ACTIVITY 1:

Designs with Half-squares:

Materials: A roll of stick back magnetic tape, 1 steel tray, a black marking pen, colored magic markers, rubber cement, half-square template found on Worksheet W1-1, 3 sheets of graph paper.

1. Work in teams of two people. One member of the group cuts out 25 squares from the template; the other member cuts 25 lengths of tape.

2. Glue squares to the tape segments with rubber cement,

3. Construct a few patterns on your metal tray.

4. Notice that you can make 4 different patterns with 1 square if you rotate the square through 4 quarter-turns. How many patterns can you make with 2 squares ? _____ Sketch the patterns on the sheet of graph paper. Which of these patterns has a plane of mirror symmetry (one half of the pattern reproduces the other half when placed against the mirror)? Indicate the position of the mirror on the graph paper. Which patterns are identical when you rotate the tray through a half-turn?

5. How many 2x2 patterns can you make with 4 squares? Sketch a few patterns on your graph paper (there are too many to record them all). Include at least two patterns that have mirror symmetry, 1 pattern that is identical after a half-turn, and 1 pattern that is identical after a quarter-turn.

6. Although there are many 2x2 patterns, they cannot be considered to be distinctly different. They can be separated into either symmetric (identical after either a rotation of 1/4 or 1/2-turn or a reflection) or nonsymmetric patterns. For each nonsymmetric pattern, there are exactly 8 patterns that are either rotations or reflections of this pattern. These 8 transformations constitute what is known in mathematics as a group. The group is called a Klein group. The 8 transformations of a square are illustrated in Figure 1-1.

7. Construct an interesting 4x4 or 5x5 pattern.

The Baravelle Spiral

1. Figure 1-2 shows how to create a Baravelle spiral. Use Worksheet W1-2 to create the spiral. Use magic markers to color the appropriate half-squares. Try to find an interesting color scheme.

2. Worksheet W1-3 presents space to create a design with up to nine Baravelle spirals similar to the one shown in Figure 1-3.

THE EIGHT TRANSFORMATIONS OF A SQUARE:
THE KLEIN GROUP.

