LTI ODE Continued

Lecture #4

Introduction of the p Operator

Let's start with this 2nd order differential equation

to represent some system.

$$a\frac{d^2y(t)}{dt^2}+b\frac{dy(t)}{dt}+cy(t)=e\frac{dx(t)}{dt}+fx(t)$$

where $x(t)$ is the input or source,

$y(t)$ is the solution or the response of the differential equation,

and $a,\ b,\ c,\ d,\ f$ are the coefficeints.

Let's define the p operators:

$$p\cong\frac{d}{dt},p^2\cong\frac{d^2}{dt^2},\ldots,p^n\cong\frac{d^n}{dt^n}$$

and rewrite the differential equation as

$$ap^2y(t)+bpy(t)+cy(t)=epx(t)+fx(t)$$

Introduction of the p Operator

And now for some mathematical blasphemy!!!

$$ap^2y(t) + bpy(t) + cy(t) = epx(t) + fx(t)$$

$$[ap^2 + bp + c]y(t) = [ep + f]x(t)$$

$$y(t) = \frac{ep + f}{ap^2 + bp + c}x(t)$$

This is **<u>not</u>** a solution for $y(t)$ but a way of reducing an equation into a simpler form.

Introduction of the p Operator

Note that $[ap^2 + bp + c]$ and $[ep + f]$ are polynomials in p and we can rewrite these polynomials as $[ap^2 + bp + c] = A(p)$ and $[ep + f] = B(p)$ and we get

$$A(p)y(t) = B(p)x(t)$$

$$y(t) = \frac{B(p)}{A(p)}x(t)$$

This is **<u>not</u>** a solution for y(t) but a way of reducing an equation into a simpler form.

Components of the Solution of LODE

$$A(p)y(t) = B(p)x(t)$$

Let's define $y(t)$ which is

the total response of the system

$$y(t) = y_s(t) + y_{sf}(t)$$

where $y_s(t)$ is called the response due to the source

and is the solution to this

$$A(p)y_s(t) = B(p)x(t)$$

and $y_{sf}(t)$ is called the source free response or the transient response.

and is the soltuion to this equation

$$A(p)y_{sf}(t) = 0$$

We can now use suposition, solve two simpler equations,

and add them up to get the total solution.

Components of the Solution of LODE Source Response

- $y_s(t)$ - Source Response has the same form as the source and is a solution of:

$$A(p)y_s(t)=B(p)x(t)$$

- If $x(t)$ is a constant then $y_s(t)$ is a constant, if $x(t)$ is a polynomial then $y_s(t)$ is a polynomial, if $x(t)$ is a sinusoid then $y_s(t)$ is a sinusoid, etc.

Components of the Solution of LODE Source-Free Response

- $y_{sf}(t)$ - Source-free Response is also called the Natural Mode Response, Transient Response, Homogeneous, or Characteristic Response and is a solution of

$$A(p)y_{sf}(t)=0 \text{ Homogeneous Equation}$$
$$A(p)=0 \text{ Characteristic Equation}$$

- Functions which satisfy the Homogeneous Equation are called eigenfunctions (e.g., Ke^{at})
- The values of p which are solutions to the Characteristic Equation are called eigenvalues.

Components of the Solution of LODE
Source-Free Response
Stability

- The source-free response is independent of the source and always appears. It is a function of the system under examination.
- It determines the stability of the system.
- For a stable system,
  1. it is expected that the source-free response is also know as the transient response since it is expected that this component will effectively "end".
  2. When the source-free response ends, the response due to the source or steady state remains.
- For an unstable system, the source-free response may never end.

Components of the Solution of LODE: Source Response

Let us look back at the RL circuit 1st Order ODE: $Vs = i(t)R + L\frac{di(t)}{dt}$

Source Response: $i_s(t)$

Since the source is a constant, Vs, then source response is a constant

$$i_s(t) = K$$

$$\frac{di_s(t)}{dt} + \frac{R}{L} i_s(t) = \frac{Vs}{L}$$

Substituting $i_s(t)$ into the differential equation, we have

$$K = \frac{Vs}{R}$$

$$i_s(t) = \frac{Vs}{R}$$

Components of the Solution of LODE: Source Free Response: Using the Homogenous Equation

Source Free Response: $i_{sf}(t)$

The Homogenous equation: $L\dfrac{di(t)}{dt}+i(t)R=0$

Solutions of the Homogenous equation (the eigenfunctions for a 1st order ODE) are

$$i_{sf}(t)=Ae^{at}$$

Substituting $i_{sf}(t)$ into the homogenous equation, we have

$$\frac{di_{sf}(t)}{dt}=aAe^{at}$$

$$aAe^{at}+\frac{R}{L}Ae^{at}=0$$

$$a=-\frac{R}{L}$$

$$\therefore i_{sf}(t)=Ke^{-\frac{R}{L}t}$$

Components of the Solution of LODE: Source Free Response: Using the Characteristic Equation

