Signal Analysis

Lecture #7

5CT.1-2,4

How to Analyze Different Classes of Signals

- Classes of Signals
  - Periodic vs. Non Periodic
  - Continuous vs. Discrete
  - Bounded vs. Non Bounded
  - Symmetries
- Use Mathematical Transformations
  - such as Fourier Series and Fourier, Laplace, & Z transforms
  - to analyze Signal Properties
    - Frequencies which make up signal: Spectrum
    - Energy Content
  - to analyze and design systems which process these signals
    - Filters
    - etc

Fourier Series

- A method for approximating a signal
- A means to analyze a signal
- Applies to either continuous or discrete signals
- Need to understand/review some background, foundations, and assumptions

Related Sources Theorem

- If we know the response to a source, then the response to the derivative/integral of the source is the derivative/integration of the response to the source.
- An intuitive proof:

$$x(t) \rightarrow y(t)$$

$$\frac{dx(t)}{dt} \rightarrow \frac{dy(t)}{dt}$$

$$A(p)y(t) = B(p)x(t)$$

$$\frac{d[A(p)y(t)]}{dt} = \frac{d[B(p)x(t)]}{dt}$$

$$A(p)\frac{dy(t)}{dt} = B(p)\frac{dx(t)}{dt}$$

$$\int x(t)dt \rightarrow \int y(t)dt$$

$$\int A(p)y(t)dt = \int B(p)x(t)dt$$

$$A(p)\int y(t)dt = B(p)\int x(t)dt$$

Taylor Series Approximation of a Signal

- From calculus, if we have a single-valued function that is continuous and has continuous derivatives, it can be approximated as

$$f(t) \approx f_a(t) = f(t_o) + \frac{df(t)}{dt}\Big|_{t=t_0}(t-t_o) + \frac{d^2 f(t)}{dt^2}\Big|_{t=t_0}(t-t_o)^2 + \cdots$$
$$\cdots + \frac{d^{n-1} f(t)}{dt^{n-1}}\Big|_{t=t_0}(t-t_o)^{n-1} + R_n(t)$$

- Assuming that $f(t)$ is the source function, using the related sources theorem, we know the response to a constant source, then we can get the response to any function t^n by successful integration and then use superposition to get the full response due to $f_a(t)$
- $R_n(t)$ can be considered to be the error between $f(t)$ and $f_a(t)$ and gets smaller as more terms are added

An Example

$$f(t) = \cos\frac{\pi t}{2};\ \ f_a(t) = a_0 + a_1 t + a_2 t^2$$

- How do we choose the coefficients, the a_i' s, to get best approximation of *f(t)* within the interval –1 < *t* < +1?

$$f(t) - f_a(t) = 0$$

- Let' s choose them that at t = -1,0,+1,

$$f_a(-1) = a_o - a_1 + a_2 = \cos\frac{-\pi}{2} = 0$$

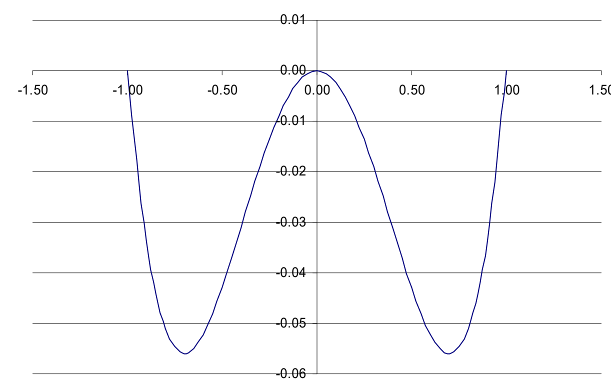
$$f_a(0) = a_o = 1$$

$$f_a(1) = a_o + a_1 + a_2 = \cos\frac{\pi}{2} = 0$$

$$a_o = 1, a_1 = 0, a_2 = -1$$

$$f_a(t) = 1 - t^2$$

$$\varepsilon(t) = \cos\frac{\pi t}{2} - (1 - t^2)$$



The Error Between f(t) & $f_a(t)$

- Object: Choose the a_i' s to minimize the error $\varepsilon(t) = f(t) - f_a(t)$ over the interval of the approximation, but
  - Average error is not a good criterion since we can have large deviations which cancel each other out. Example: $\varepsilon(t) = sin\ t$ over the period 0 to 2π .
- Instead try to minimize the average value of

