Convolution

Lecture #6

2CT.3 – 8

Definition

Convolution is an operation on two functions of time.

The following integral is the definition of convolving $f_1(t)$ with $f_2(t)$:

$$g(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau$$

This says, first take $f_2(t)$ and convolve it doing the following:

1. flip it in time : $f_2(t) \Rightarrow f_2(-t)$
2. displace (shift) it in time by the amount τ seconds: $f_2(-t) \Rightarrow f_2(\tau - t)$
3. take the convolved $f_2(t)$ and multiple it by $f_1(t)$
4. integrate with respect to τ over all time which will produce another function, $g(t)$, of time, t.

Properties of Convolution

- First some shorthand:

$$\int f_1(\tau) f_2(t-\tau) d\tau \Rightarrow f_1(t) \otimes f_2(t)$$

- Commutative:

$$f_1(t) \otimes f_2(t) = f_2(t) \otimes f_1(t)$$

- Associative:

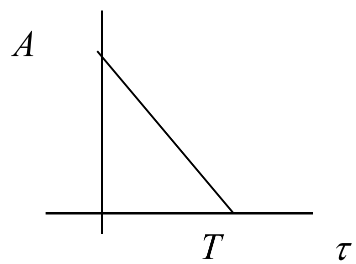
$$f_1(t) \otimes [f_2(t) \otimes f_3(t)] = [f_1(t) \otimes f_2(t)] \otimes f_3(t)$$

- Distributive:

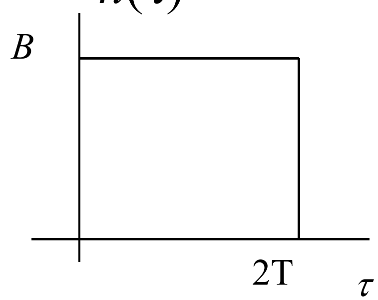
$$f_1(t) \otimes [f_2(t) + f_3(t)] = [f_1(t) \otimes f_2(t)] + [f_1(t) \otimes f_3(t)]$$

A Graphical Example of How to Perform a Convolution

$f(\tau) = A(1- \tau / T)$

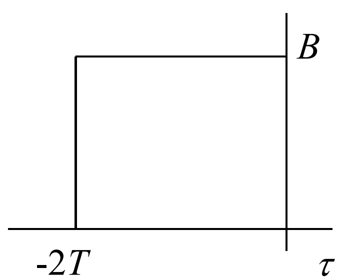


$h(\tau)$

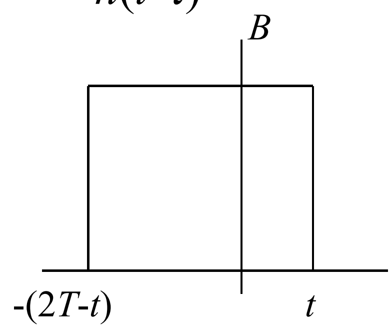


$$C = \int f(\tau)h(t-\tau)\,d\tau$$

$h(-\tau)$



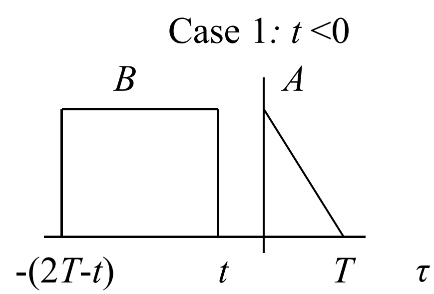
$h(t-\tau)$



A Graphical Example of How to Perform a Convolution

We need to look at 5 cases:

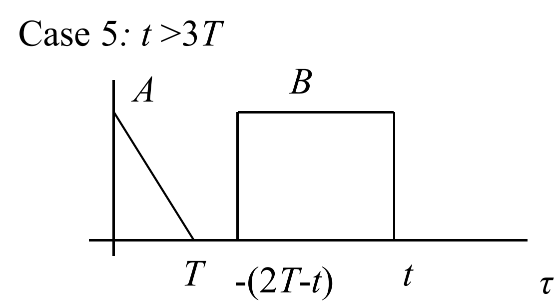
1) $t < 0$ For this case 1, C=0 since there is no overlap.