Source Free Response: $i_{sf}(t)$

Note that the solution of the Characteristic equation (the eigenvalue) is

$$(p+\frac{R}{L})=0$$

$$p=-\frac{R}{L}$$

and, therefore, same solution: $i_{sf}(t)=Ae^{-\frac{R}{L}t}$

Components of the Solution of LODE: Total Solution

Then the total response is:

$$i(t) = i_s(t) + i_{sf}(t)$$

$$= \frac{Vs}{R} + Ae^{-\frac{R}{L}t}$$

The constant A can be found from initial conditions of $i(t)$

$$i(0) = \frac{Vs}{R} + Ae^{-\frac{R}{L}0} = \frac{Vs}{R} + A$$

$$A = i(0) - \frac{Vs}{R}$$

$$i(t) = \frac{Vs}{R} + (i(0) - \frac{Vs}{R})e^{-\frac{R}{L}t}$$

Components of the Solution of LODE

In our problem $i(0) = 0$:

$$i(t) = \frac{Vs}{R} + (i(0) - \frac{Vs}{R})e^{-\frac{R}{L}t}$$

$$i(t) = \frac{Vs}{R} - \frac{Vs}{R}e^{-\frac{R}{L}t}$$

$$= \frac{Vs}{R}(1 - e^{-\frac{R}{L}t})$$

Free Response of a 2nd ODE
Solutions of the Characteristic Equation

$$ap^2 + bp + c = 0$$

Let p_1,p_2 be the roots of the characteristic equation, then

$$p_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The 4 Free Response Cases of a 2nd ODE Solutions of the Homogeneous Equation

Case	Roots		Solution	Type
$b^2-4ac>0$	$p_{1,2}=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$	Real, Unequal, Negative	$y_{sf}(t)=C_1e^{p_1t}+C_2e^{p_2t}$ where C_1 and C_2 are real	Overdamped Stable
$b^2-4ac=0$	$p_{1,2}=p=\dfrac{-b}{2a}$	Real, Equal, Negative	$y_{sf}(t)=(C_1t+C_2)e^{pt}$ where C_1 and C_2 are real	Critically damped Stable
$b^2-4ac<0$	$\boldsymbol{p}_{1,2}=\dfrac{-b\pm j\sqrt{4ac-b^2}}{2a}$ $=\alpha\pm j\omega$	Complex conjugates, Unequal	$y_{sf}(t)=e^{-\alpha t}(C_1e^{j\omega t}+C_2e^{-j\omega t})$ where $p_{1,2}=-\alpha\pm j\omega$ $C_2=C_1{*};C_1=Ce^{j\theta}$ are complex conjugates $y_{sf}(t)=e^{-\alpha t}C(e^{j(\omega t+\theta)}+e^{-j(\omega t+\theta)})$ $=e^{-\alpha t}2C\cos(\omega t+\theta)$	Underdamped Stable
$b^2-4ac<0$ & $b=0$	$\boldsymbol{p}_{1,2}=\pm j\dfrac{\sqrt{4ac}}{2a}=\pm j\omega$	Imaginary	$y_{sf}(t)=C_1e^{j\omega t}+C_2e^{-j\omega t}$ $=2C\cos(\omega t+\theta)$ where $p_{1,2}=\pm j\omega$ & $C_2=C_1{*};C_1=Ce^{j\theta}$	Undamped or Oscillatory Unstable

Solutions to the Source Free Response of 2^{nd} Order ODE

Overdamped $y_{sf}(t) = C_1 e^{p_1 t} + C_2 e^{p_2 t}$; note that p_1 and p_2 are negative

Critically Damped $y_{sf}(t) = (C_1 t + C_2) e^{pt}$; note that p is negative

Underdamped $y_{sf}(t) = C_1 e^{p_1 t} + C_2 e^{p_2 t} = e^{-\alpha t}(C_1 e^{j\omega t} + C_2 e^{-j\omega t})$ where $p_{1,2} = -\alpha \pm j\omega$

but we can show that

$C_2 = C_1{}^*$; and assuming

$C_1 = a + jb = Ce^{j\theta}$; then $C_2 = a - jb = Ce^{-j\theta}$

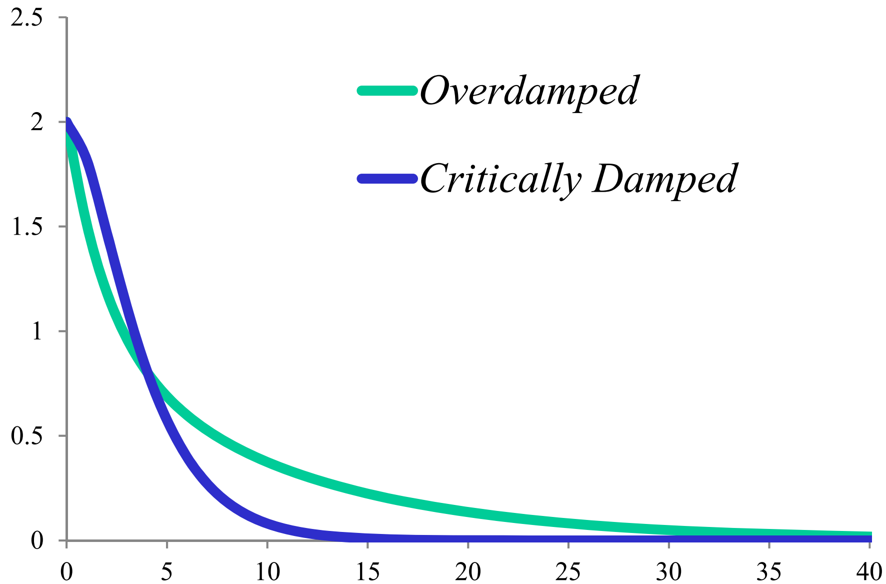
$$y_{sf}(t) = e^{-\alpha t}(Ce^{j\theta}e^{j\omega t} + Ce^{-j\theta}e^{-j\omega t}) = e^{-\alpha t}(Ce^{j(\omega t+\theta)} + Ce^{-j(\omega t+\theta)})$$