$$E^2 = \frac{1}{t_1 - t_2}\int \varepsilon^2 dt = \frac{1}{t_1 - t_2}\int (f(t) - f_a(t))^2 dt$$

which is known as the mean squared error.

An Example

$\varepsilon(t) = f(t) - (a_0 + a_1 t + a_2 t^2)$ over the interval $-1 < t < +1$

$$E^2 = \frac{1}{2}\int_{-1}^{+1}[f(t)]^2 dt - \int_{-1}^{+1}(a_0 + a_1 t + a_2 t^2) f(t) dt + \frac{1}{2}\int_{-1}^{+1}(a_0 + a_1 t + a_2 t^2)^2 dt$$

To choose the a_k's to minimize the mean squared error, we must have: $\dfrac{\partial E^2}{\partial a_k} = 0, \ \dfrac{\partial^2 E^2}{\partial a_k^{\ 2}} > 0$

$$\frac{\partial E^2}{\partial a_o} = -\int_{-1}^{+1} f(t)dt + \frac{1}{2}\frac{\partial}{\partial a_o}[\int_{-1}^{+1}(a_0 + a_1 t + a_2 t^2)^2 dt = -\int_{-1}^{+1} f(t)dt + 2a_0 + \frac{2}{3}a_2 = 0 \qquad \frac{\partial^2 E^2}{\partial a_o^{\ 2}} = 2$$

$$\frac{\partial E^2}{\partial a_1} = -\int_{-1}^{+1} tf(t)dt + \frac{1}{2}\frac{\partial}{\partial a_1}[\int_{-1}^{+1}(a_0 + a_1 t + a_2 t^2)^2 dt = -\int_{-1}^{+1} tf(t)dt + \frac{2}{3}a_1 = 0 \qquad \frac{\partial^2 E^2}{\partial a_1^{\ 2}} = \frac{2}{3}$$

$$\frac{\partial E^2}{\partial a_2} = -\int_{-1}^{+1} t^2 f(t)dt + \frac{1}{2}\frac{\partial}{\partial a_2}[\int_{-1}^{+1}(a_0 + a_1 t + a_2 t^2)^2 dt = -\int_{-1}^{+1} t^2 f(t)dt + \frac{2}{3}a_0 + \frac{2}{5}a_2 = 0 \qquad \frac{\partial^2 E^2}{\partial a_2^{\ 2}} = \frac{2}{5}$$

Since the second partials are positive we will have a minimum. The minimum is E^2 =.017. But can we do better?

Can we do better?

- Yes, choose more terms, $f_a(t)=a_0+a_1t+a_2t^2+a_3t^3+a_4t^4$
- Or better yet, choose different approximating functions that are orthogonal in the interval, i.e., choose

$$f_a(t)=A_0 g_0(t) + A_1 g_1(t) + A_2 g_2(t) + \cdots + A_n g_n(t)$$

such that

$$\int_{t_1}^{t_2} g_k(t) g_j(t)\, dt = 0 \text{ for } k \neq j$$

$$\int_{t_1}^{t_2} g_k(t) g_j(t)\, dt = G_k \text{ for } k = j$$

Orthogonal Functions

Using $f_a(t)=A_0\,g_0(t)+A_1\,g_1(t)+A_2\,g_2(t)+\cdots+A_n\,g_n(t)$ over the interval T and choose the A_n*'s* to minimize E^2, we have:

$$E^2=\frac{1}{T}\int_T[f(t)-f_a(t)]^2\,dt$$

$$=\frac{1}{T}\left[\int_T f(t)^2\,dt-2\int_T f_a(t)f(t)\,dt+\int_T f_a(t)^2\,dt\right]$$

$$\frac{\partial E^2}{\partial A_k}=\frac{1}{T}\frac{\partial}{\partial A_k}\left[-2\int_T f_a(t)f(t)\,dt+\int_T f_a(t)^2\,dt\right]=0$$