2) 2) $0 < t < T$

3) 3)$T < t < 2T$,

4) $2T < t < 3T$

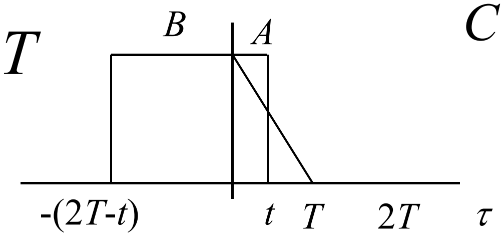


5) $t > 3T$ For this case 1, C=0 since there is no overlap.

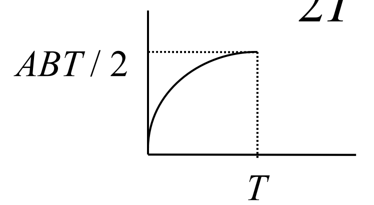
Graphical Convolution Example Continued

Case 2: $0 < t < T$

$f(t)\,h(t\text{-}t),\ 0 < t < T$

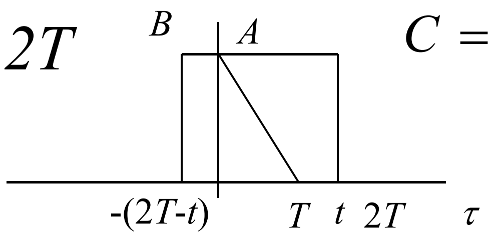


$$C = \int_0^t AB\left(1 - \frac{\tau}{T}\right)d\tau = AB\left(t - \frac{t^2}{2T}\right)$$

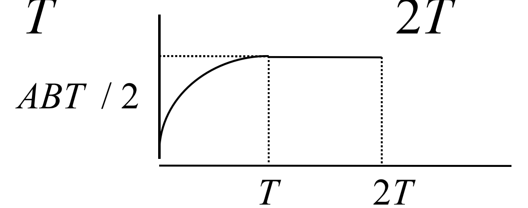


Case 3: $T < t < 2T$

$f(t)\,h(t\text{-}\tau),\ T < t < 2T$

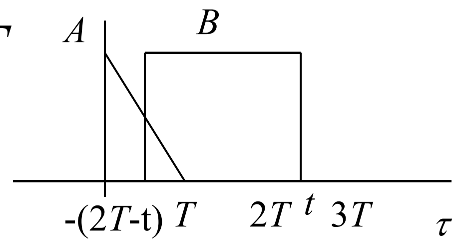


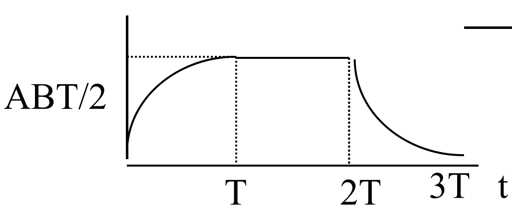
$$C = \int_0^T AB\left(1 - \frac{\tau}{T}\right)d\tau = AB\left(T - \frac{T^2}{2T}\right) = \frac{ABT}{2}$$



