Fourier Series for Periodic Functions

Lecture #8

5CT3,4,6,7

Fourier Series for Periodic Functions

- Up to now we have solved the problem of approximating a function $f(t)$ by $f_a(t)$ within an interval T.
- However, if $f(t)$ is periodic with period T, i.e., $f(t)=f(t+T)$, then the approximation is true for all t.
- And if we represent a periodic function in terms of an infinite Fourier series, such that the frequencies are all integral multiples of the frequency $1/T$, where $k=1$ corresponds to the fundamental frequency of the function and the remainder are its harmonics.

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{\frac{-j2\pi kt}{T}} dt$$

$$f(t) = a_0 + \sum_{k=1}^{\infty} [\mathbf{a}_k e^{\frac{j2\pi kt}{T}} + \mathbf{a}_k * e^{\frac{-j2\pi kt}{T}}] = \sum_{k=-\infty}^{\infty} \mathbf{a}_k e^{\frac{j2\pi kt}{T}} = a_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}[\mathbf{a}_k e^{\frac{j2\pi kt}{T}}]$$

$$f(t) = a_0 + \sum_{k=\mathbf{1}}^{\infty} C_k \cos(\frac{2\pi kt}{T} + \psi_k), \text{ where } 2\mathbf{a}_k = C_k e^{j\psi_k} \text{ and } \mathbf{a}_0 = C_0$$

Another Form for the Fourier Series

$$f(t) = C_0 + \sum_{\mathbf{k=1}}^{\infty} C_k \cos(\frac{2\pi kt}{T} + \psi_k)$$

$$= A_0 + \sum_{\mathbf{k=1}}^{\infty} A_k \cos(\frac{2\pi kt}{T}) + B_k \sin(\frac{2\pi kt}{T})$$

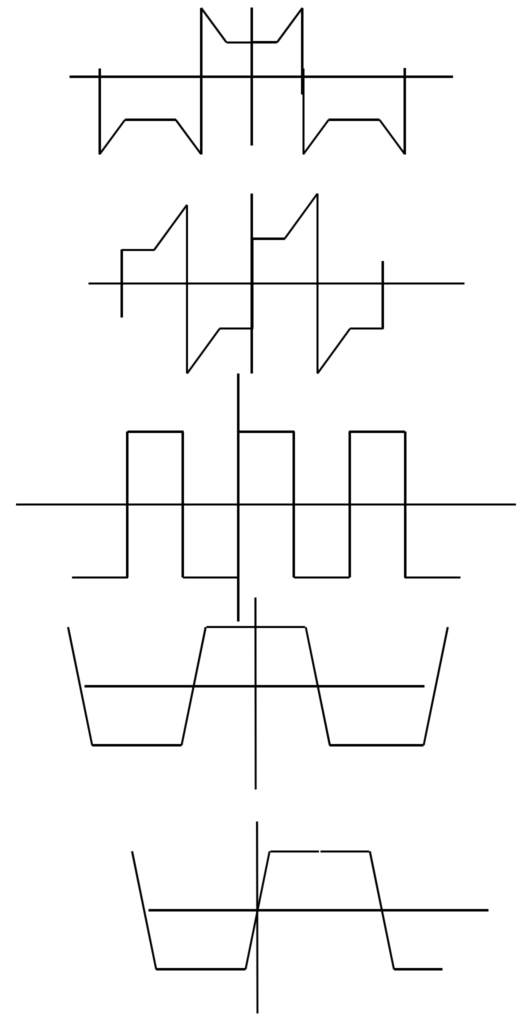
where $A_k = C_k \cos\psi_k$ and $B_k = -C_k \sin\psi_k$ and $A_0 = C_0$

Fourier Series Theorem

- Any periodic function $f(t)$ with period T which is integrable ($\int f(t)\,dt < \infty$) can be represented by an infinite Fourier Series
- If $[f(t)]^2$ is also integrable, then the series converges to the value of $f(t)$ at every point where $f(t)$ is continuous and to the average value at any discontinuity.

