Fourier Transforms

Lecture 9

4CT.1-4

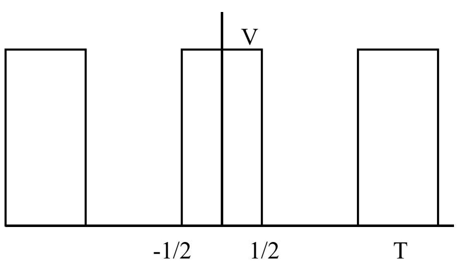
Fourier Transforms

- Recall that a Fourier Series of a periodic function with period *T* we have:

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{\frac{-j2\pi kt}{T}}dt$$

$$f(t) = a_0 + \sum_{\mathbf{k}=1}^{\infty}[a_k e^{\frac{j2\pi kt}{T}} + a_k{*}e^{\frac{-j2\pi kt}{T}}] = a_0 + \sum_{\mathbf{k=1}}^{\infty} C_k \cos(\frac{j2\pi kt}{T} + \psi_k)$$

- More specially for a periodic pulse:

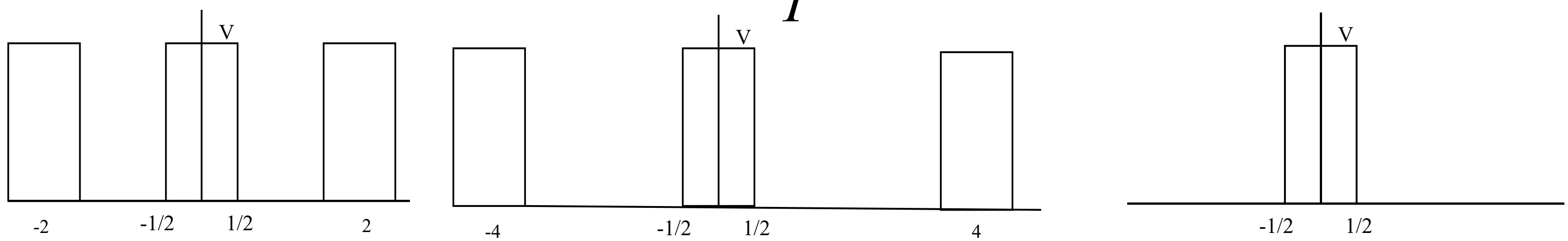


$$a_k = \frac{1}{T}\int_{-1/2}^{1/2} Ve^{\frac{-j2\pi kt}{T}}dt$$

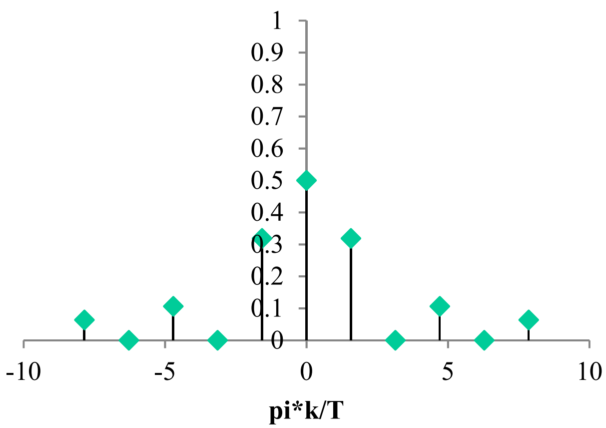
$$= \frac{V}{T}\frac{1}{\frac{-j2\pi k}{T}}e^{\frac{-j2\pi kt}{T}}\Big|_{-1/2}^{1/2} = \frac{V}{T}\frac{\sin \pi k/T}{\frac{\pi k}{T}}$$

Pulse Example Continued

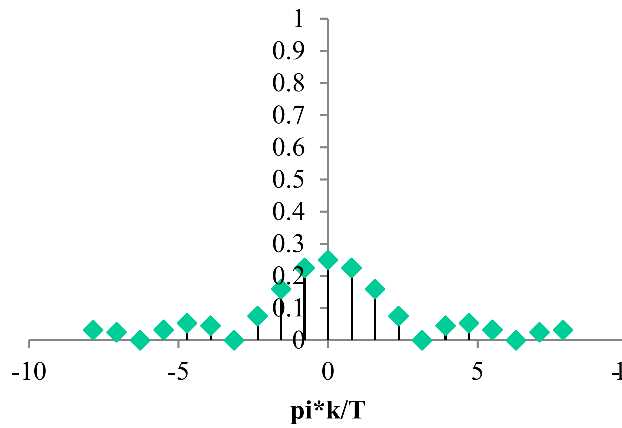
$$a_k = \frac{V}{T}\frac{\sin \pi k/T}{\frac{\pi k}{T}}$$



