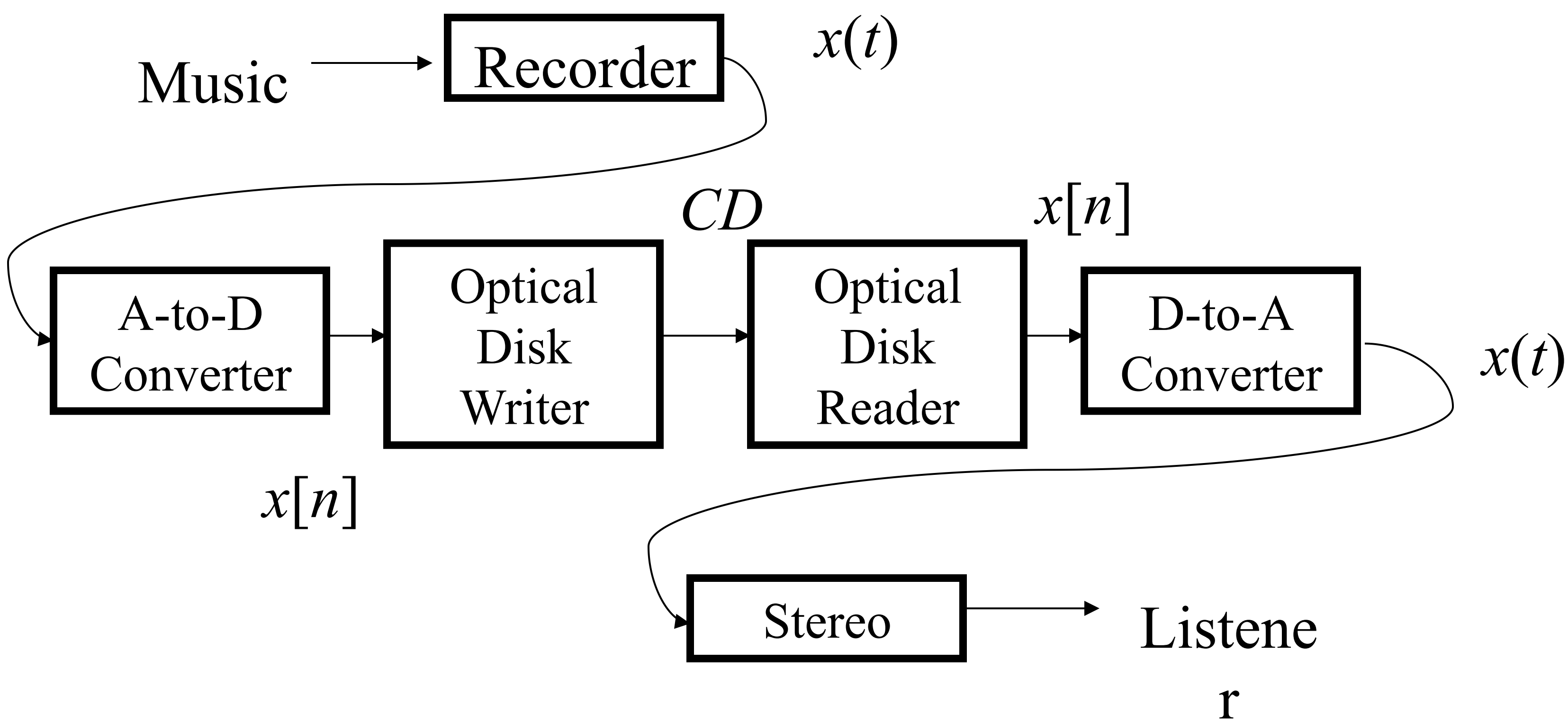
  – Example: a sampler: $y[n]=x(nT_s)$
    - This is also called a A-to-D converter
- Discrete-to-Continuous systems

$$y(t) = \mathcal{T}\{x[n]\}$$

  – Example: An D-to-A converter
    - The opposite of a sampler
    - Takes the samples a recreates the Continuous Signal