Case 4: $2T < t < 3T$

$f(t)\,h(t\text{-}t),\ 2T < t < 3T$

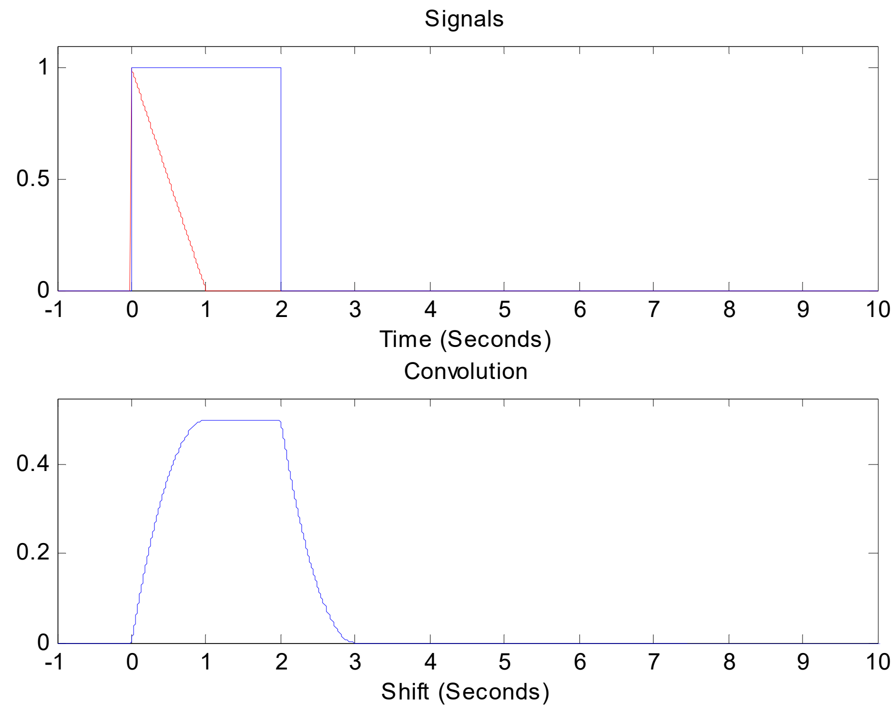




$$C = \int_{-(2T-t)}^{T} AB\left(1 - \frac{\tau}{T}\right)d\tau =$$

$$AB\left[\frac{T}{2} + (2T - t) + \frac{(2T - t)^2}{2T}\right]$$

Matlab Code




```
clear all;
endpulse=2;
ts=.001;
endpoint=10;
n=-endpoint:ts:endpoint;
nn=-endpoint*2:ts:endpoint*2;
pulser=(n>=0)&(n<=endpulse);
pulse=1*pulser;
tripulse=(n>=0)&(n<=1);
tri=(1-n).*tripulse;
subplot(2,1,1)
plot(n,tri,'r',n,pulse,'b');
title('Signals');
xlabel('Time (Seconds)');
axis([-1 10 min([min(tri) min(pulse)]) 1.1*max([max(tri) max(pulse)])]);
c=conv(pulse,tri)*ts;
subplot(2,1,2)
plot(nn,c);
title('Convolution');
xlabel('Shift (Seconds)');
axis([-1 10 min(c) 1.1*max(c)]);
```

Integration of Convolution Integral

Case 2: $0 < t < T$

$$C = \int_0^t AB(1-\frac{\tau}{T})d\tau = AB(-T)\frac{1}{2}(1-\frac{\tau}{T})^2 \Big|_0^t = AB(-T)\frac{1}{2}\{(1-\frac{t}{T})^2 - 1\}$$

$$= AB(-T)\frac{1}{2}\{(1-\frac{2t}{T}+\frac{t^2}{T^2})-1\} = AB(-T)\frac{1}{2}\{-\frac{2t}{T}+\frac{t^2}{T^2}\}$$

$$= AB(-T)\frac{1}{2}\{-\frac{2t}{T}+\frac{t^2}{T^2}\} = AB\{t-\frac{t^2}{2T}\}$$