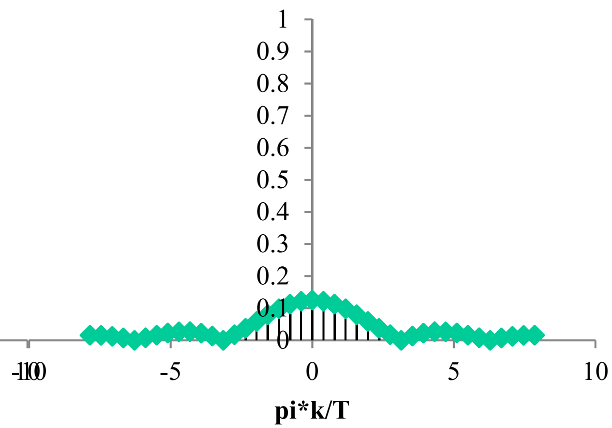
T=2; a_k spacing=1.57 radians



T=4; a_k spacing=0.785 radians

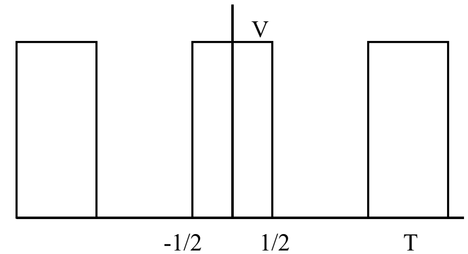


T=8; a_k spacing=0.39 radians

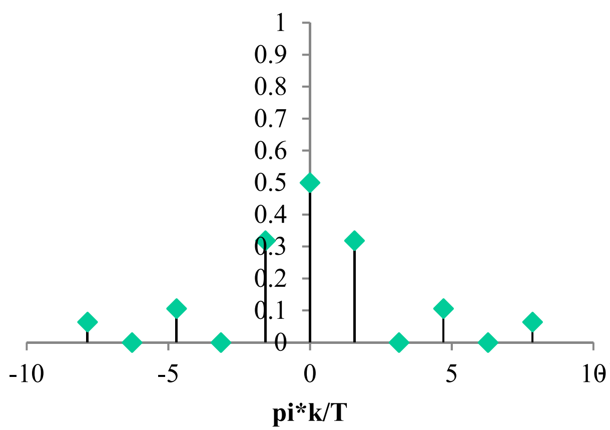


Pulse Example Continued

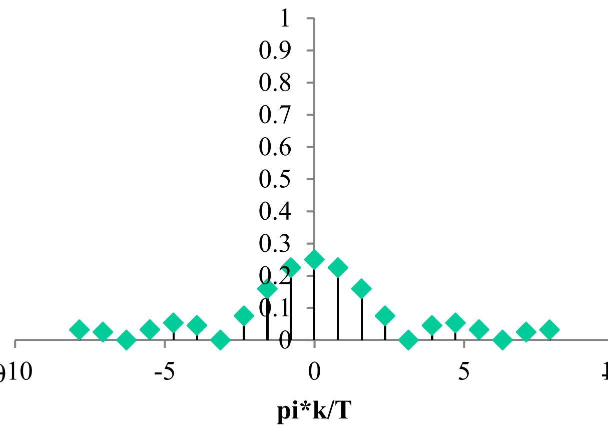
$$a_k = \frac{V}{T}\frac{\sin \pi k/T}{\frac{\pi k}{T}}$$