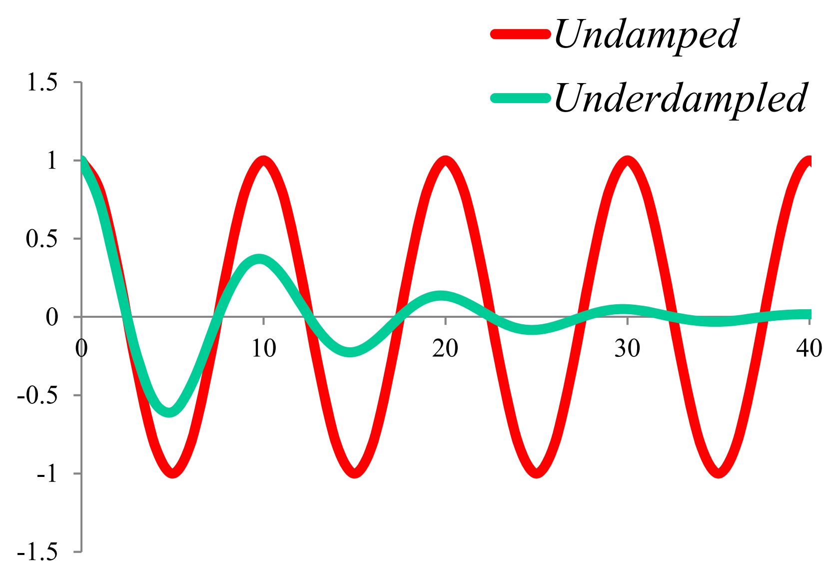
$$= e^{-\alpha t}C(e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}) = e^{-\alpha t}2C\cos(\omega t + \theta)$$

Undamped

$y_{sf}(t) = C_1 e^{j\omega t} + C_2 e^{-j\omega t} = 2C\cos(\omega t + \theta)$; where $p_{1,2} = \pm j\omega$; $C_2 = C_1{}^*$; $C_1 = Ce^{j\theta}$

Source Free Responses of 2nd Order ODE





Proof of the Complex Conjugate Constants Underdamped Case

$$y(t) = e^{-\alpha t}(C_1 e^{j\omega t} + C_2 e^{-j\omega t}) = C_1 e^{s_1 t} + C_2 e^{s_1^* t}; \text{where } s_1 = -\alpha + j\omega$$

$$y(0) = C_1 + C_2$$

$$\dot{y}(t) = s_1 C_1 e^{s_1 t} + s_1^* C_2 e^{s_1^* t}$$

$$\dot{y}(0) = s_1 C_1 + s_1^* C_2$$

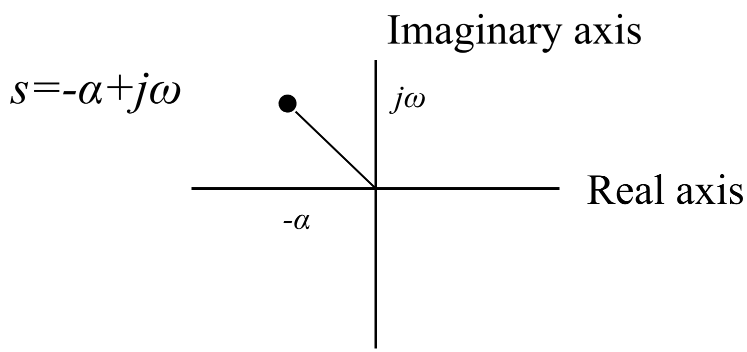
$$C_1 = y(0) - C_2$$

$$\dot{y}(0) = s_1[y(0) - C_2] + s_1^* C_2$$

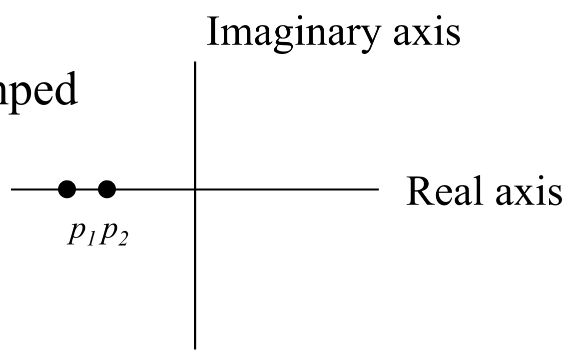
$$C_2 = \frac{\dot{y}(0) - s_1 y(0)}{s_1^* - s_1} = \frac{\dot{y}(0) - (-\alpha + j\omega)y(0)}{-j2\omega} = \frac{y(0)}{2} + j\frac{\dot{y}(0) + \alpha y(0)}{2\omega}$$

$$C_1 = y(0) - \frac{\dot{y}(0) - s_1 y(0)}{s_1^* - s_1} = \frac{s_1^* y(0) - \dot{y}(0)}{s_1^* - s_1} = \frac{(-\alpha - j\omega)y(0) - \dot{y}(0)}{-j2\omega} = \frac{y(0)}{2} - j\frac{\dot{y}(0) + \alpha y(0)}{2\omega}$$