# *An Example*

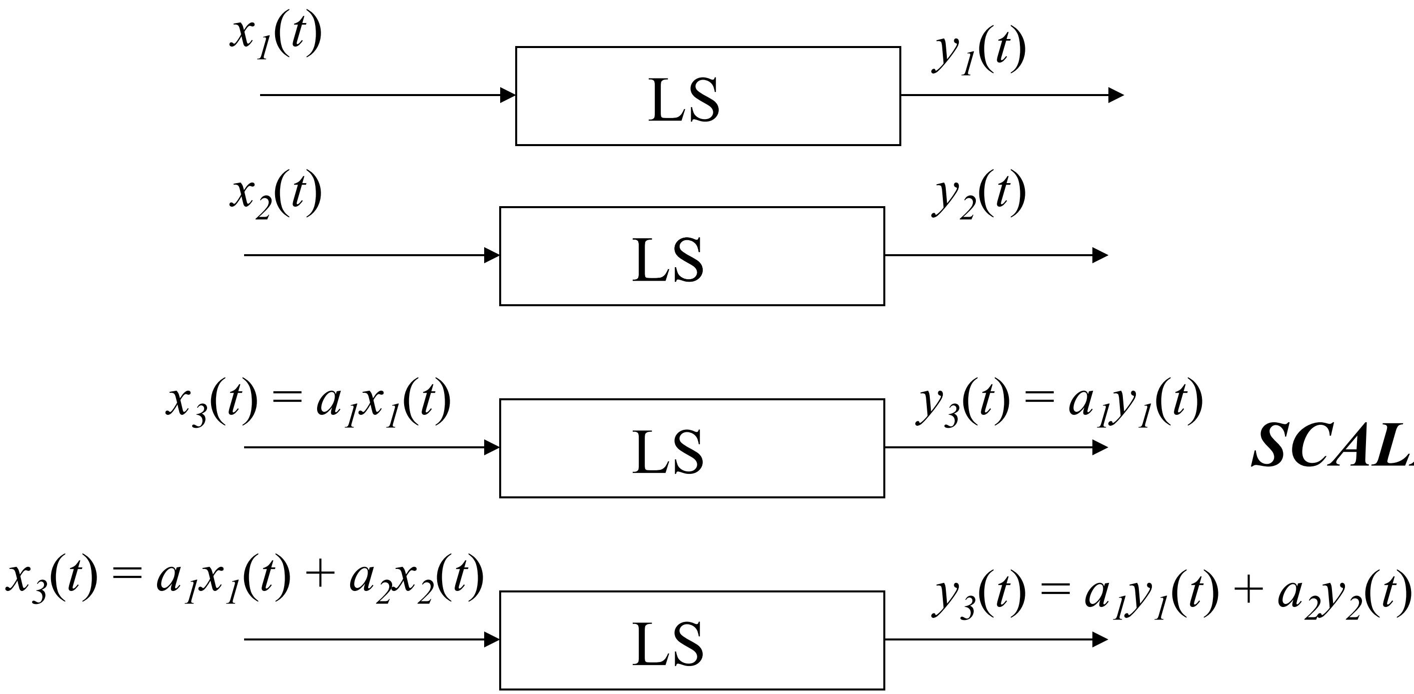
- Example: A music CD



## ***Some Basic Properties of Linear Systems***

- If a system is Linear, or better yet Linear and Time Invariant (LTI), it is easier to analyze and understand than systems that are non-linear and/or vary with time.
- All LTI systems must be
  - Linear and support superposition
  - Causal
  - Time Invariant