Where $\frac{\partial E^2}{\partial A_k}$ represents a set $k+1$ simultaneous equations

Note: $\int_T f(t)^2\,dt$ is sometimes called the quadratic content or energy associated with $f(t)$ in interval T

Coefficients of Orthogonal Functions

It can be shown that the first integral of each set of the *k+1* equations is:

$$\frac{1}{T}\frac{\partial}{\partial A_k}(-2\int_T f_a(t)f(t)\,dt)$$

$$=\frac{1}{T}\frac{\partial}{\partial A_k}[-2\int_T (A_0 g_0(t)+A_1 g_1(t)+\cdots+A_n g_n(t))f(t)\,dt]$$

$$=\frac{-2}{T}[\int_T g_k(t)f(t)\,dt]$$

And applying the orthogonal property to the second integral, we have :

$$\frac{1}{T}\frac{\partial}{\partial A_k}\int_T f_a(t)^2\,dt=\frac{1}{T}\frac{\partial}{\partial A_k}\int_T (A_0 g_0(t)+A_1 g_1(t)+\cdots+A_n g_n(t))^2\,dt$$

$$=\frac{1}{T}2\int_T (A_0 g_0(t)+A_1 g_1(t)+\cdots+A_n g_n(t))g_k(t)\,dt=A_k\frac{1}{T}2\int_T g_k(t)^2\,dt=A_k\frac{2}{T}G_k$$

Coefficients of Orthogonal Functions

$$\frac{1}{T}\frac{\partial}{\partial A_k}[-2\int_T f_a(t)f(t)\,dt + \int_T f_a(t)^2\,dt]$$

$$= \frac{-2}{T}[\int_T g_k(t)f(t)\,dt] + A_k\frac{1}{T}2\int_T g_k(t)^2\,dt$$

$$= \frac{-2}{T}[\int_T g_k(t)f(t)\,dt] + A_k\frac{2}{T}G_k = 0$$

And, at last we have: $$A_k = \frac{\int_T g_k(t)f(t)dt}{\int_T [g_k(t)]^2\,dt} = \frac{\int_T g_k(t)f(t)dt}{G_k}$$

What Functions are Orthogonal

- There is a class of polynomials which form an orthogonal set
- But a better choice are the sinusoidal functions:

$$f_a(t) = C_0 + \sum_{k=1}^{N}[A_k \cos(\frac{2\pi kt}{T}) + B_k \sin(\frac{2\pi kt}{T})]$$

$$= C_0 + \sum_{k=1}^{N} C_k \cos(\frac{2\pi kt}{T} + \psi_k)$$

where $C_k = \sqrt{A_k^2 + B_k^2}$

$$\psi_k = \tan^{-1}(\frac{-B_k}{A_k})$$

Some Properties of Sinusoids Which Make Things Neater

Recall that $e^{jt} = \cos t + j \sin t$

$$\cos t = ½(e^{jt} + e^{-jt})$$

$$\sin t = 1/(2j)(e^{jt} - e^{-jt})$$

And for the complex number $\mathbf{s} = \alpha + j\omega,$ there is its conjugate $\mathbf{s}^* = \alpha - j\omega$. Furthermore, $\mathbf{s} + \mathbf{s}^* = 2\mathrm{Re}[\mathbf{s}] = 2\alpha$

Therefore, let's rewrite $f_a(t)$ in terms of complex series of $e^{j\omega t}$ functions and their conjugates.

We now call this the <u>Fourier Series</u> of a function within an interval of *T*.