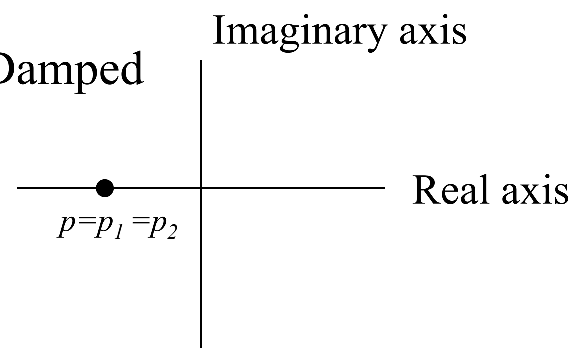
Complex Plane



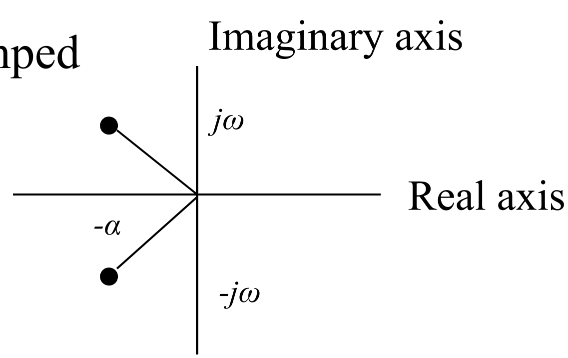
Overdamped



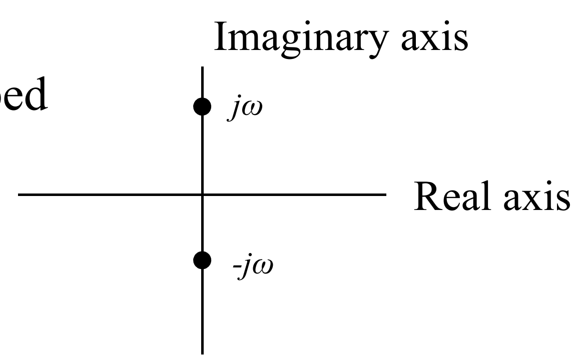
Critically Damped



Underdamped



Undamped



Complete Response of 2nd Order ODE An Example

$\ddot{y} + 4\dot{y} + 3y = 4e^{-2t}; y(0) = 2; \dot{y}(0) = 4$ System with initial conditions

$p^2 + 4p + 3 = 0$ Characteristic Equation

$p_{1,2} = -3, -1$ Eigenvalues - Overdamped

$y_{sf}(t) = A_1 e^{-3t} + A_2 e^{-t}$ Eigenfunctions and source free response

$y_s(t) = A_3 e^{-2t}$ Source free response

$y(t) = A_1 e^{-3t} + A_2 e^{-t} + A_3 e^{-2t}$ Total response

$$\dot{y}(t) = -3A_1 e^{-3t} - A_2 e^{-t} - 2A_3 e^{-2t}$$

$$\ddot{y}(t) = 9A_1 e^{-3t} + A_2 e^{-t} + 4A_3 e^{-2t}$$

Complete Response of 2nd Order ODE Solution

Substitute the solution, the first and second derivatives into the system

$$(9A_1e^{-3t} + A_2e^{-t} + 4A_3e^{-2t}) + 4(-3A_1e^{-3t} - A_2e^{-t} - 2A_3e^{-2t}) + 3(A_1e^{-3t} + A_2e^{-t} + A_3e^{-2t}) = 4e^{-2t}$$

Resort the equation into like terms

$$(9-12+3)A_1e^{-3t} + (1-4+3)A_2e^{-t} + (4-8+3)A_3e^{-2t} = 4e^{-2t}$$

Notice that the coefficents of the eigenfunctions are zero since they must satisfy the Homogeneous Equation.

What is left is the source term, the response due to the sources:

$$(0)A_1e^{-3t} + (0)A_2e^{-t} - A_3e^{-2t} = 4e^{-2t}$$

From this A_3 is determined:

$$-A_3e^{-2t} = 4e^{-2t} \Rightarrow A_3 = -4$$

Using the initial conditions of the response and the first derivative, A_1 and A_2 are determined.

$$y(0) = A_1 + A_2 - 4 = 2 \Rightarrow A_1 + A_2 = 6$$

$$\dot{y}(0) = -3A_1 - A_2 + 8 = 4 \Rightarrow -3A_1 - A_2 = -4$$

$$-2A_1 = 2 \Rightarrow A_1 = -1; A_2 = 7$$

The total solution is:

$$\therefore y(t) = -e^{-3t} + 7e^{-t} + -4_3e^{-2t}$$

Natural Frequency and Damping Ratio

$ap^2 + bp + c = 0$

$p^2 + \frac{b}{a}p + \frac{c}{a} = 0$

$p_{1,2} = -\frac{b}{2a} \pm \sqrt{(\frac{b}{2a})^2 - \frac{c}{a}}$

If $b = 0$, then $p_{1,2} = \pm j\sqrt{\frac{c}{a}} = \pm j\omega_o$

Define $\frac{c}{a} = \omega_o{}^2$

Define $\frac{b}{2a} = \xi\omega_o \Rightarrow \xi = \frac{b}{2a\omega_o} = \frac{b}{2a\sqrt{\frac{c}{a}}} = \frac{b}{2\sqrt{ac}}$

We call ω_o the undamped natural frequency and ξ the damping ratio

$p^2 + 2\xi\omega_o p + \omega_o{}^2 = 0 \Rightarrow p_{1,2} = -\frac{b}{2a} \pm \sqrt{(\frac{b}{2a})^2 - \frac{c}{a}} \Rightarrow -\xi\omega_o \pm \sqrt{(\xi\omega_o)^2 - \omega_o{}^2}$

$p_{1,2} = -\xi\omega_o \pm \sqrt{(\xi\omega_o)^2 - \omega_o{}^2} = -\xi\omega_o \pm \omega_o\sqrt{\xi^2 - 1}$