Properties of Fourier Series

- Symmetries
  - If $f(t)$ is even, $f(t)=f(-t)$, then the Fourier Series contains only cosine terms
  - If $f(t)$ is odd, $f(t)=-f(-t)$, then the Fourier Series contains only sine terms
  - If $f(t)$ has half-wave symmetry, $f(t) = -f(t+T/2)$, then the Fourier Series will only have odd harmonics
  - If $f(t)$ has half-wave symmetry and is even, even quarter-wave, then the Fourier Series will only have odd harmonics and cosine terms
  - If $f(t)$ has half-wave symmetry and is odd, odd quarter-wave, then the Fourier Series will only have odd harmonics and sine terms



Properties of FS Continued

- Superposition holds, if $f(t)$ and $g(t)$ have coefficients $\boldsymbol{f_k}$ and $\boldsymbol{g_k}$, respectively, then $Af(t)+Bg(t) \Rightarrow A\boldsymbol{f_k}+B\boldsymbol{g_k}$
- Time Shifting: $f(t)=\sum_{k=-\infty}^{\infty} a_k e^{\frac{j2\pi kt}{T}}$

$$f(t-t_1)=\sum_{k=-\infty}^{\infty} a_k e^{\frac{j2\pi kt}{T}} e^{\frac{-j2\pi kt_1}{T}} = \sum_{k=-\infty}^{\infty} a_k e^{\frac{-j2\pi kt_1}{T}} e^{\frac{j2\pi kt}{T}}$$

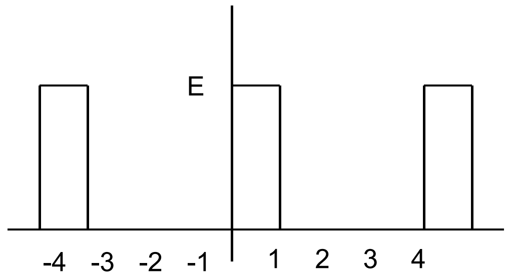
$$(a_k)_{\text{delayed}} = (a_k)_{\text{original}} e^{\frac{-j2\pi kt_1}{T}}$$

- Differentiation and Integration

$$(a_k)_{\text{derivative}} = \frac{j2\pi k}{T}(a_k)_{\text{original}}$$

$$(a_k)_{\text{integral}} = \frac{T}{j2\pi k}(a_k)_{\text{original}}$$

FS Coefficients Calculation Example



$$f(t) = 0, -2 < t < 0$$
$$f(t) = E, 0 < t < 1$$
$$f(t) = 0, 1 < t < 2$$
$$f(t) = f(t \pm 4)$$

$$a_k = \frac{1}{4}[\int_{-2}^{0} 0e^{\frac{-j2\pi kt}{4}}dt + \int_{0}^{1} Ee^{\frac{-j2\pi kt}{4}}dt + \int_{1}^{2} 0e^{\frac{-j2\pi kt}{4}}dt]$$

