Sinusoidal Response and Discrete Systems

Lecture #5

3CT.2

Steady State Sinusoidal Response

- When the source-free or transient response ends in a stable system, what remains is the Steady State Response.
- For systems where the source is a Sinusoid, it is called the Steady State Sinusoidal Response.
- Therefore, we look at the solution of the system for the source response.

Steady State Sinusoidal Response

$$a\ddot{y}+b\dot{y}+cy = A\cos\omega t$$

Source response is sinusoidal since the Source is Sinusoidal:

$$y(t) = A_1 \sin\omega t + B_1 \cos\omega t$$

First Derivative: $\dot{y}(t) = A_1\omega\cos\omega t - B_1\omega\sin\omega t$

Second Derivative: $\ddot{y}(t) = -A_1\omega^2 \sin\omega t - B_1\omega^2 \cos\omega t$

Substitute the source response and it's derivatives into the system equation:

$$a(-A_1\omega^2 \sin\omega t - B_1\omega^2 \cos\omega t) + b(A_1\omega\cos\omega t - B_1\omega\sin\omega t) + c(A_1 \sin\omega t + B_1 \cos\omega t) = A\cos\omega t$$

Sort the coefficients of the sine and cosine time dependancy functions

$$([c - a\omega^2]A_1 - b\omega B_1)\sin\omega t + ([c - a\omega^2]B_1 + b_1\omega A_1)\cos\omega t = A\cos\omega t$$

Compare the coeffieients of the left side of the equation to the right side.

Steady State Sinusoidal Response

The coefficients of the the cosine function

$$A = (c - a\omega^2)B_1 + b\omega A_1 \qquad \text{Eqn. 1}$$

The coefficients of the the sine function

$$0 = (c - a\omega^2)A_1 - b\omega B_1 \qquad \text{Eqn. 2}$$

Solve for the unknown coefficients of the source response, A_1 and B_1

$$\text{From Eqn. 2} \Rightarrow B_1 = \frac{(c - a\omega^2)}{b\omega}A_1$$

$$\text{Substituting this into Equ. 1 and solve for } A_1 \Rightarrow A_1 = \frac{b\omega}{[\left((c - a\omega^2)\right)^2 + \left(b\omega\right)^2]}A$$

$$\text{Then for } B_1 \Rightarrow B_1 = \frac{(c - a\omega^2)}{b\omega}\frac{b\omega}{[\left((c - a\omega^2)\right)^2 + \left(b\omega\right)^2]}A = \frac{(c - a\omega^2)}{[\left((c - a\omega^2)\right)^2 + \left(b\omega\right)^2]}A$$

Substituting A_1 and B_1 into $y(t)$

$$y(t) = \frac{A}{[\left((c - a\omega^2)\right)^2 + \left(b\omega\right)^2]}[b\omega \sin\omega t + (c - a\omega^2)\cos\omega t]$$

And combine $y(t)$

$$y(t) = \frac{A}{\sqrt{[\left((c - a\omega^2)\right)^2 + \left(b\omega\right)^2]}}\cos[\omega t - \tan^{-1}(\frac{b\omega}{(c - a\omega^2)})]$$

A Simpler Approach to Steady State Sinusoidal Systems – Frequency Response

- For systems where the source is a sinusoid, we can replace p with $j\omega$ in the system function $H(p)$ to yield in a complex function of $j\omega$, $\mathbf{H}(j\omega)$**,** or phasor form of $j\omega$,

$$\mathbf{H}(j\omega) = A(\omega) + jB(\omega)$$

$$= \sqrt{A(\omega)^2 + B(\omega)^2}\angle \tan^{-1}[\frac{B(\omega)}{A(\omega)}]$$

- We call $\mathbf{H}(j\omega)$ the **<u>Frequency Response</u>**.

$$\{j\omega \rightarrow j2\pi F; \mathbf{H}(j\omega) \rightarrow \mathbf{H}(F)\}$$