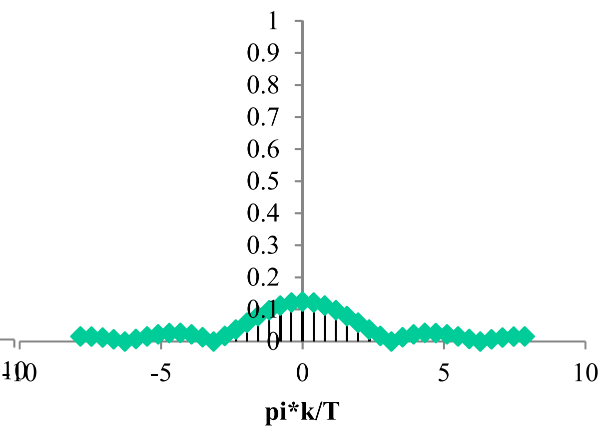
T=2; a_k spacing=1.57 radians



T=4; a_k spacing=0.785 radians



T=8; a_k spacing=0.39 radians



- Let's look at the FS coefficients as T approaches ∞
- As T increases, the duration of each pulse is the same, the next pulse moves farther away from each other, the harmonics move closer to each other, and their amplitude reduces. Note that the harmonics are always $1/T$ apart from each other.
- As T ➔ ∞, we would expect that the duration of each pulse is still the same, there is only a single pulse, the harmonics move infinitesimally closer together, and their amplitude is infinitesimally small.

Fourier Transform Analysis

- We need to investigate the class of non-periodic signals whose quadratic content is finite, i.e., $\int_{-\infty}^{\infty} f(t)^2\, dt < \infty$
- Going back to the definition of a Fourier Series of a periodic function, let's take the limit as T ➔ ∞ and see what we get:

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{\frac{-j2\pi kt}{T}}dt$$

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{\frac{j2\pi kt}{T}}$$

$$a_k = \Delta f\int_{-1/(2\Delta f)}^{1/(2\Delta f)} f(t)e^{-jk\Delta\omega t}dt$$

$$f(t) = \sum_{\mathbf{k}=-\infty}^{\infty} a_k e^{jk\Delta\omega t}$$

- The fundamental frequency is *1/T* which is the spacing between harmonics in the spectrum. Let's define $\Delta f = 1/T$ and $\Delta\omega = 2\pi\Delta f$.
- As T ➔ ∞ , Δf ➔ 0, a_k ➔ 0 but $a_k / \Delta f$ may remain finite. It will be finite for sure $\int_{-\infty}^{\infty} f(t)e^{-jk\Delta\omega t}\, dt$ remains finite.
- It can be shown that if the quadratic content is finite then $\int_{-\infty}^{\infty} f(t)e^{-jk\Delta\omega t}\, dt$ and $a_k / \Delta f$ remain finite.

Fourier Transform Analysis Continued

- Let's define $\omega=k\Delta\omega$ which means that ω is the k^{th} harmonic of the fundamental frequency (radians) $\Delta\omega$. Therefore, a_k is the amplitude of this k^{th} harmonic.
- As $T \rightarrow \infty$, $\Delta f \rightarrow 0$, k must approach ∞ to keep $a_k/\Delta f$ finite and so $\omega=k\Delta\omega$ will become a continuous variable.
- In the limit, instead of having discrete harmonics corresponding to k, we have a continuous variable ω and we can replace a_k with $\boldsymbol{a}(\omega)$.
- We now define this limit to be the complex Fourier Transform $F(j\omega)$ of $f(t)$

$$\lim_{\Delta f \to 0} \frac{a(\omega)}{\Delta f} = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = F(j\omega)$$

The Inverse Fourier Transform

- Going back to our Fourier Series formation for $f(t)$, substituting our new definition of $a_k = a(\omega) = F(j\omega)\,\Delta f$, and taking the limit as $T \rightarrow \infty$, we get:

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk2\pi t/T}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} F(j\omega) e^{jk\Delta\omega t} \Delta\omega$$

$$\lim_{\Delta\omega \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} F(j\omega) e^{jk\Delta\omega t} \Delta\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega$$

The Fourier Transform and It's Inverse

$$\Im[x(t)] = X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$X(F) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi Ft}\,dt \text{ The text's version.}$$

$$\Im^{-1}[X(j\omega)] = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(F)e^{j2\pi Ft}d(2\pi F) = \int_{-\infty}^{\infty} X(F)e^{j2\pi Ft}\,dF \text{ The text's version.}$$

- The difference between a_k and $F(j\omega)$ is that $F(j\omega)$ has units of a_k divided by frequency and therefore *F(jω)* is called the harmonic **density** function.
- So if a_k has the units of volts, then $F(j\omega)$ has the units of volts/(radians/sec)

Why Do We Use Transforms?