$$= \frac{1}{4}\frac{E}{\frac{-j2\pi k}{4}}e^{\frac{-j2\pi kt}{4}}\Big|_0^1$$

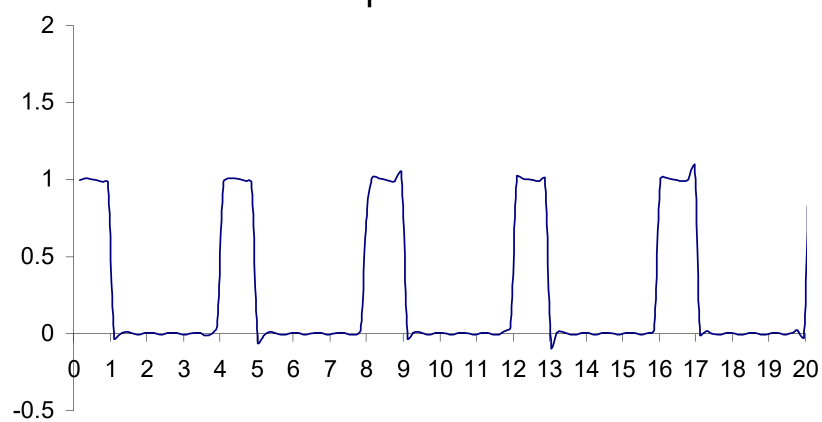
$$= \frac{E}{-j2\pi k}[e^{\frac{-j\pi k}{2}} - 1] = \frac{Ee^{\frac{-j\pi k}{4}}}{\pi k}[\frac{e^{\frac{j\pi k}{4}} - e^{\frac{-j\pi k}{4}}}{j2}]$$

$$= \frac{E}{4}\frac{\sin(\frac{\pi k}{4})}{\frac{\pi k}{4}}e^{\frac{-j\pi k}{4}}, \quad k \neq 0$$

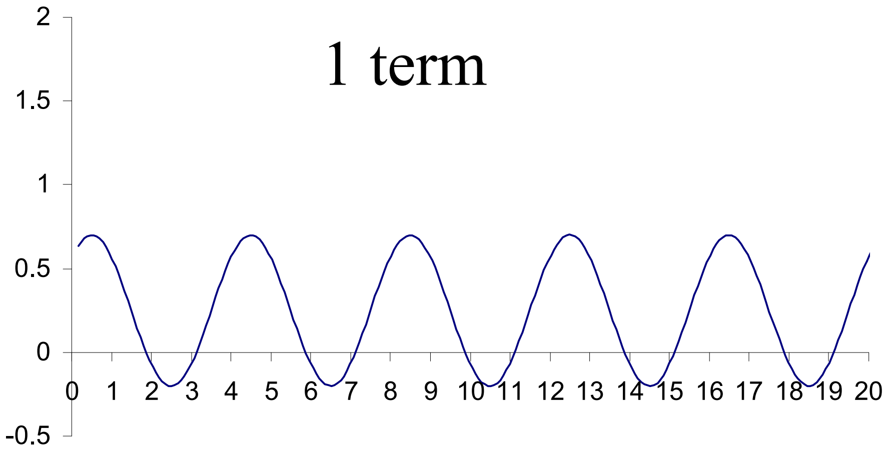
$$\mathrm{a}_0 = \frac{E}{4}\int_{0}^{1} dt = \frac{E}{4}$$

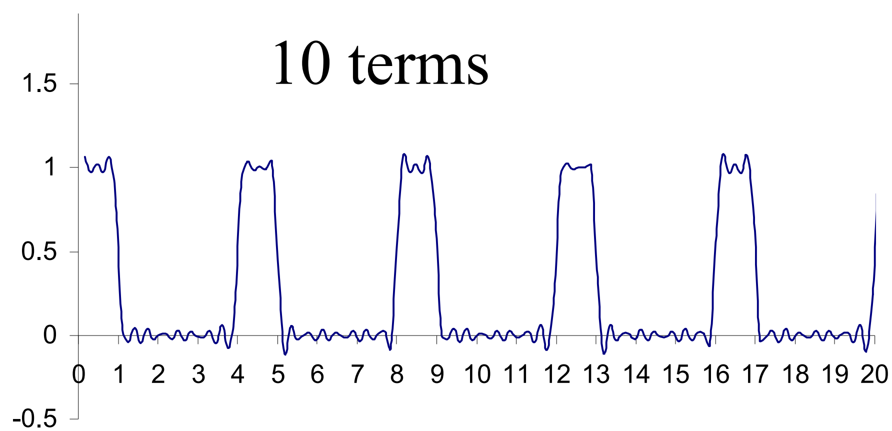
Since $2a_k = C_k\angle\psi$, then

$$f(t) = \frac{E}{4} + \frac{E}{2}\sum_{k=1}^{\infty}\frac{\sin\frac{k\pi}{4}}{\frac{k\pi}{4}}\cos(\frac{2\pi kt}{4} - \frac{k\pi}{4})$$