Frequency Response

We have $A(p)y(t) = B(p)x(t)$

And if $x(t) = X(\omega)\cos\omega t = \Re e\{\mathbf{X}(j\omega)e^{j\omega t}\}$

where $e^{j\omega t} = \cos\omega t + j\sin\omega t$

then $y(t) = \Re e\{\mathbf{Y}(j\omega)e^{j\omega t}\}$

If we put $x(t)$ in the system we can get $y(t)$

but instead we can use $\mathbf{Y}(j\omega)e^{j\omega t}$ and $\mathbf{X}(j\omega)e^{j\omega t}$

Therefore, $A(p)\mathbf{Y}(j\omega)e^{j\omega t} = B(p)\mathbf{X}(j\omega)e^{j\omega t}$

Frequency Response

Therefore, $A(p)\mathbf{Y}(j\omega)e^{j\omega t} = B(p)\mathbf{X}(j\omega)e^{j\omega t}$

which becomes $\mathbf{Y}(j\omega)A(j\omega)e^{j\omega t} = \mathbf{X}(j\omega)B(j\omega)e^{j\omega t}$

$A(j\omega)\mathbf{Y}(j\omega) = B(j\omega)\mathbf{X}(j\omega)$

Or

$$\mathbf{Y}(j\omega) = \frac{B(j\omega)}{A(j\omega)}\mathbf{X}(j\omega)$$

$$\mathbf{Y}(j\omega) = \mathbf{H}(j\omega)\mathbf{X}(j\omega)$$

$$\mathbf{H}(j\omega) = \frac{B(j\omega)}{A(j\omega)}$$

[The text uses $\omega = 2\pi F \;\; y(t) = \Re e\{\mathbf{H}(F)\mathbf{X}(F)e^{j2\pi Ft}\}$]

Frequency Response Using Phasors

Example:

In our example: $x(t) = A\cos(\omega t) \Rightarrow \mathbf{X}(j\omega) = A\angle 0$

$(ap^2 + bp + c)y(t) = A\cos\omega t \Rightarrow (a(j\omega)^2 + bj\omega + c)\mathbf{Y}(j\omega) = A\angle 0$

$$\mathbf{Y}(j\omega) = \frac{A\angle 0}{c - a\omega^2 + jb\omega} = \frac{A\angle 0}{\sqrt{(c-a\omega^2)^2 + (b\omega)^2}\angle \tan^{-1}(\frac{b\omega}{c-a\omega^2})}$$

$$\mathbf{Y}(j\omega) = \frac{A}{\sqrt{(c-a\omega^2)^2 + (b\omega)^2}}\angle -\tan^{-1}(\frac{b\omega}{c-a\omega^2})$$

$$y(t) = \frac{A}{\sqrt{(c-a\omega^2)^2 + (b\omega)^2}}\cos[\omega t - \tan^{-1}(\frac{b\omega}{c-a\omega^2})]$$