Case 3: $T < t < 2T$ (From the integration from Case 2)

$$C = \int_0^T AB(1-\frac{\tau}{T})d\tau = AB(-T)\frac{1}{2}(1-\frac{\tau}{T})^2 \Big|_0^T = AB\{T-\frac{T^2}{2T}\}$$

$$= AB\{T-\frac{T}{2}\} = AB\frac{T}{2}$$

Case 4: $2T < t < 3T$ (From the integration from Case 2)

$$C = \int_{-(2T-t)}^T AB(1-\frac{\tau}{T})d\tau = AB(-T)\frac{1}{2}(1-\frac{\tau}{T})^2 \Big|_{-(2T-t)}^T = AB(-T)\frac{1}{2}\{(1-\frac{T}{T})^2 - (1-\frac{-(2T-t)}{T})^2\}$$

$$= AB(-T)\frac{1}{2}\{(0)^2 - (1+\frac{(2T-t)}{T})^2\} = AB(T)\frac{1}{2}\{(1+\frac{(2T-t)}{T})^2\}$$

$$= AB(T)\frac{1}{2}\{(1+\frac{2(2T-t)}{T}+\frac{(2T-t)^2}{T^2})\} = AB\{(\frac{T}{2}+(2T-t)+\frac{(2T-t)^2}{2T})\}$$

Convolution and Systems

- For an LTI system, let's define $h(t)$ as the system response to a unit impulse source, $\delta(t)$.
- Then the following must be true:

$$x(t) \overset{LTI}{\rightarrow} y(t) \quad x(t) \text{ input to the system yields output } y(t)$$

$$\delta(t) \overset{LTI}{\rightarrow} h(t) \quad \delta(t) \text{ input to the system yields output } h(t)$$

$$\delta(t-k\Delta) \overset{LTI}{\rightarrow} h(t-k\Delta) \text{ time invariance, time shift by } k\Delta$$

$$x(k\Delta)\delta(t-k\Delta) \overset{LTI}{\rightarrow} x(k\Delta)h(t-k\Delta) \text{ scalar, multiply by } x(k\Delta)$$

$$\sum_k x(k\Delta)\delta(t-k\Delta) \overset{LTI}{\rightarrow} \sum_k x(k\Delta)h(t-k\Delta) \text{ superposition, add up for all value of time shift } k$$

$$\sum_k x(k\Delta)\delta(t-k\Delta)\Delta \overset{LTI}{\rightarrow} \sum_k x(k\Delta)h(t-k\Delta)\Delta \text{ scalar, multiply by } \Delta$$

Convolution and Systems Continued

Construct $x(t)$ as the sum of k unit impulse slices. The first expression represents the k slices of the source totaled as k approaches infinity.

Left Side Equals

$$\lim_{\substack{\Delta\to 0\\ k\to\infty\\ k\Delta\to\tau}} \sum_k x(k\Delta)\delta(t-k\Delta)\Delta \ \text{ approaches} \int_\tau x(\tau)\delta(t-\tau)\,d\tau = x(t)$$

$$\text{Recall}: \int_{-\infty}^{\infty} f(t)\delta(t-\tau)\,dt = f(\tau)$$

Construct $y(t)$ as the sum of k slices of the response due to an unit impulse function: $x(k\Delta)h(t\text{-}k\Delta)$. The integral on the right is the **convolution of $x(t)$ and $h(t)$**.

Right Side Equals

$$\lim_{\substack{\Delta\to 0\\ k\to\infty\\ k\Delta\to\tau}} \sum_k x(k\Delta)h(t-k\Delta)\Delta \ \text{ approaches} \int_\tau x(\tau)h(t-\tau)\,d\tau$$

$$\sum_k x(k\Delta)\delta(t-k\Delta)\Delta \to \sum_k x(k\Delta)h(t-k\Delta)\Delta$$