- A transform of a time function is a representation of the function.
- $F(j\omega)$ conveys information about the function $f(t)$, in particular, harmonic density information.
- For Fourier Transforms, we can analyze the function in terms of sinusoidal functions.
- We can also formulate a relationship of a response to a source into a single algebraic equation.

Properties of Fourier Transform

- If a function is even, then

$$\Im[f(t)] = 2\int_{0}^{\infty} f(t)\cos\omega t\,dt$$

- If a function is odd, then

$$\Im[f(t)] = -2j\int_{0}^{\infty} f(t)\sin\omega t\,dt$$

- Superposition: If $\Im[f_1(t)] = F_1(j\omega)$ and
- $\Im[f_2(t)] = F_2(j\omega)$, then

$$\Im[f_1(t) + f_2(t)] = F_1(j\omega) + F_2(j\omega)$$

Properties of Fourier Transform

- Differentiation: If $\Im[f_1(t)] = F_1(j\omega)$, then

$$\Im[\frac{df(t)}{dt}] = j\omega F(j\omega)$$

- Integratation: If $\Im[f(t)] = F(j\omega)$ then

$$\Im[\int f(t)dt] = \frac{F(j\omega)}{j\omega}$$

Proof of Fourier Transform for even f(t)

$$\Im[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} f(t)(\cos\omega t + j\sin\omega t)\,dt = \int_{-\infty}^{0} f(t)(\cos\omega t + j\sin\omega t)\,dt + \int_{0}^{\infty} f(t)(\cos\omega t + j\sin\omega t)\,dt$$

$$\int_{-\infty}^{0} f(t)(\cos\omega t + j\sin\omega t)\,dt; t \Rightarrow -\tau; dt = -d\tau; -\infty < t < 0 \Rightarrow \infty < \tau < 0$$

$$\int_{-\infty}^{0} f(t)(\cos\omega t + j\sin\omega t)\,dt = \int_{\infty}^{0} f(-\tau)(\cos(-\omega\tau) + j\sin(-\omega\tau)) - d\tau = \int_{0}^{\infty} f(-\tau)(\cos(-\omega\tau) + j\sin(-\omega\tau))\,d\tau$$

$$NOTE: f(-\tau) = f(\tau); \cos(-\omega\tau) = \cos(\omega\tau); \sin(-\omega\tau) = -\sin(\omega\tau)$$

$$= \int_{0}^{\infty} f(-\tau)(\cos(-\omega\tau) + j\sin(-\omega\tau))\,d\tau = \int_{0}^{\infty} f(\tau)(\cos(\omega\tau) - j\sin(\omega\tau))\,d\tau$$

$$= \int_{-\infty}^{0} f(t)(\cos\omega t + j\sin\omega t)\,dt + \int_{0}^{\infty} f(t)(\cos\omega t + j\sin\omega t)\,dt = \int_{0}^{\infty} f(\tau)(\cos(\omega\tau) - j\sin(\omega\tau))\,d\tau + \int_{0}^{\infty} f(t)(\cos\omega t + j\sin\omega t)\,dt$$

$$\int_{0}^{\infty} f(t)(\cos\omega t - j\sin\omega t)\,dt + \int_{0}^{\infty} f(t)(\cos\omega t + j\sin\omega t)\,dt = 2\int_{0}^{\infty} f(t)\cos\omega t\,dt$$

Integration and Differentiation Properties for the FT

Differentiation Property

Let $\dfrac{df(t)}{dt} = g(t)$

$$\Im\{\frac{df(t)}{dt}\} = \Im\{g(t)\} = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}dt = G(j\omega)$$

And $\Im\{f(t)\} = F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$

But $f(t) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega$

$$g(t) = \frac{df(t)}{dt} = \frac{d}{dt}\{\frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d}{dt}\{F(j\omega)e^{j\omega t}\}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega F(j\omega)e^{j\omega t}d\omega$$

Since $g(t) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} G(j\omega)e^{j\omega t}d\omega$

$$\therefore G(j\omega) = j\omega F(j\omega)$$

Integration Property

Let $\int f(t)dt = g(t)$

$$\Im\{\int f(t)dt\} = \Im\{g(t)\} = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}dt = G(j\omega)$$