Bode Plots

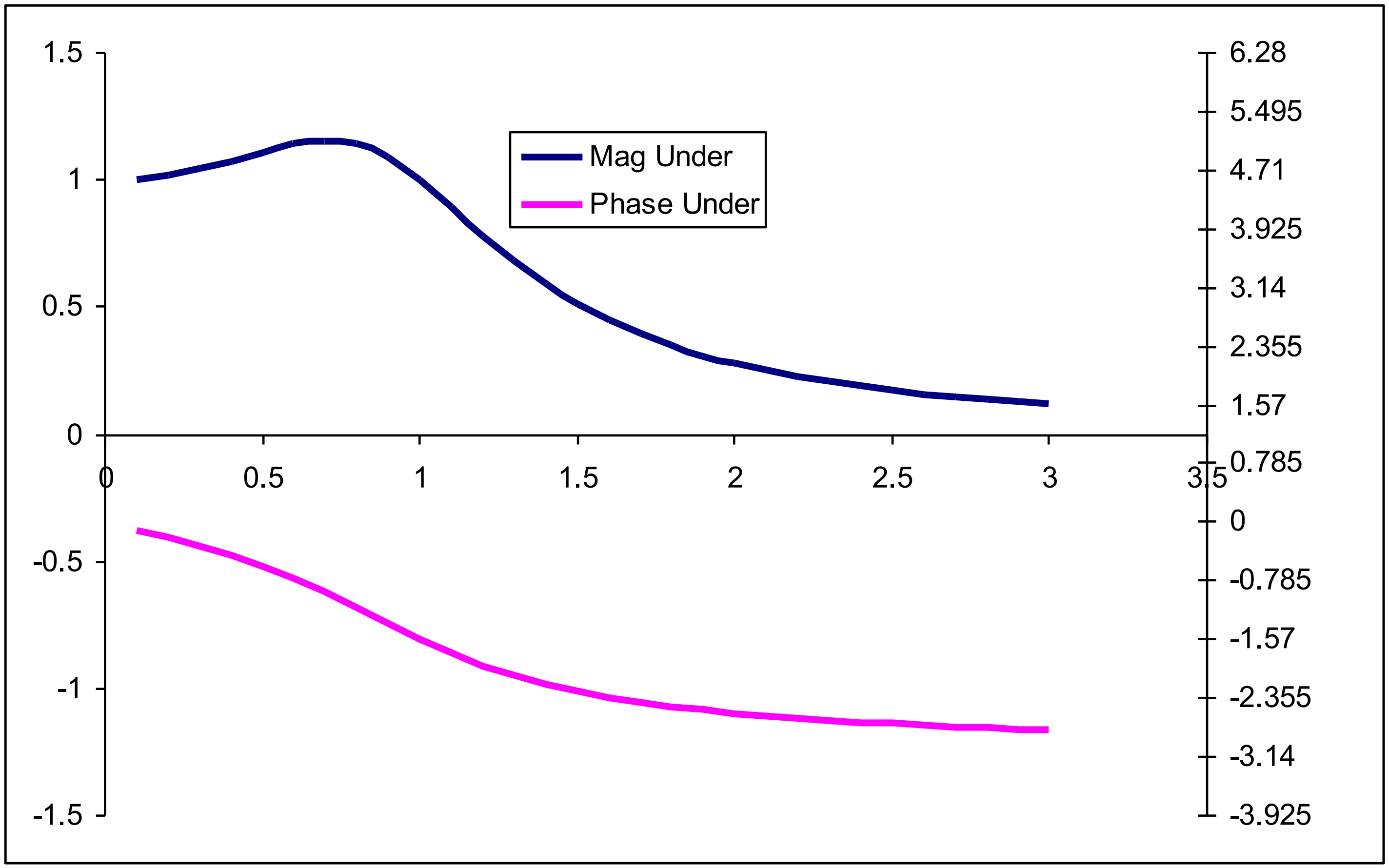
- Plot of the log of the magnitude and angle of the frequency response *H(jω)* on a single logarithmic chart
- Sanity Checks: at $\omega = 0$, $\omega \rightarrow \infty$, at other ω' s (e.g., at poles or zero break frequencies or resonance frequencies)
- From the previous second order example:

$$H(j\omega) = \frac{1}{(c - a\omega^2) + jb\omega}$$

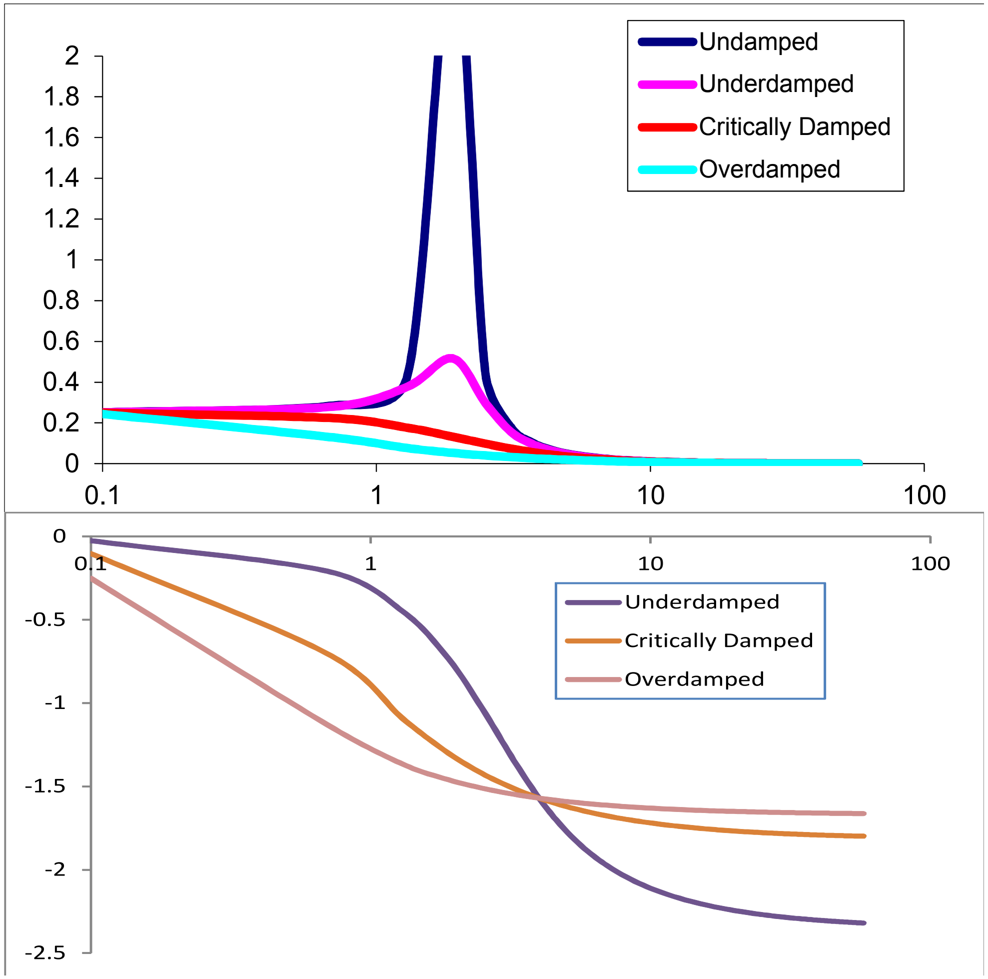
$$|H(j\omega)| = \frac{1}{\sqrt{(c - a\omega^2)^2 + (b\omega)^2}}$$

$$\phi(j\omega) = -\tan^{-1}\left(\frac{b\omega}{c - a\omega^2}\right)$$

2^{nd} Order ODE Bode Plot



2^{nd} Order ODE Bode Plot



Our old Example

Let us look back again at our RL circuit:

$$V\cos\omega t = i(t)R + L\frac{di(t)}{dt}$$

$$L\frac{di(t)}{dt} + Ri(t) = V\cos\omega t$$

$$(pL + R)i(t) = V\cos\omega t$$

$$(j\omega L + R)I(j\omega) = V\angle 0$$

$$I(j\omega) = \frac{V\angle 0}{(j\omega L + R)} = \frac{V}{\sqrt{R^2 + (\omega L)^2}}\angle -\tan^{-1}\frac{\omega L}{R}$$

Discrete Time Equations

- Source Response:

$$a\,y[n+2] + b\,y[n+1] + c\,y[n] = x[n]$$

- Characteristic/Homogeneous Response

$$a\,y[n+2] + b\,y[n+1] + c\,y[n] = 0$$

- Eigenfunctions: $y[n]=A\,z^n$
  Eigenvalues: z

A Discrete Time Example: Mortgage Loan Calculation

- Assumptions:
  - Let $P[i]$ = remaining principal at period i
  - Let r = the interest rate per period
  - N = point at which the loan is paid off (i.e., $P[N] = 0$)
  - Pc = constant periodic payment = $\Delta_P^i + \Delta_I^i$
  - where Δ_P^i is the portion of Pc associated with the payout of the principal for period i
  - where $\Delta_I^i = rP[i]$ is the portion of Pc associated with the payout of the interest for period i

Mortgage Loan Calculation Problem Formulation

- The principal remaining at period *i* equals the principal at period *i-1* less the principal payout at period *i-1*