# *Linearity for Continuous Signals*




***SCALAR***

***SUPERPOSITION***

# *Shorthand*

$$x_k(t) \rightarrow y_k(t)$$

$$\sum_k a_k x_k(t) \rightarrow \sum_k a_k y_k(t)$$

# *Same for Discrete Signals*

$$x_k[n] \rightarrow y_k[n]$$

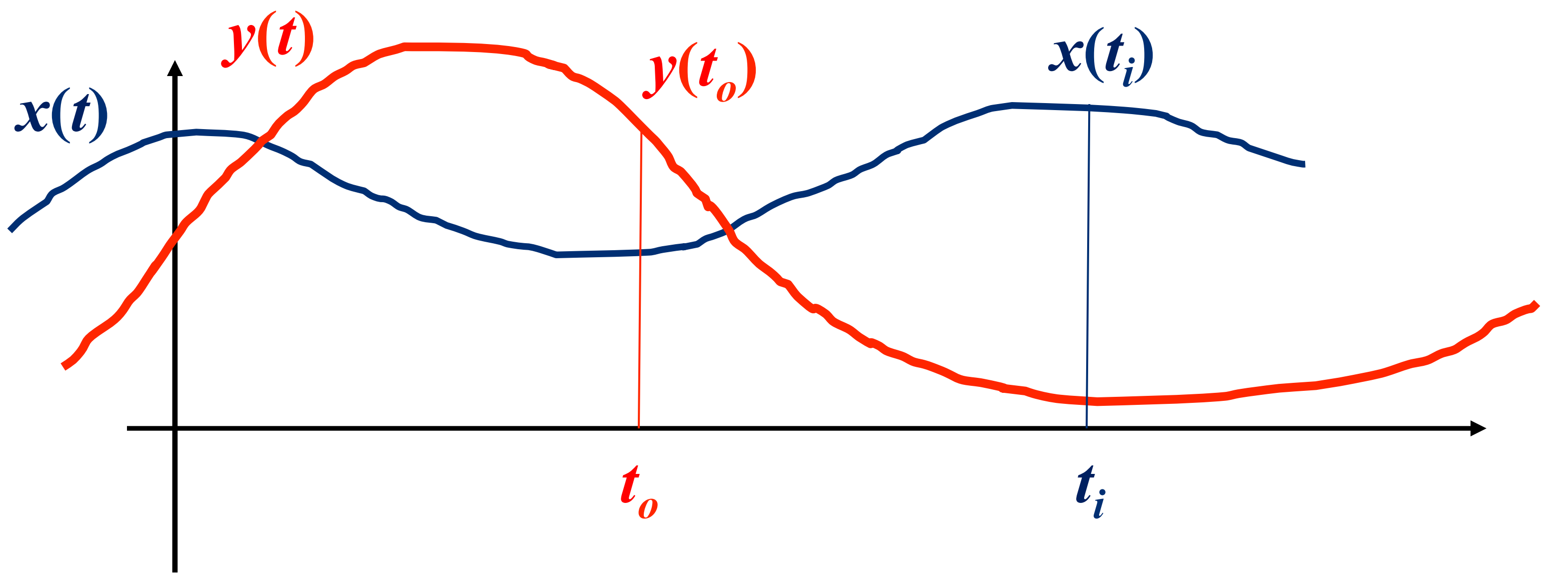
$$\sum_k a_k x_k[n] \rightarrow \sum_k a_k y_k[n]$$

# ***Causality***

- A system is causal if the output at any time depends **<u>only</u>** on the input values up to that time
- $y(t_o)$ does not depend on $x(t_i)$ that occur at times after $t_o$, $t_i > t_o$.
- True for all real time physical systems
- Not true for system-processed recorded signals or spatial varying signal
  - Such systems can look ahead or left, right, up & down
  - E.g., a Morphing System