Fourier Series

$$f_a(t) = C_0 + \sum_{k=1}^{N} C_k \cos(\frac{2\pi kt}{T} + \psi_k)$$

$$= C_0 + C_1 \cos(\frac{2\pi 1t}{T} + \psi_1) + \cdots + C_k \cos(\frac{2\pi kt}{T} + \psi_k) \cdots + C_N \cos(\frac{2\pi Nt}{T} + \psi_N)$$ Expanding the sum

$$= C_0 + \frac{C_1}{2} e^{j(\frac{2\pi 1t}{T} + \psi_1)} + \frac{C_1}{2} e^{-j(\frac{2\pi 1t}{T} + \psi_1)} + \cdots + \frac{C_k}{2} e^{j(\frac{2\pi kt}{T} + \psi_k)} + \frac{C_k}{2} e^{-j(\frac{2\pi kt}{T} + \psi_k)} +$$

$$+ \cdots + \frac{C_N}{2} e^{j(\frac{2\pi Nt}{T} + \psi_N)} + \frac{C_N}{2} e^{-j(\frac{2Nkt}{T} + \psi_N)}$$ Using Euler's formula.

$$= C_0 + \frac{C_1}{2} e^{j\psi_1} e^{j\frac{2\pi 1t}{T}} + \frac{C_1}{2} e^{-j\psi_1} e^{-j\frac{2\pi 1t}{T}} + \cdots + \frac{C_k}{2} e^{j\psi_k} e^{j\frac{2\pi kt}{T}} + \frac{C_k}{2} e^{-j\psi_k} e^{-j\frac{2\pi kt}{T}} +$$

$$+ \cdots + \frac{C_N}{2} e^{j\psi_N} e^{j\frac{2\pi Nt}{T}} + \frac{C_N}{2} e^{-j\psi_N} e^{-j\frac{2\pi Nt}{T}}$$ Formulation of phasors

Let $\mathbf{g}_k(\mathbf{t}) = e^{\frac{j2\pi kt}{T}}$ and then $\mathbf{g}_k(\mathbf{t})^* = e^{\frac{-j2\pi kt}{T}}$ and $\mathbf{a}_k = \frac{C_k}{2} e^{j\psi_k}$ and then $\mathbf{a}_k^* = \frac{C_k}{2} e^{-j\psi_k}$ where $\mathbf{a}_0 = C_0$

Fourier Series

$$f_a(t) = a_0 + \mathbf{a}_1\mathbf{g}_1(\mathbf{t}) + [\mathbf{a}_1\mathbf{g}_1(\mathbf{t})]^* + \cdots + \mathbf{a}_k\mathbf{g}_k(\mathbf{t}) + [\mathbf{a}_k\mathbf{g}_k(\mathbf{t})]^* + \cdots + \mathbf{a}_N\mathbf{g}_N(\mathbf{t}) + [\mathbf{a}_N\mathbf{g}_N(\mathbf{t})]^*$$

Recasting in terms of general orthogonal functions.

$$= a_0 + \sum_{k=1}^{N} \mathbf{a}_k\mathbf{g}_k(\mathbf{t}) + [\mathbf{a}_k\mathbf{g}_k(\mathbf{t})]^*$$ Simplifying the sum.

where $\mathbf{g}_k(\mathbf{t}) = e^{\frac{j2\pi kt}{T}}$, $\mathbf{g}_k(\mathbf{t})^* = e^{\frac{-j2\pi kt}{T}}$, $\mathbf{a}_k = \frac{C_k}{2}e^{j\psi_k}$, $\mathbf{a}_k^* = \frac{C_k}{2}e^{-j\psi_k}$, $\mathbf{a}_0 = C_0$

and $$\mathbf{a}_k = \frac{\int_{t_1}^{t_1+T} f(t)\mathbf{g}_k(\mathbf{t})^*\,dt}{\int_{t_1}^{t_1+T} \mathbf{g}_k(\mathbf{t})\mathbf{g}_k(\mathbf{t})^*\,dt} = \frac{1}{T}\int_{t_1}^{t_1+T} f(t)e^{\frac{-j2\pi kt}{T}}\,dt$$

$$f_a(t) = a_0 + \sum_{k=1}^{N}[\mathbf{a}_k e^{\frac{j2\pi kt}{T}} + \mathbf{a}_k^* e^{\frac{-j2\pi kt}{T}}] = \sum_{k=-N}^{N} \mathbf{a}_k e^{\frac{j2\pi kt}{T}} = C_0 + \sum_{k=1}^{\mathbf{N}} 2\,\mathrm{Re}[\mathbf{a}_k e^{\frac{j2\pi kt}{T}}]$$