$$x(t) \to \int_\tau x(\tau)h(t-\tau)\,d\tau = y(t)$$

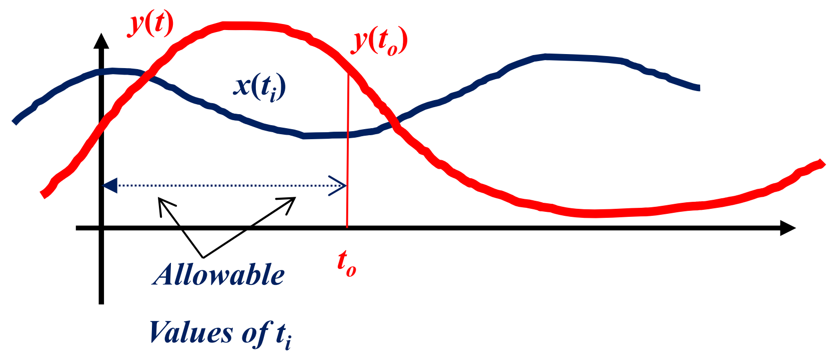
Convolution and Systems Continued

$$y(t) = \int_{\tau} x(\tau)h(t-\tau)\,d\tau$$

This result is very important since it says that if one knows the impulse response of a system then the output response for any given input source can be found by convolving the input with the impulse response.

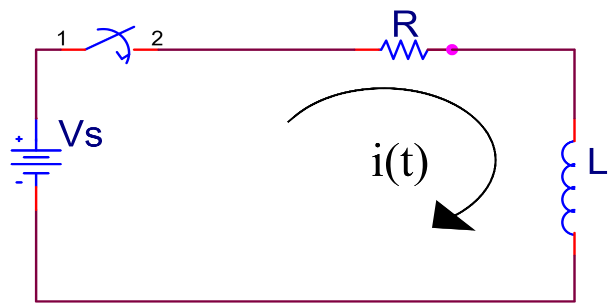
Impulse Response and Causality

- $y(t_o)$ does not depend on $x(t_i)$ that occur at times after t_o, $t_i > t_o$.



- $y(t_o) = \int_{-\infty}^{\infty} h(\tau)x(t_o - \tau)d\tau = \int_{-\infty}^{0^-} h(\tau)x(t_o - \tau)d\tau + \int_{0}^{\infty} h(\tau)x(t_o - \tau)d\tau$ break up into 2 integrals: one for positive τ and one for negative τ
- For the positive τ integral $\int_{0}^{\infty} h(\tau)x(t_o - \tau)d\tau$, if τ is positive, then $t_o - \tau < t_o$, and for $x(t_i = t_o - \tau) = x(t_o - \tau)$ we have $t_i < t_o$
- For the negative τ integral $\int_{-\infty}^{0^-} h(\tau)x(t_o - \tau)d\tau$, if τ is negative, then $t_o - \tau > t_o$, and for $x(t_i = t_o - \tau) = x(t_o - \tau)$ we have $t_i > t_o$ THIS CAN'T HAPPEN IN A REAL CAUSAL SYSTEM.
- Therefore for $y(t)$ to be causal this $\int_{-\infty}^{0^-} h(\tau)x(t_o - \tau)d\tau$ must be zero and for this to happen $h(t) = 0, t < 0$.

Calculating the Unit Impulse Response, h(t)

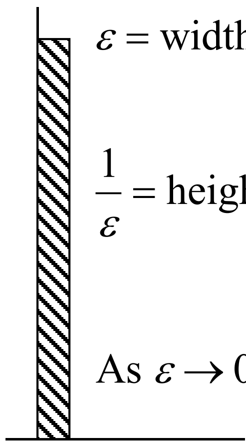


Let's first look at 2 methods:

1. Narrow Pulse approximation
2. Differentiating $u(t)$

Better Methods to Come

Narrow Pulse Approximation



ε = width of the pulse

$\frac{1}{\varepsilon}$ = height of the pulse

As $\varepsilon \to 0$, the narrow pulse $= \frac{1}{\varepsilon}\{u(t) - u(t-\varepsilon)\} \to \delta(t)$

So then let's first look at the response to $u(t)$:

$$Vs\, u(t) = i(t)R + L\frac{di(t)}{dt}$$

$$i(t) = Vs/R\ (1 - e^{-(R/L)t})u(t)$$

Now we construct the narrow pulse response:

$$i(t) = \frac{Vs}{R}\frac{1}{\varepsilon}\{(1 - e^{-(\frac{R}{L})t})u(t) - (1 - e^{-(\frac{R}{L})(t-\varepsilon)})u(t-\varepsilon)\}$$