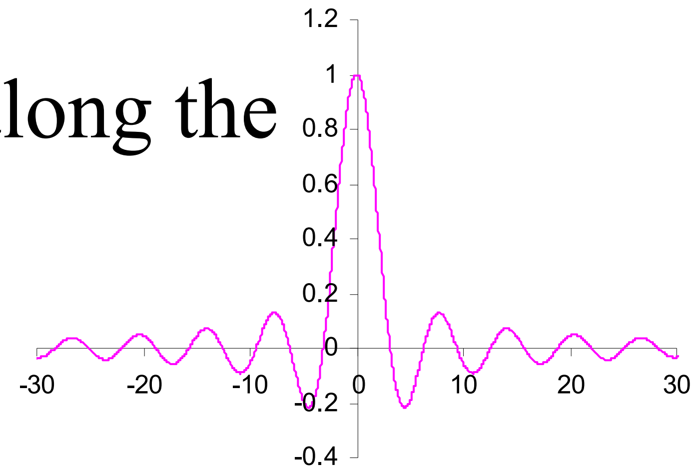
Example Continued





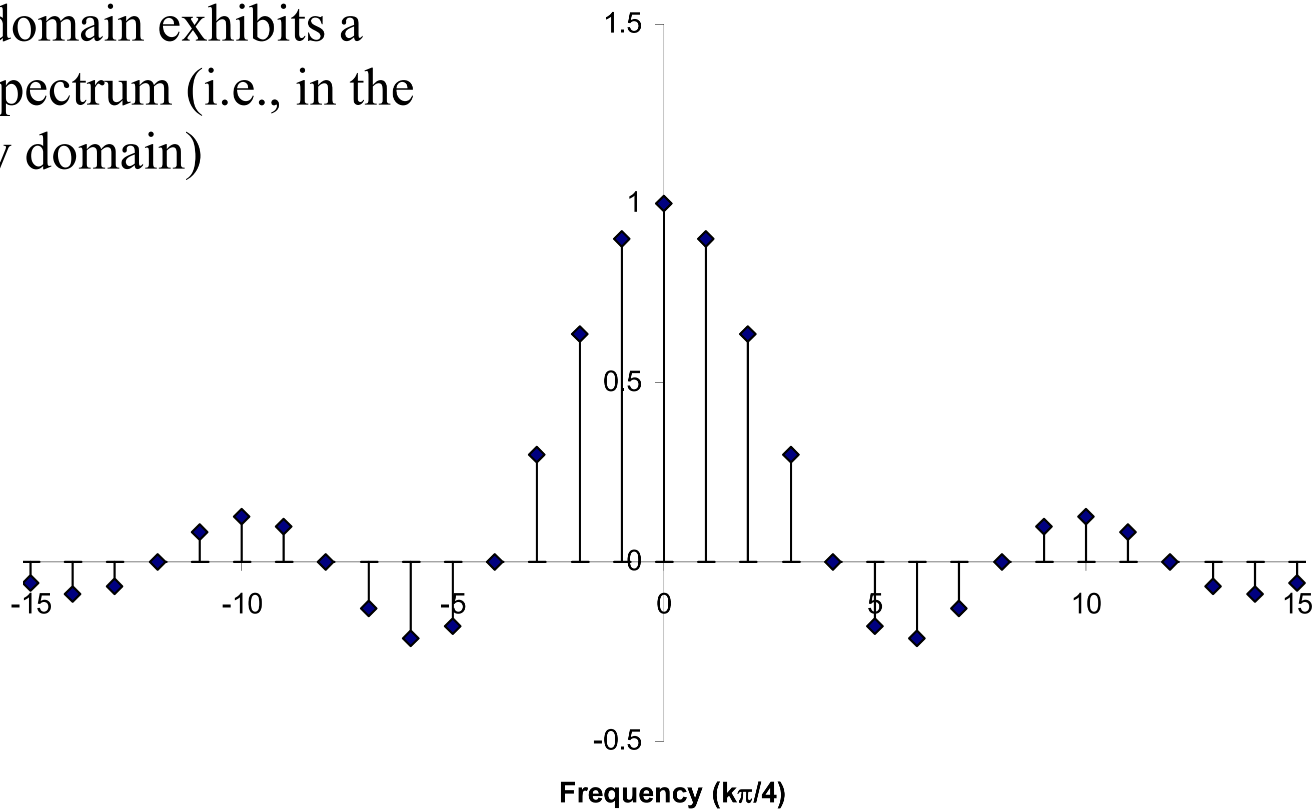
Frequency Spectrum of the Pulse Function