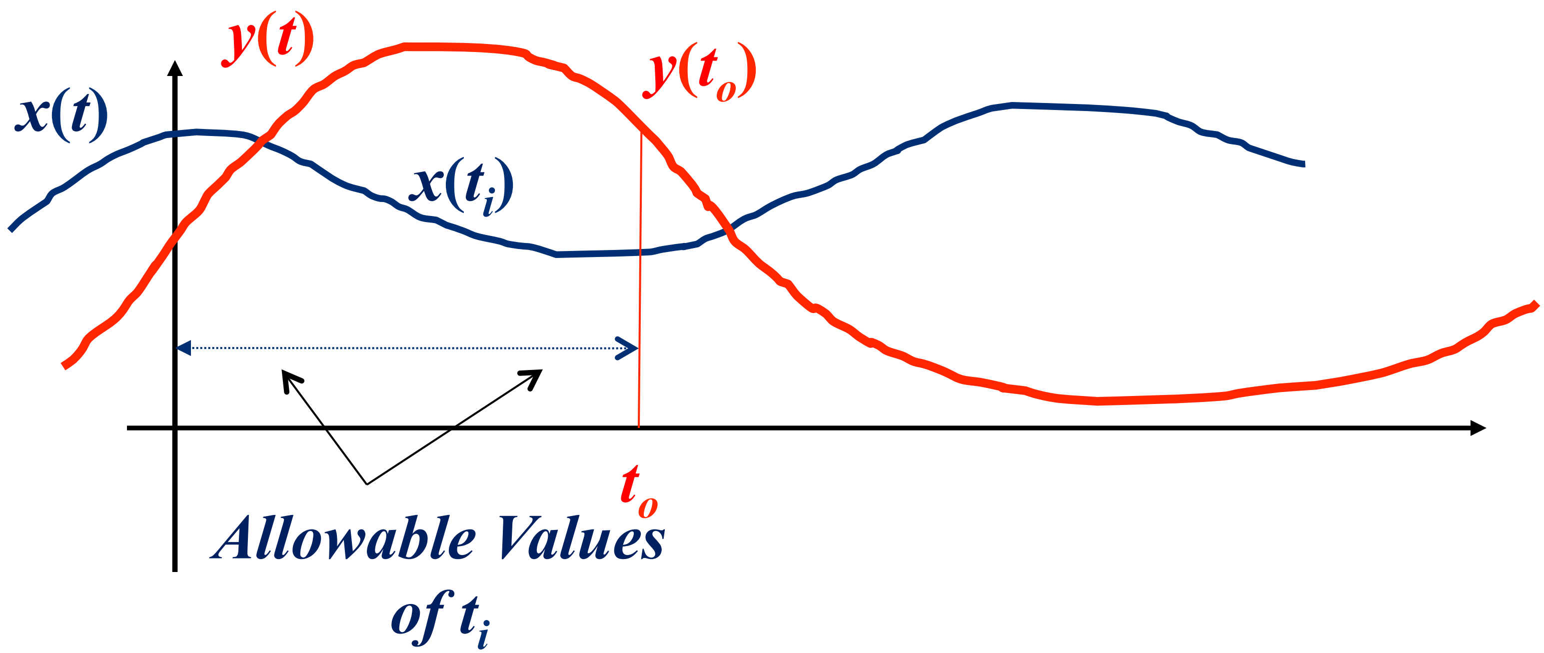
# *Causality*

- Not Causal



# *Causality*

- Causal



# *Time Invariance*

Continuous Signals

$$x_k(t) \longrightarrow y_k(t)$$

Delay $x(t)$ by $t_0$ yields same response only later

$$x_k(t-t_0) \longrightarrow y_k(t-t_0)$$

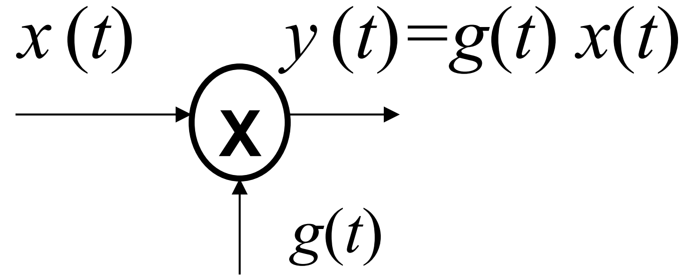
Discrete Signals

$$x_k[n] \longrightarrow y_k[n]$$

$$x_k[n-n_0] \longrightarrow y_k[n-n_0]$$

# *A Non-LTI System*

A multiplier which is a function of time




Check Superposition:

$x_1(t)$ yields $y_1(t)=g(t)\, x_1(t)$

$x_2(t)$ yields $y_2(t)=g(t)\, x_2(t)$

let $x_3(t)= a_1 x_1(t)+a_2 x_2(t)$ then

$y_3(t)=g(t)\, x_3(t) = g(t)\, [a_1 x_1(t)+a_2 x_2(t)]$

$= a_1 y_1(t)+a_2 y_2(t)$

OK

Check Time Invariance:

$x_1(t)=x(t)$ yields $y(t)=g(t)\, x(t)$

$x_2(t) = x(t\text{-}t)$ yields $y_2(t)=g(t)\, x_2(t)$

$=g(t)x(t\text{-}t)$

But to be TI

$x_2(t)= x(t\text{-}t)$ yields $y_2(t) = y(t\text{-}t)$

$= g(t\text{-}t)\, x(t\text{-}t)$

Not OK

# *Another Non-LTI System*

A system with an additive constant

$$y(t) = x(t) + K$$

Check Superposition:

For Superposition to hold, we need to have:

let $x(t) = a_1 x_1(t) + a_2 x_2(t)$ then $y(t) = a_1 x_1(t) + a_2 x_2(t) + K$

But for this system:

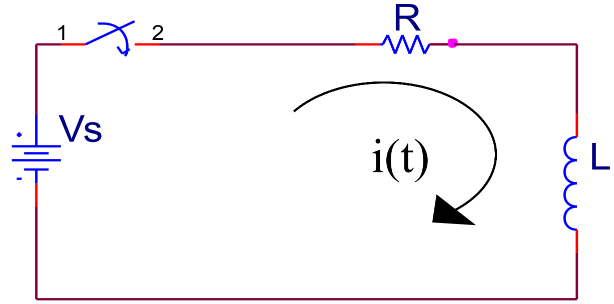
$y(t) = y_1(t) + y_2(t) = a_1 x_1(t) + K + a_2 x_2(t) + K$

Not OK

## *How Does One Describe LTI Systems*

- For Continuous Systems – By Using Ordinary Differential Equations (ODE)
- For Discrete Systems – By Using Difference Equations

# *1st Order Linear ODE: Simple Electrical Circuit*




R = Resistance
L = Inductance
Vs = Voltage

$$Vs = i(t)R + L\frac{di(t)}{dt}$$

$$\frac{di}{dt} + \frac{R}{L}i = \frac{Vs}{L}$$

1st Order Linear ODE

Solve for i(t) assuming: $i(t) = K_1 e^{-At} + K_2$ with the initial condition that i(0)=0. The 2 terms are need due to the following: Since the source Vs is a constant (battery), we assume that the output must a component which is a constant, $K_2$. Since the differential equation is requires that the output and its derivative be proportional to each other, we assume that the output must have a component which is proportional to an exponential function, $K_1 e^{-At}$.