OR

$$P[i] = P[i-1] - \Delta_P^{i-1} = P[i-1] - \{Pc - rP[i-1]\}$$

OR

$$(1+r)P[i-1] - P[i] = Pc$$

Mortgage Loan Calculation Solution

- Using the eigenfunction = a^i, we test the solution: $P[i] = A_1 a^i + A_2$ and we have

$$(1+r)\{A_1 a^{i-1} + A_2\} - \{A_1 a^i + A_2\} = Pc$$

$$\{(1+r)A_1 a^{i-1} - A_1 a^i\} + (1+r)A_2 - A_2 = Pc$$

$$1)\ (1+r)A_1 a^{i-1} - A_1 a^i = 0 \Rightarrow a = (1+r)$$

$$2)\ \{(1+r)\text{-}1\}A_2 = Pc \Rightarrow A_2 = \frac{Pc}{r}$$

$$P[i] = A_1(1+r)^i + \frac{Pc}{r};\ P[N] = 0 \Rightarrow A_1 = -\frac{Pc}{r(1+r)^N}$$

$$\therefore P[i] = \frac{Pc}{r(1+r)^N}\{(1+r)^N - (1+r)^i\}$$

Mortgage Loan Calculation Solution

- z operators

$$z^{1} \Rightarrow \text{ advance of 1 sample time}$$
$$z^{2} \Rightarrow \text{advance of 2 sample times}$$
$$\vdots$$
$$z^{-1} \Rightarrow \text{ delay of 1 sample time}$$
$$z^{-2} \Rightarrow \text{ delay of 2 sample times}$$
$$\vdots$$

Mortgage Loan Calculation Solution

$$(1+r)P[i-1]-P[i]=Pc$$

$[(1+r)z^{-1}-1]P[i]=Pc$: Source equation

$[(1+r)z^{-1}-1]P[i]=0$: Source Free Homogeneous equation

$(1+r)z^{-1}-1=0$: Characteristic equation

$$\therefore z_1=1+r$$

$$P[i]=K_1+K_2 z_1{}^i=K_1+K_2(1+r)^i$$

Mortgage Loan Calculation Solution

Source equation

$$[(1+r)z^{-1}-1]K_1 = Pc$$

Note: $z^{-1}K_1 = K_1$ since K_1 is a constant

$$[(1+r)-1]K_1 = Pc$$

$$K_1 = \frac{Pc}{r}$$

$$P[i] = \frac{Pc}{r} + K_2(1+r)^i$$

Note: Final condition: $P[N] = 0$

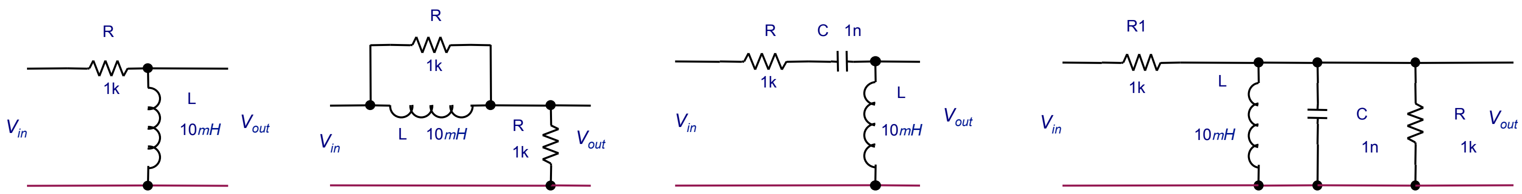
$$P[N] = \frac{Pc}{r} + K_2(1+r)^N = 0$$

$$K_2 = -\frac{Pc}{r(1+r)^N}$$

$$P[i] = \frac{Pc}{r} - \frac{Pc}{r(1+r)^N}(1+r)^i = \frac{Pc}{r(1+r)^N}[(1+r)^N - (1+r)^i]$$

Homework

- Sinusoidal Steady State
  - Calculate the Sinusoidal Steady State Response of the network function for the following networks:



- Bode Plots
  - Draw the Bode Plots for these networks.
  - Use Matlab to plot the Bode Plot, submit your code.
- Discrete ODE
  - Calculate the monthly payment Pc
- 3CT.3.1, 3CT.3.2, 3CT.3.4