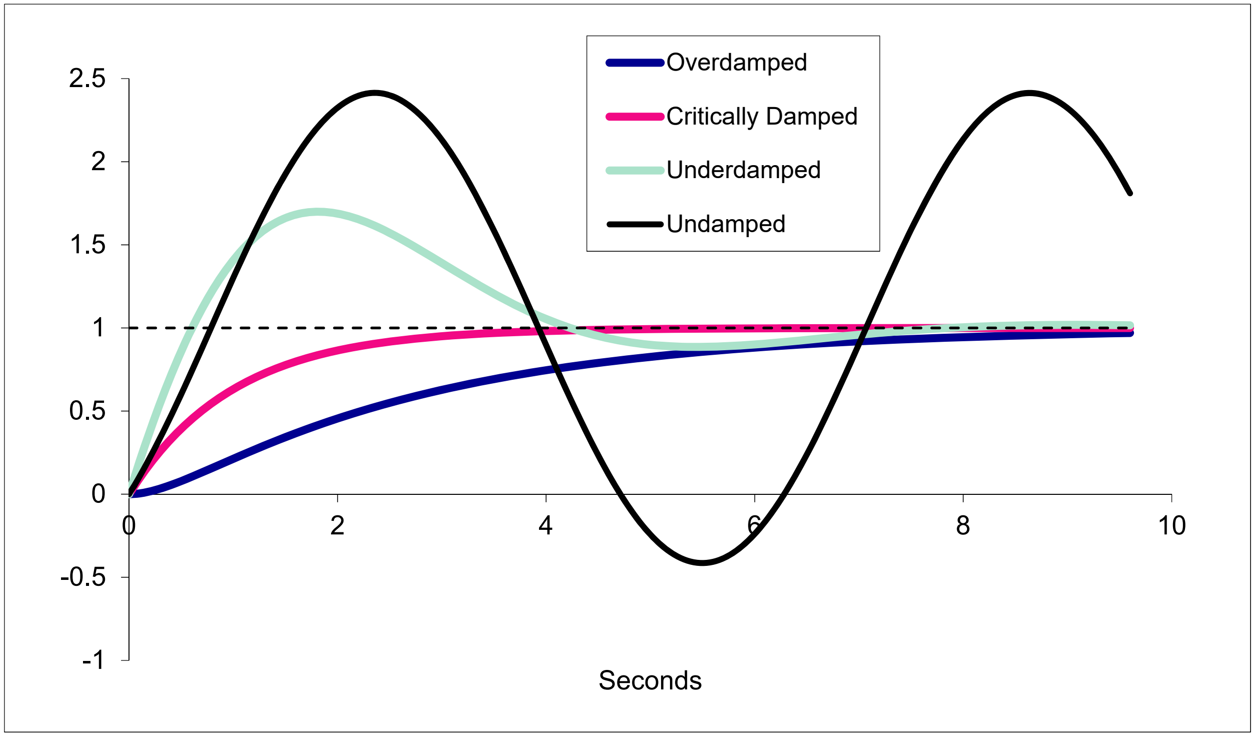
For $\xi > 1$, overdamped, $p_{1,2} = -\xi\omega_o \pm \omega_o\sqrt{\xi^2 - 1} = \omega_o(-\xi \pm \sqrt{\xi^2 - 1})$

For $\xi = 1$, critically damped, $p_{1,2} = -\omega_o$

For $\xi < 1$, underdamped, $p_{1,2} = -\xi\omega_o \pm j\omega_o\sqrt{1 - \xi^2}$

For $\xi = 0$, undamped, $p_{1,2} = \pm j\omega_o$

Natural Frequency and Damping Ratio Unit Step Response



Unit Step Response Overdamped

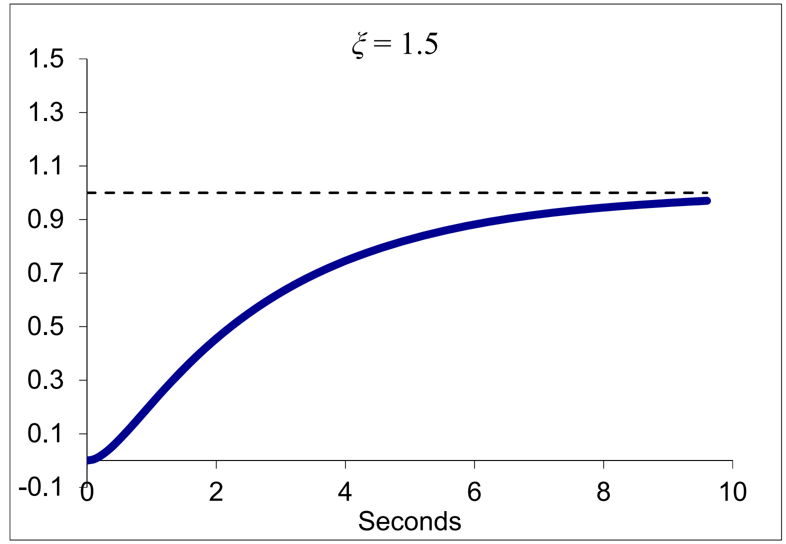
$$p^2 y(t) + \frac{b}{a} p y(t) + \frac{c}{a} y(t) = \frac{1}{a} x(t)$$

$$p^2 y(t) + 2\xi\omega_o p y(t) + \omega_o{}^2 y(t) = Kx(t) = 1$$

Source response is $y_s(t) = A = \dfrac{1}{\omega_o{}^2}$

For $\xi > 1$, overdamped, $p_{1,2} = (-\xi \pm \sqrt{\xi^2 - 1})\omega_o$

Source free response is $y_{sf}(t) = C_1 e^{(-\xi - \sqrt{\xi^2-1})\omega_o t} + C_2 e^{(-\xi + \sqrt{\xi^2-1})\omega_o t}$