Narrow Pulse Approximation Continued

$$i(t) = (Vs/R)(1/\varepsilon)\ [(1\text{-}e^{-(R/L)t})u(t) - (1\text{-}e^{-(R/L)(t-\varepsilon)})u(t\text{-}\varepsilon)]$$

OR

$$i(t) = \begin{cases} 0, & \text{for } t < 0 \\ (Vs/R)(1/\varepsilon)\ [(1\text{-}e^{-(R/L)t})] & \text{for } 0 \leq t < \varepsilon \\ (Vs/R)(1/\varepsilon)\ [(1\text{-}e^{-(R/L)t})] - (1\text{-}e^{-R/L(t-\varepsilon)})], & \text{for } t > \varepsilon \end{cases}$$

OR

$$i(t) = \begin{cases} 0, & \text{for } t < 0 \\ (Vs/R)(1/\varepsilon)\ [(1\text{-}e^{-(R/L)t})], & \text{for } 0 \leq t < \varepsilon \\ (Vs/R)(1/\varepsilon)\ [(e^{R/L\varepsilon} - 1)e^{-(R/L)t})], & \text{for } t > \varepsilon \end{cases}$$

Taylor Series Approximation for e^x

$$e^{ax} = \frac{(ax)^0}{0!} + \frac{(ax)^1}{1!} + \frac{(ax)^2}{2!} + \frac{(ax)^3}{3!} + \ldots$$

for x ≈ 0, we can drop the higher order terms:

$$e^{ax} \approx \frac{(ax)^0}{0!} + \frac{(ax)^1}{1!} = 1 + ax$$

$$e^{-\frac{R}{L}t} = 1 - \frac{R}{L}t$$

$$1 - e^{-\frac{R}{L}t} = 1 - (1 - \frac{R}{L}t) = \frac{R}{L}t$$

$$e^{\frac{R}{L}\varepsilon} - 1 = 1 + \frac{R}{L}\varepsilon - 1 = \frac{R}{L}\varepsilon$$

Narrow Pulse Approximation Continued

Applying the approximation for e^x, Vs=1 (the unit impulse function) and α = R/L

$$i(t)=\begin{cases} 0, & \text{for } t<0 \\ (Vs/R)(1/\varepsilon)\,[(1-e^{-R/Lt})] = (Vs/R)(1/\varepsilon)(R/Lt) = (Vs/R)(R/L\ t/\varepsilon), & \text{for } 0 \le t < \varepsilon \\ (Vs/R)(1/\varepsilon)\,[(e^{+R/L\varepsilon}-1)e^{-R/Lt}] = (Vs/R)(1/e)(R/L\ \varepsilon)e^{-R/Lt}, & \text{for } t > \varepsilon \end{cases}$$

$$i(t)=\begin{cases} 0, & \text{for } t<0 \\ (1/R)(R/L\ t/\varepsilon) = R/L\ /R = 1/L, & \text{for } 0 \le t < \varepsilon \\ (1/R)\ R/L\ e^{-R/L\,t}, & \text{for } t > \varepsilon \end{cases}$$

$$h(t) = \lim_{\varepsilon \to 0} i(t) = \frac{R/L}{R} e^{-R/Lt} u(t) = \frac{1}{L} e^{-R/Lt} u(t)$$

Differentiating the Unit Step function

The response due to a Unit Step function is $i(t) = Vs/R\ (1\text{-}e^{-\alpha t})u(t)$ and since