- In the preceding example, the coefficients for each of the cosine terms was proportional to $\sin(k\pi/4)/(k\pi/4)$
- We call the function $Sa(x) = \dfrac{\sin x}{x}$ the Sampling Function
- If we plot these coefficients along the frequency axis we have the frequency spectrum of *f(t)*

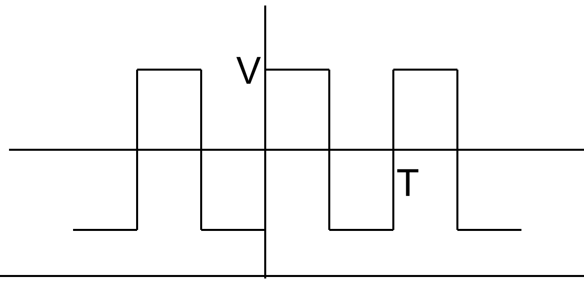


Frequency Spectrum

Note that the periodic signal in the time domain exhibits a discrete spectrum (i.e., in the frequency domain)



Another Example



This signal is odd, H/W symmetrical, and mean=0; this means no cosine terms, odd harmonics and mean=0.

$$a_k = \frac{1}{T}[\int_{-\frac{T}{2}}^{0} -Ve^{\frac{-j2\pi kt}{T}}dt + \int_{0}^{\frac{T}{2}} Ve^{\frac{-j2\pi kt}{T}}dt]$$

$$= \frac{V}{T}[\frac{-1}{\frac{-j2\pi k}{T}}e^{\frac{-j2\pi kt}{T}}\Big|_{-\frac{T}{2}}^{0} + \frac{1}{\frac{-j2\pi k}{T}}e^{\frac{-j2\pi kt}{T}}\Big|_{0}^{\frac{T}{2}}]$$

$$= \frac{V}{T}[\frac{1}{\frac{-j2\pi k}{T}}\{[-e^{\frac{-j2\pi k0}{T}} + e^{\frac{j2\pi kT}{T2}}] + [e^{\frac{-j2\pi kT}{T2}} - e^{\frac{-j2\pi k0}{T}}]\}$$

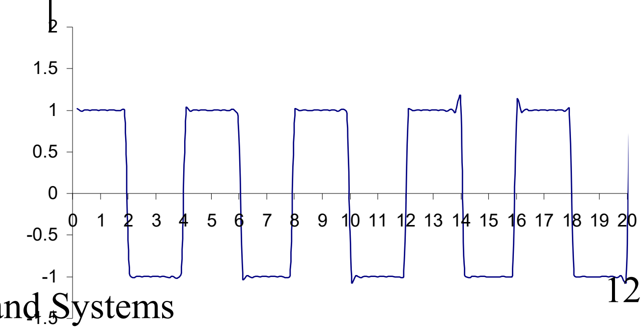
$$= \frac{V}{-j2\pi k}\{[-e^{\frac{-j2\pi k0}{T}} + e^{\frac{j2\pi kT}{T2}}] + [e^{\frac{-j2\pi kT}{T2}} - e^{\frac{-j2\pi k0}{T}}]\}$$

$$a_k = \frac{V}{j\pi k}[1-\cos \pi k] = \begin{cases} \frac{2V}{j\pi k} \text{ for k odd,} \\ 0 \text{ for k even.} \end{cases}$$