And $\Im\{f(t)\} = F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$

But $f(t) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega$

$$\int f(t)dt = \int\{\frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega\}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\{\int F(j\omega)e^{j\omega t}dt\}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\{\int e^{j\omega t}dt\}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\frac{e^{j\omega t}}{j\omega}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F(j\omega)}{j\omega}e^{j\omega t}d\omega = g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(j\omega)e^{j\omega t}d\omega$$

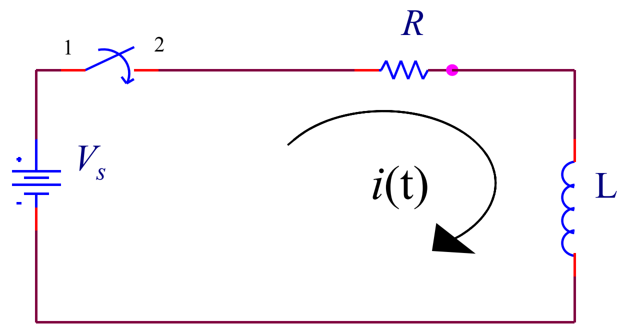
$$\therefore G(j\omega) = \frac{F(j\omega)}{j\omega}$$

OR

$\dfrac{dg(t)}{dt} = f(t)$; We just saw that $\Im\{\dfrac{dg(t)}{dt}\} = j\omega G(j\omega) = \Im\{f(t)\} = F(j\omega)$

$$\therefore G(j\omega) = \frac{F(j\omega)}{j\omega}$$

Our Old Example



R = Resistance
L = Inductance
V_s = Voltage

$$V_s(t) = i(t)R + L\frac{di(t)}{dt}$$

$$\Im[Vs(t)] = \Im[i(t)R] + \Im[L\frac{di(t)}{dt}]$$

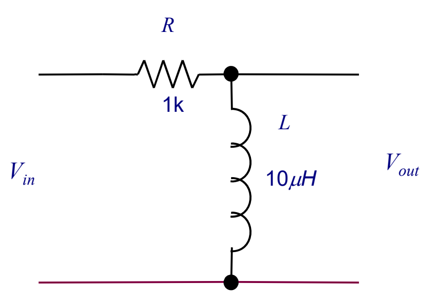
$$\mathbf{V}_s(j\omega) = \mathbf{I}(j\omega)R + j\omega L\mathbf{I}(j\omega)$$

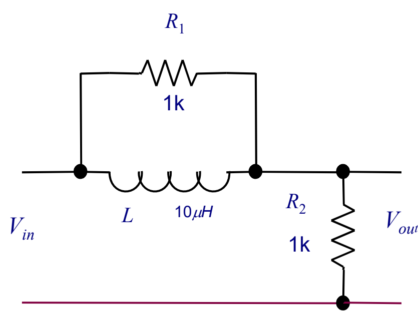
$$\mathbf{I}(j\omega) = \frac{\mathbf{V}_s(j\omega)}{R + j\omega L}$$

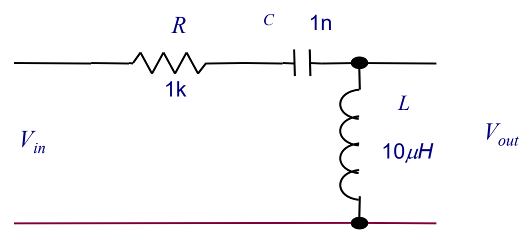
This is similar to what we did when replaced the p operator with $j\omega$.

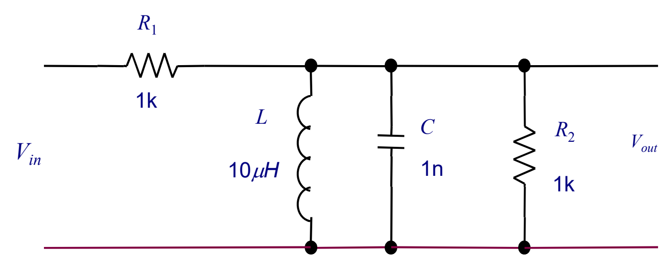
Homework

- Calculate the Fourier Transform of the network function for the following networks:









- 4CT.2.2
- 4CT.2.3