$$y(t) = \frac{1}{\omega_o{}^2} + C_1 e^{(-\xi - \sqrt{\xi^2-1})\omega_o t} + C_2 e^{(-\xi + \sqrt{\xi^2-1})\omega_o t}$$

$$y(0) = 0 = \frac{1}{\omega_o{}^2} + C_1 + C_2; C_1 = -(\frac{1}{\omega_o{}^2} + C_2)$$

$$\dot{y}(0) = 0 + (-\xi - \sqrt{\xi^2-1})\omega_o C_1 + (-\xi + \sqrt{\xi^2-1})\omega_o C_2$$

$$\dot{y}(0) = -(-\xi - \sqrt{\xi^2-1})\omega_o[(\frac{1}{\omega_o{}^2} + C_2)] + (-\xi + \sqrt{\xi^2-1})\omega_o C_2 = (\xi + \sqrt{\xi^2-1})\omega_o[(\frac{1}{\omega_o{}^2} + C_2)] + (-\xi + \sqrt{\xi^2-1})\omega_o C_2$$

$$= -(-\xi - \sqrt{\xi^2-1})\omega_o \frac{1}{\omega_o{}^2} + (\xi + \sqrt{\xi^2-1})\omega_o C_2 + (-\xi + \sqrt{\xi^2-1})\omega_o C_2 = (\xi + \sqrt{\xi^2-1})\frac{1}{\omega_o} + (2\sqrt{\xi^2-1})\omega_o C_2$$

$$\dot{y}(0) = -(-\xi - \sqrt{\xi^2-1})\frac{1}{\omega_o} + (2\sqrt{\xi^2-1})\omega_o C_2 \Rightarrow C_2 = \frac{\dot{y}(0) + (-\xi - \sqrt{\xi^2-1})\frac{1}{\omega_o}}{(2\sqrt{\xi^2-1})\omega_o}$$

$$C_2 = \frac{\dot{y}(0)\omega_o + (-\xi - \sqrt{\xi^2-1})}{2\omega_o{}^2\sqrt{\xi^2-1}}; C_1 = (\frac{\xi - \sqrt{\xi^2-1} - \dot{y}(0)\omega_o}{2\omega_o{}^2\sqrt{\xi^2-1}})$$

$$y(t) = \frac{1}{\omega_o{}^2} + \frac{-\dot{y}(0)\omega_o + \xi - \sqrt{\xi^2-1}}{2\omega_o{}^2\sqrt{\xi^2-1}} e^{(-\xi - \sqrt{\xi^2-1})\omega_o t} + \frac{\dot{y}(0)\omega_o - \xi - \sqrt{\xi^2-1}}{2\omega_o{}^2\sqrt{\xi^2-1}} e^{(-\xi + \sqrt{\xi^2-1})\omega_o t}$$

Unit Step Response Critically Damped

$$p^2 y(t) + \frac{b}{a} py(t) + \frac{c}{a} y(t) = \frac{1}{a} x(t)$$

$$p^2 y(t) + 2\xi\omega_o py(t) + \omega_o^2 y(t) = Kx(t) = 1$$

Source response is $y_s(t) = A = \dfrac{1}{\omega_o^2}$

For $\xi = 1$, critically damped, $p_{1,2} = -\omega_o$

Source free response is $y_{sf}(t) = (C_1 t + C_2)e^{-\omega_o t}$

$$y(t) = \frac{1}{\omega_o^2} + (C_1 t + C_2)e^{-\omega_o t}$$

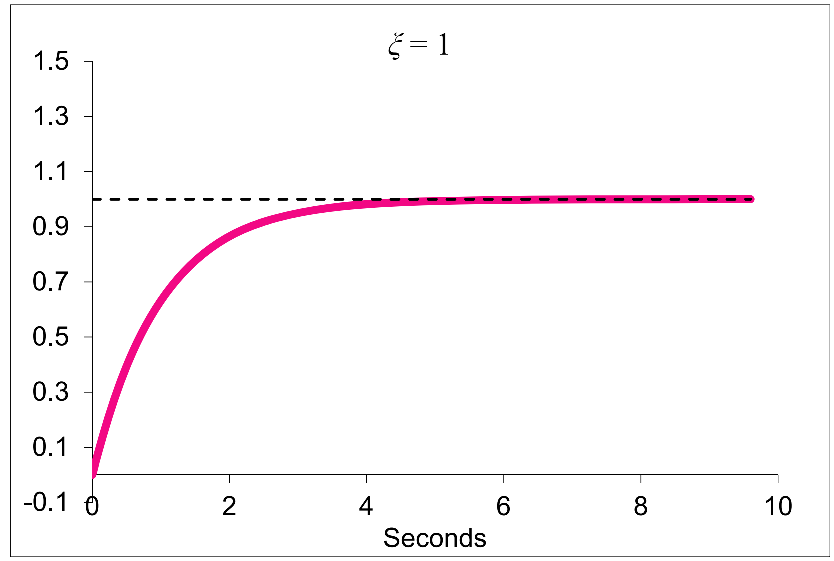
$$y(0) = 0 = \frac{1}{\omega_o^2} + C_2$$

$$C_2 = -\frac{1}{\omega_o^2}$$

$$\dot{y}(0) = 0 + C_1 - \omega_o C_2$$

$$C_1 = \dot{y}(0) + \omega_o C_2 = \dot{y}(0) - \frac{1}{\omega_o}$$

$$y(t) = \frac{1}{\omega_o^2} + ([\dot{y}(0) - \frac{1}{\omega_o}]t - \frac{1}{\omega_o^2})e^{-\omega_o t}$$