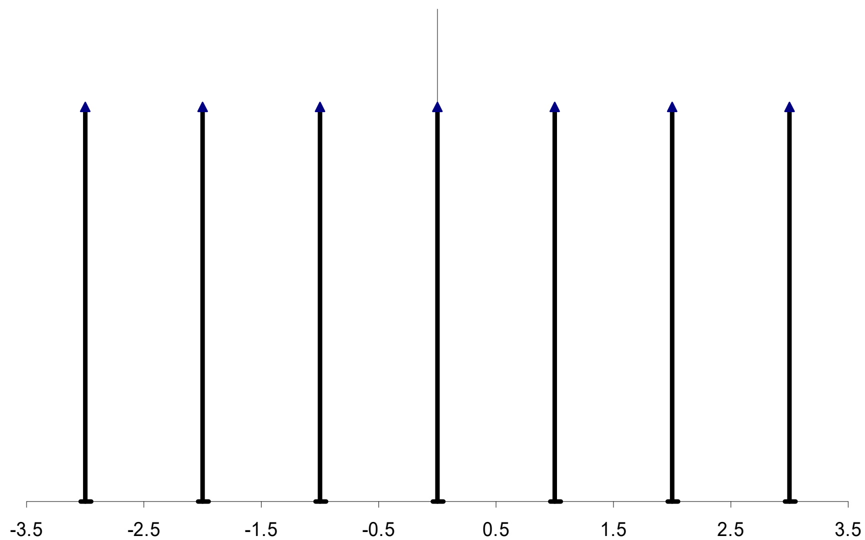
$$= \frac{2V}{\pi k}\angle -\frac{\pi}{2} \text{ for k odd.}$$

$$\therefore f(t) = 2\sum_{k=odd}^{\infty}\frac{2V}{\pi k}\cos(\frac{2\pi kt}{T} - \frac{\pi}{2})$$

$$f(t) = \sum_{k=odd}^{\infty}\frac{4V}{\pi k}\sin(\frac{2\pi kt}{T})$$



Fourier Series of an Impulse Train or Sampling Function



$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{\frac{-j2k\pi t}{T}} dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{\frac{-j2k\pi t}{T}} dt$$

$$= \frac{1}{T} \text{ for all } k$$

$$x(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{\frac{j2k\pi t}{T}}$$

Fourier Series for Discrete Periodic Functions

$$x[n] = a_0 + \sum_{k=1}^{\infty} a_k e^{\frac{j2\pi kn}{T}} + \sum_{k=1}^{\infty} a_k^* e^{\frac{-j2\pi kn}{T}} = \sum_{k=-\infty}^{\infty} a_k e^{\frac{j2\pi kn}{T}}$$

$$a_k = \frac{1}{T} \sum_{n=-T/2}^{T/2} x[n] e^{\frac{-j2\pi kn}{T}}$$

Since $x[n]$ is discrete, this becomes a summation

Homework

- Problem (1)
  - Compute the Fourier Series for the periodic functions

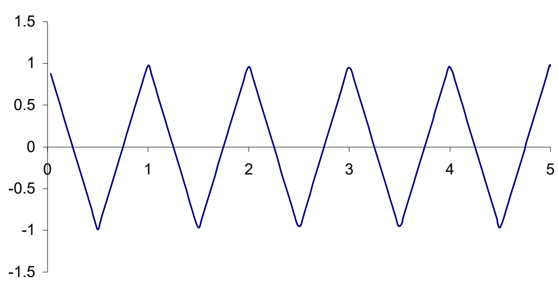
    a) $f(t) = 1$ for $0<t<\pi$, $f(t)=0$ for $\pi<t<2\pi$