# 1st Order Linear ODE: Simple Electrical Circuit

$$Vs = i(t)R + L\frac{di(t)}{dt}$$

$$\frac{di}{dt} + \frac{R}{L}i = \frac{Vs}{L}$$

Substituting $i(t) = K_1e^{-At} + K_2$ in the equation, we get

Note that the first derivative, equals

$$\frac{di}{dt} = -AK_1e^{-At} + 0$$

$$-AK_1e^{-At} + 0 + \frac{R}{L}(K_1e^{-At} + K_2) = \frac{Vs}{L}$$

Resorting we have

$$-AK_1e^{-At} + \frac{R}{L}K_1e^{-At} + \frac{R}{L}K_2 = \frac{Vs}{L}$$

This implies

$$-AK_1e^{-At} + \frac{R}{L}K_1e^{-At} = 0$$

$$\frac{R}{L}K_2 = \frac{Vs}{L}$$

$$-AK_1e^{-At} + \frac{R}{L}K_1e^{-At} = 0$$

$$-A + \frac{R}{L} = 0; A = \frac{R}{L}$$

$$\frac{R}{L}K_2 = \frac{Vs}{L}; K_2 = \frac{Vs}{R}$$

Therefore,

$$i(t) = K_1e^{-\frac{R}{L}t} + \frac{Vs}{R}$$

But the initial condition states that $i(0) = 0$

$$i(0) = K_1e^{-\frac{R}{L}0} + \frac{Vs}{R} = K_1 + \frac{Vs}{R} = 0$$

$$K_1 = -\frac{Vs}{R}$$

$$i(t) = \frac{Vs}{R}(1 - e^{-\frac{R}{L}t})$$

# *1st Order Linear ODE: Simple Electrical Circuit*


$$i(t) = \frac{Vs}{R}(1 - e^{-\frac{R}{L}t}) = \frac{Vs}{R}(1 - e^{-\frac{t}{L/R}})$$

$\frac{L}{R}$ is called the time constant and we see that within 3 time constants 95% of its final value is reached.

# *Another 1st Order LODE : Drug Concentration in Blood Being Removed by the Liver*

$$\dot{D} + K_L D = \frac{R_D}{V_C}$$

Where $K_L$ = drug loss rate

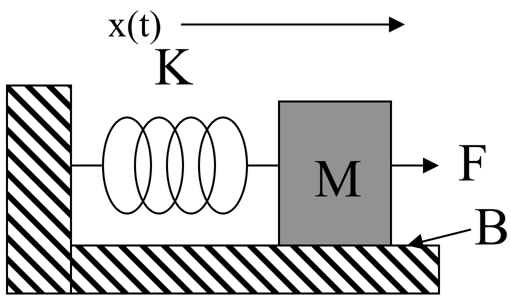
Vc = Volume of circulatory system in liters

$R_D$ is the rate of drug input (mg/min)

In a similar way as in the RL circuit, we can solve this for

$$D(t) = \frac{R_D}{V_C K_L}(1 - e^{-K_L t})$$

# $2^{nd}$ Order LODE

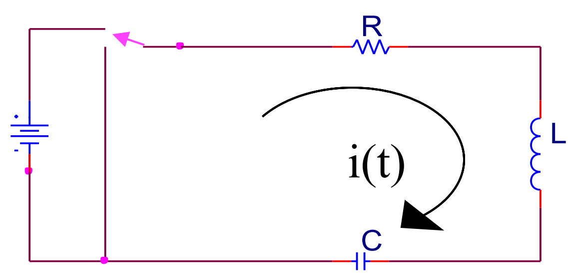



$$M\ddot{x}+B\dot{x}+Kx=F(t)$$

M = Mass

B = Friction

K = Spring constant




$$L\ddot{i}+R\dot{i}+\frac{1}{C}i=0$$

R = Resistance

L = Inductance

C = Capacitance

# *Homework*

- Linear Systems
  - Is $y(t)=x(t)^2$ a linear system? Prove your point.
  - Is $y(t)=t^2$ a linear system? Prove your point.
  - CT.1.3.1
- ODE
  - Solve and plot the solution to the equation: $dx/dt + 6\ x = 0$; $x(0) = 5$; use Matlab to obtain the plot
  - Solve and plot the solution to the equation : $dx/dt + 6\ x = 6$ ; $x(0) = 0$ ; use Matlab to obtain the plot