Unit Step Response Underdamped

$$p^2 y(t) + \frac{b}{a} py(t) + \frac{c}{a} y(t) = \frac{1}{a} x(t)$$

$$p^2 y(t) + 2\xi\omega_o py(t) + \omega_o^{\,2} y(t) = Kx(t) = 1$$

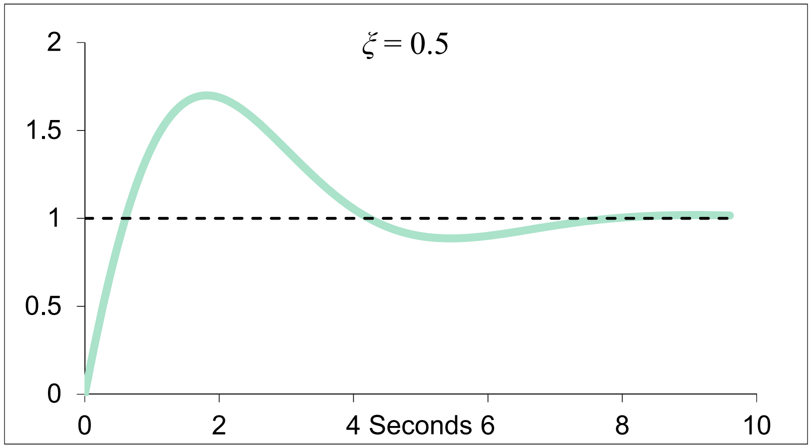
Source response is $y_s(t) = A = \dfrac{1}{\omega_o^{\,2}}$

For $\xi < 1$, underdamped, $p_{1,2} = -\xi\omega_o \pm j\omega_o\sqrt{1-\xi^2}$

Source free response is $y_{sf}(t) = e^{-\xi\omega_o t} K\cos(\omega_o\sqrt{1-\xi^2}\,t + \theta)$

$$y(t) = \frac{1}{\omega_o^{\,2}} + e^{-\xi\omega_o t} K\cos(\omega_o\sqrt{1-\xi^2}\,t + \theta)$$

$$y(0) = 0 = \frac{1}{\omega_o^{\,2}} + K\cos(\theta); K\cos(\theta) = -\frac{1}{\omega_o^{\,2}}$$



$$\dot{y}(0) = 0 + -\xi\omega_o K\cos(\theta) - \omega_o\sqrt{1-\xi^2}K\sin(\theta) = \xi\omega_o\frac{1}{\omega_o^{\,2}} - \omega_o\sqrt{1-\xi^2}K\sin(\theta); K\sin(\theta) = -\frac{\dot{y}(0) - \dfrac{\xi}{\omega_o}}{\omega_o\sqrt{1-\xi^2}}$$

$$\theta = \tan^{-1}\left(\frac{\dot{y}(0)\omega_o - \xi}{\sqrt{1-\xi^2}}\right); K = -\frac{1}{\omega_o^{\,2}\cos\left(\tan^{-1}\left(\dfrac{\dot{y}(0)\omega_o - \xi}{\sqrt{1-\xi^2}}\right)\right)}$$

$$y(t) = \frac{1}{\omega_o^{\,2}} - e^{-\xi\omega_o t}\frac{1}{\omega_o^{\,2}\cos\left(\tan^{-1}\left(\dfrac{\dot{y}(0)\omega_o - \xi}{\sqrt{1-\xi^2}}\right)\right)}\cos\left(\omega_o\sqrt{1-\xi^2}\,t + \tan^{-1}\left(\frac{\dot{y}(0)\omega_o - \xi}{\sqrt{1-\xi^2}}\right)\right)$$

Unit Step Response
Undamped

$$p^2 y(t) + \frac{b}{a} p y(t) + \frac{c}{a} y(t) = \frac{1}{a} x(t)$$

$$p^2 y(t) + 2\xi\omega_o p y(t) + \omega_o^2 y(t) = Kx(t) = 1$$

Source response is $y_s(t) = A = \frac{1}{\omega_o^2}$

For $\xi = 0$, undamped, $p_{1,2} = \pm j\omega_o$

Source free response is $y_{sf}(t) = C\cos(\omega_o t + \theta)$

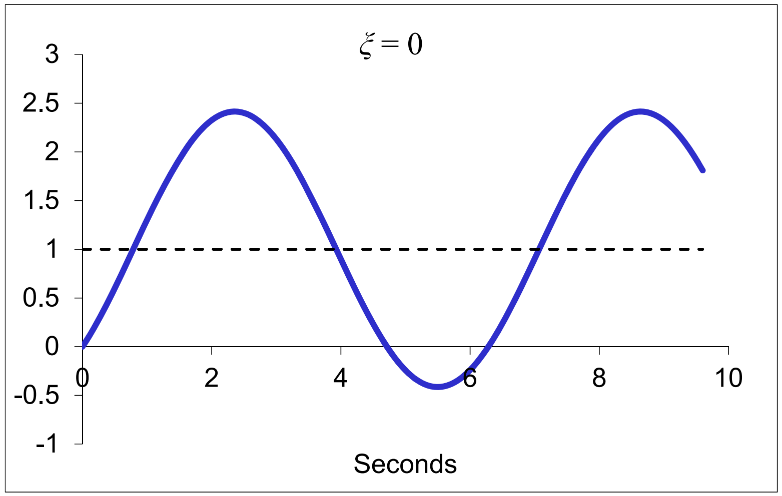
$$y(t) = \frac{1}{\omega_o^2} + C\cos(\omega_o t + \theta)$$

$$y(0) = 0 = \frac{1}{\omega_o^2} + C\cos(\theta); C\cos(\theta) = -\frac{1}{\omega_o^2}$$

$$\dot{y}(0) = 0 - \omega_o C\sin(\theta); C\sin(\theta) = -\frac{\dot{y}(0)}{\omega_o}$$

$$\theta = \tan^{-1}(\dot{y}(0)\omega_o); C = -\frac{1}{\omega_o^2\cos(\tan^{-1}(\dot{y}(0)\omega_o))}$$

$$y(t) = \frac{1}{\omega_o^2} - \frac{1}{\omega_o^2\cos(\tan^{-1}(\dot{y}(0)\omega_o))}\cos(\omega_o t + \tan^{-1}(\dot{y}(0)\omega_o))$$