Note that since the magnitude of the $\boldsymbol{a}_k$ coefficients are 1/2 the value of the C_k coefficients, 2 real part is required.

$$f_a(t) = a_0 + \sum_{k=1}^{\mathbf{N}} C_k \cos(\frac{j2\pi kt}{T} + \psi_k),$$ where $2\mathbf{a}_k = C_k e^{j\psi_k}$ and $\mathbf{a}_0 = C_0$

Homework

- Fourier Series
  - Problem (3)
    - Compute the Fourier Series for the function using 3 terms in the series:

$$f(t) = 1 \text{ for } 0 < t < \pi \text{ and } f(t) = 0 \text{ for } \pi < t < 2\pi$$

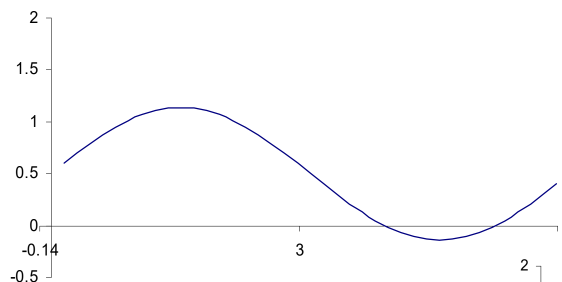
$$a_k = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-jkt}dt = \frac{1}{2\pi}\int_0^{\pi} 1e^{-jkt}dt = (\frac{1}{2\pi})(\frac{1}{-jk})\,e^{-jkt}\Big|_0^{\pi} = \frac{1}{-2\pi kj}(e^{-jk\pi} - 1)$$

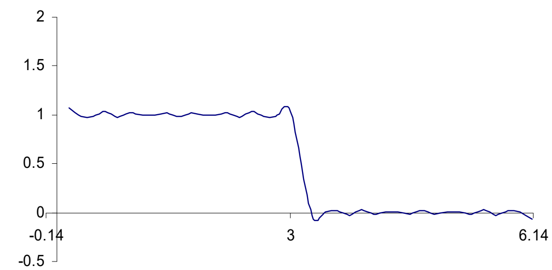
$$= \frac{1}{-2\pi kj}e^{-jk\pi/2}(e^{-jk\pi/2} - e^{+jk\pi/2})$$

$$= \frac{\sin\frac{k\pi}{2}}{\pi k}e^{-jk\pi/2};\ \text{for } k \neq 0$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t)dt = \frac{1}{2\pi}\int_0^{\pi} 1dt = \frac{1}{2}$$

$$f(t) = \frac{1}{2} + 2\sum_1^N \frac{\sin\frac{k\pi}{2}}{\pi k}\cos(kt - k\frac{\pi}{2})$$





Homework

- Mean Squared Error
  - Problem (1)
    - For our example in class, prove that $E^2=0.017$ for $f(t) = \cos(\pi t/2)$
  - Problem (2)
    - It is desired to approximate $f(t) = \sin(t)$ in the interval $0<t<\pi/2$ by the straight line $f_a(t)=mt+b$. Determine the values of m and b for a least mean square error approximation and calculate the corresponding MSE.
- Fourier Series
  - Problem (3)
    - Compute the Fourier Series for the function using 3 terms in the series:

      , $f(t) = 1$ for $0 < t < \pi, f(t)=0$ for $\pi<t<2\pi$
  - Problem (4)
    - Compute the Fourier Series for the function using 4 terms in the series:

      $f(t) = t$ for $0<t<3$
- 5CT.1.1, 5CT.1.2