$\delta(t) = \dfrac{du(t)}{dt}$, then $h(t) = \dfrac{di(t)}{dt}$

$$i(t) = \frac{1}{R}(1 - e^{-\alpha t})u(t)$$

$$h(t) = \frac{di(t)}{dt} = \frac{1}{R}\frac{d}{dt}(1 - e^{-\alpha t})u(t) = \frac{1}{R}\frac{d}{dt}[u(t) - e^{-\alpha t}u(t)] = \frac{1}{R}[\frac{d}{dt}u(t) - \frac{d}{dt}e^{-\alpha t}u(t)]$$

$$= \frac{1}{R}[\delta(t) - \{u(t)\frac{d}{dt}e^{-\alpha t} + e^{-\alpha t}\frac{d}{dt}u(t)\}] = \frac{1}{R}[\delta(t) - \{-\alpha e^{-\alpha t}u(t) + e^{-\alpha t}\delta(t)\}]$$

$$h(t) = \frac{1}{R}[\delta(t) + \alpha e^{-\alpha t}u(t) - e^{-\alpha t}\delta(t)]$$

$$= \frac{1}{R}[(1 - e^{-\alpha t})\delta(t) + \alpha e^{-\alpha t}u(t)]$$

$$= \frac{1}{R}\alpha e^{-\alpha t}u(t)$$

Convolution for Discrete Systems

- For an LTI system, let's define $h[n]$ as the system response to a unit impulse source, $\delta[n]$.
- $\delta[n]=1$, $n=0$ and 0 for $n \neq 0$
- We have:

$$x[n]= \Sigma\, x[m]\delta[n\text{-}m]$$

$$y[n]=\Sigma\, x[m]\, h[n\text{-}m]$$

- In addition the same convolution properties hold:
  - Commutative $f_1[n]\otimes f_2[n] = f_2[n]\otimes f_1[n]$
  - Associative $f_1[n]\otimes\{f_2[n]\otimes f_3[n]\}=\{f_1[n]\otimes f_2[n]\}\otimes f_3[n]$
  - Distributive $f_1[n]\otimes\{f_2[n]+f_3[n]\}=\{f_1[n]\otimes f_2[n]\}+\{f_1[n]\otimes f_3[t]\}$

Stability of Systems

- If a system is stable, then if the input is bounded then the output must be bounded i.e., Bounded Input, Bounded Output (BIBO), the following must be true:

$$x(t) < \infty \;\; \{x[n] < \infty\}, \text{ then } y(t) < \infty \;\; \{y[n] < \infty\}$$

$$y(t) = \int h(\tau)x(t-\tau)d\tau < \int h(\tau)x_{\max}d\tau = x_{\max}\int h(\tau)d\tau < \infty$$

$$\{y[n] = \sum h[m]x[n-m] = x_{\max}\sum h[m] < \infty\}$$

OR

$$\int h(\tau)d\tau < \infty \;\; \{\sum h[m] < \infty\}$$

- However, this is not always the case.
  - Positive Feedback causes instability

What is needed for BIBO

- For a continuous time system, the poles of $H(p)$ must lie within the left hand complex plane and not on the imaginary axis such that Re $s_i < 0$ where s_i are the poles of $H(p)$. This will assure that the free response will be damped and not grow exponentially. ***THIS IS WHY WE STUDIED SOLUTIONS OF LINEAR ODE IN TERMS OF SOURCE-FREE AND SOURCE COMPONENTS.***

- ***THIS IMPLIES THAT H(p) AND THE IMPULSE RESPONSE, h(t), MAY BE RELATED.***

What is needed for BIBO

For a discrete time system, the eigenvalues of $h[n]$ must lie within the unit circle such that $z_i < 1$ where z_i are the eigenvalues of $h[n] = \sum_i A_i z_i^{\,n}$.

This will assure that free response will not diverge and $\sum_n h[n] = \sum_n \sum_i A_i \; z_i^{\,n} < \infty.$