Poles and Zeroes

- Source Response:

$$A(p)y(t) = B(p)x(t)$$
$$y(t) = \frac{B(p)}{A(p)}x(t)$$
$$y(t) = H(p)x(t)$$
$$H(p) = \frac{B(p)}{A(p)}$$

*$H(p)$ is known as the **system function** or **network response***

- We can think of the solutions of $B(p) = 0$ as the zeroes of the system
- We can think of the solutions of $A(p) = 0$ as the poles of the system

Poles and Zeros Continued

- If we assume that the source *x(t)* has the form $x(t)=e^{st}$ where s is a complex number, we can plot the poles and zeroes in the complex plane to graphically see the response to a particular source function. Note that *H(s)* is a complex number with magnitude and angle.

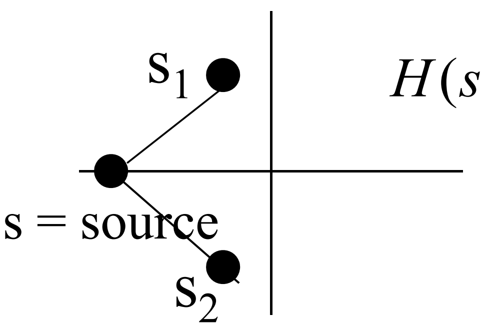
$$y(t) = H(p)x(t) = H(p)e^{st}$$

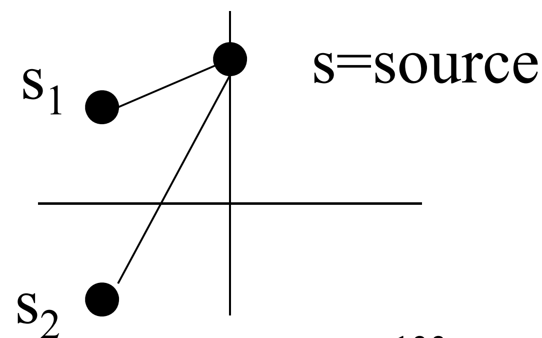
$$= H(s)e^{st}$$

$$\text{since } pe^{st} = \frac{d}{dt}e^{st} = se^{st}$$

For example, if *H(p)* have no zeroes and 2 poles, then

$$H(s) = \frac{A}{(s - s_1)(s - s_2)}$$





Homework

- 2^{st} Order ODEs
  1. Using Matlab, plot the response for following systems. Identity what type of system each is. Submit your code:

$$a)\ddot{x}+10\dot{x}+4x=0; x(0)=5; \dot{x}(0)=0$$

$$b)\ddot{x}+4\dot{x}+4x=0; x(0)=5; \dot{x}(0)=0$$

$$c)\ddot{x}+1\dot{x}+4x=0; x(0)=5; \dot{x}(0)=0$$

$$d)\ddot{x}+4x=0; x(0)=5; \dot{x}(0)=0$$

  2. 2.2 An LTI system is described by the second-order ODE:

$$\ddot{y}+7\dot{y}+10y=x(t)$$

     a. Use the p operator notation to find the roots of the characteristic equation.
     b. Assume y(0)=0, dy(0)/dt=9 and x(t)=0, find y(t).
     c. Now Let x(t) = 10 and same initial conditions, find y(t).

Homework

3. A quadratic low-pass filter is described by the second-order ODE:

$$\ddot{y} + (2\xi\omega_n)\dot{y} + \omega_n^{\ 2} y = x(t)$$

a. The characteristic equation for this ODE in terms of the p operator has complex conjugate roots. Find an algebraic expression for the position of the roots.

b. Let $x(t)$=1. Find the steady-state output.

c. Let $x(t)$=0, $y(0)$=0 and $dy(0)/dt$=10, Find and sketch $y(t)$ for ξ =.5 and ω_n=1

Homework

4. An alternate way of writing the ODE for an underdamped system is

$$\ddot{y} + 2a\dot{y} + (b^2 + a^2)y = x(t)$$

Let a=.5 and b=.86603. Repeat a, b, and c in problem 2.

Homework

5. BioSignals
   – An heart signal is sampled at the rate 250 s/s and is passed to EKG which has an input consisting of a low pass filter. The filter is a resistor and capacitor in series where the output of the filter is taken across the capacitor. What should be the value of the Capacitor if the Resistor is 1k ohms and the time constant of the filter so that the transient response is completed within 1/10 of the sample time? What is the cutoff frequency of this filter?

Homework

6. Respiration may be modeled with the following second order equation, where $y(t)$ is the respiration signal and $x(t)$ is the additional load above the resting respiration on the body:

$$\ddot{y} + \omega_n^2 y = x(t)$$

a. Calculate the resting respiration rate. Assume $y(0)=0$ and $dy(0)/dt=\omega_n$.

b. Calculate the respiration rate when $x(t)=\cos(\omega_n t)$. Assume the same initial conditions as part a.

c. Use Matlab to graph the signals for both parts. Assume $\omega_n=2\pi$.