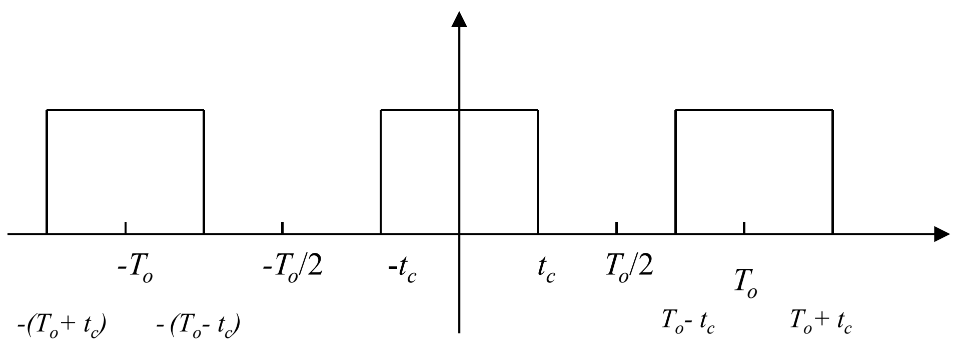
    b) $f(t) = t$ for $0<t<3$

- Problem (2)
  - Compute the Fourier series of the following Periodic Functions:
    - $f(t)= t,\ 2n\pi < t < (2n+1)\pi$ for $n \geq 0$

      $= 0,\ (2n+1)\pi < t < (2n+2)\pi$ for $n \geq 0$
    - $f(t)= e^{-t/\pi},\ 2n\pi < t < (2n+2)\pi$ for $n \geq 0$ Use Matlab to plot $f(t)$ using a_k for maximum number of components, N=5,10, 100, and1000. Show your code.

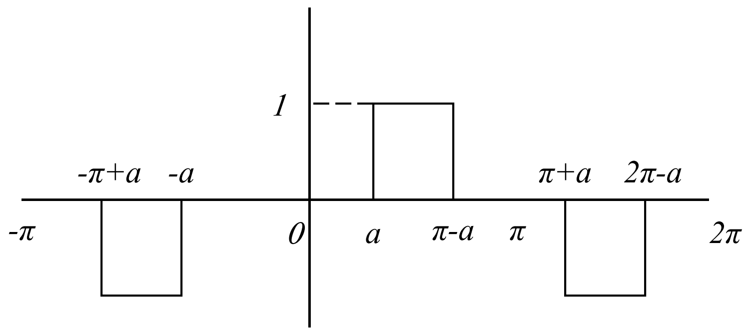
- Problem (3)
  - Problems: 4.1/3 Find the Fourier series of the following waveforms (choose $t_c=T_o/4$):

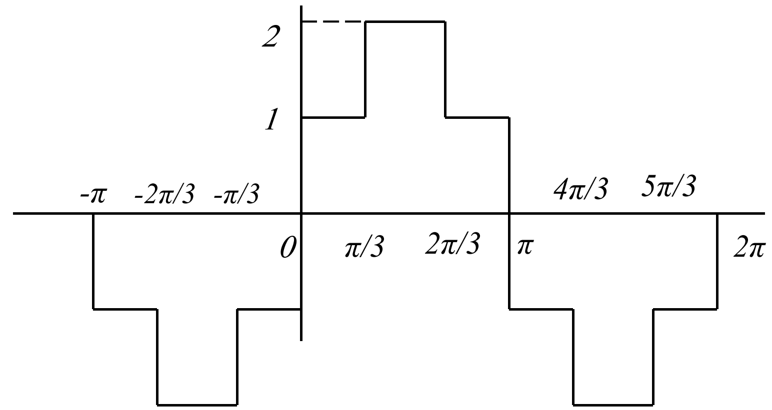




Homework

- Problem (4)
  - Deduce the Fourier series for the functions shown (hint: deduce the second one using superposition):





- 5CT.7.1