Using the formula for the partial sums of a geometric series,
where N is the number of roots of the Characteristic Equation

$$\lim_{L\to\infty} \sum_{n=0}^{L-1} h[n] = \lim_{L\to\infty} \sum_{n=0}^{L-1} \sum_{i=1}^{N} z_i^{\,n} = \lim_{L\to\infty} \sum_{i=1}^{N} \sum_{n=0}^{L-1} z_i^{\,n} = \lim_{L\to\infty} \sum_{i=1}^{N} \frac{1 - z_i^{\,L}}{1 - z_i} \to \sum_{i=1}^{N} \frac{1}{1 - z_i}; \text{ provided } |z_i| < 1$$

$$\text{if } |z_i| \ge 1, \lim_{L\to\infty} \sum_{n=0}^{L-1} \sum_{i=1}^{N} z_i^{\,n} = \lim_{L\to\infty} \sum_{i=1}^{N} \left| \frac{1 - z_i^{\,L}}{1 - z_i} \right| \to \infty$$

$$z_i = |z_i| \angle \text{angle}(z_i)$$

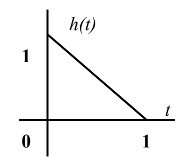
$$z_i^{\,L} = |z_i|^L \angle (L \times \text{angle}(z_i))$$

$$\lim_{L\to\infty} z_i^{\,L} = \lim_{L\to\infty} \{|z_i|^L \angle (L \times \text{angle}(z_i))\} = \lim_{L\to\infty} \{|z_i|^L\} \angle \{-\pi < L \times \text{angle}(z_i) \le \pi\}$$

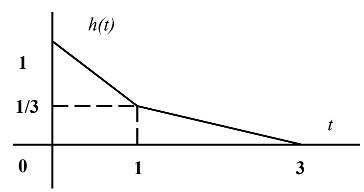
$$\lim_{L\to\infty} \{|z_i|^L\} = 0; z_i < 1 \text{ and } \lim_{L\to\infty} \{|z_i|^L\} \to \infty; z_i \ge 1$$

Homework

- Convolution Verify your all your results of these convolution problems using Matlab and its conv function.
  - Problem (1)
    - Assume that a system response is given by the following:



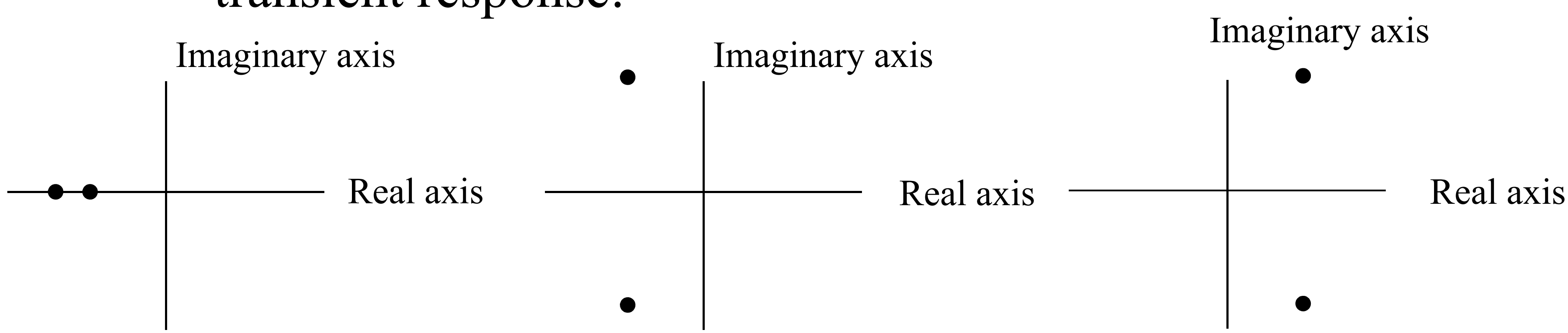
    - Sketch the response to a) *u(t),* b) *u(t)-u(t-a)* for a=0.5, a=1, and a=5, and c) evaluate $e^{-t}\,u(t)$ at *t=1* and *t=2*
  - Problem (2)
    - Assume that a system response is given by the following:



    - Evaluate the response to $te^{-t}\,u(t)$ at *t=1* and *t=3*

Homework

- Stability
  - Determine the stability of the following systems with poles in the complex plane, describe the form of the transient response:



- 2CT.3.1, 2CT.3.2