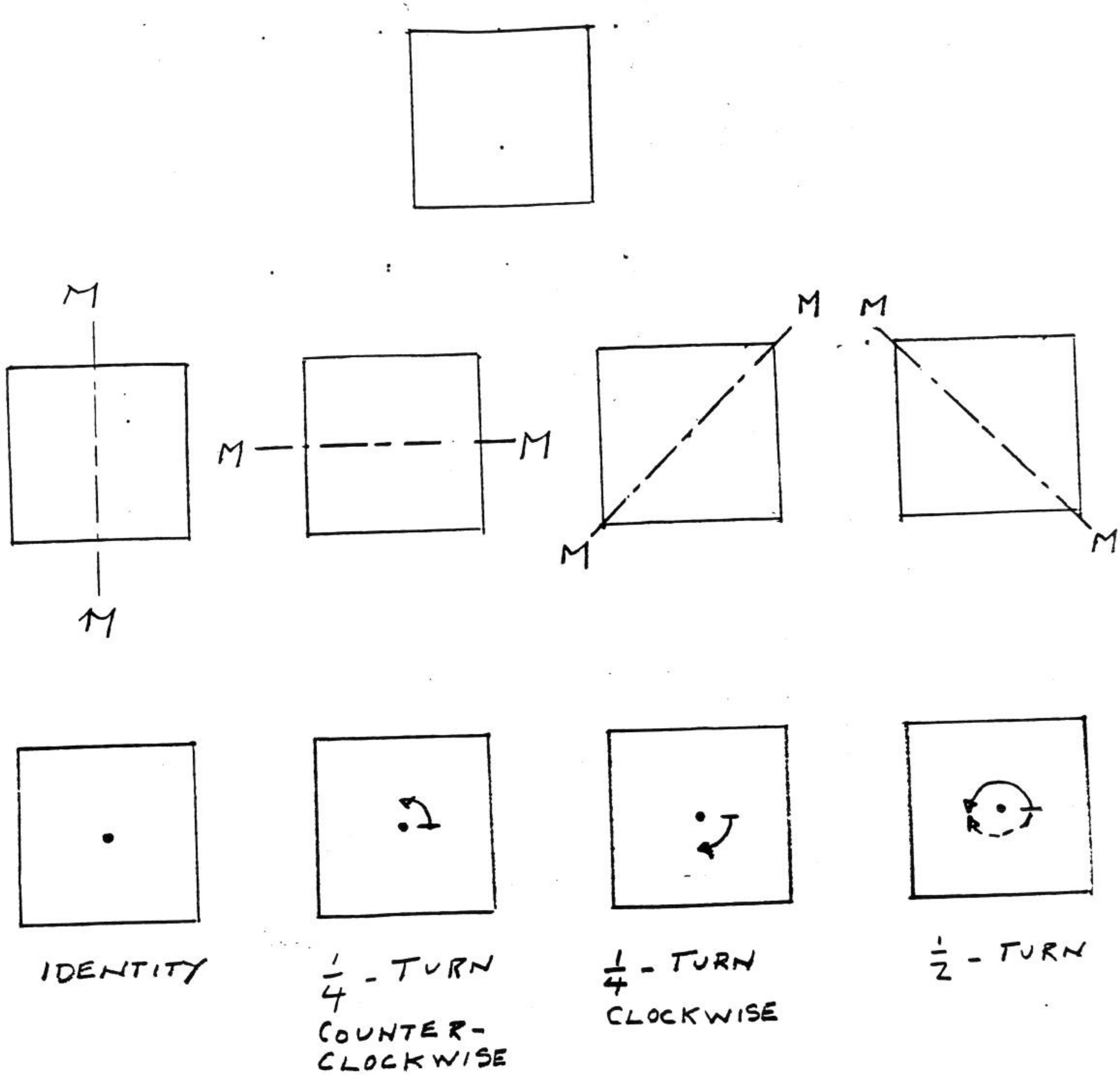
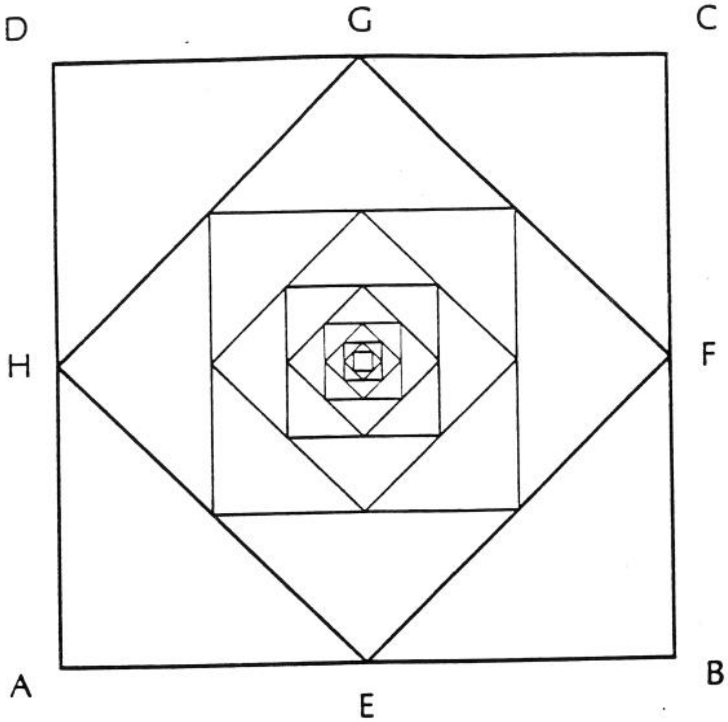


FIGURE 1-1

Construct a Baravelle Spiral within a Square



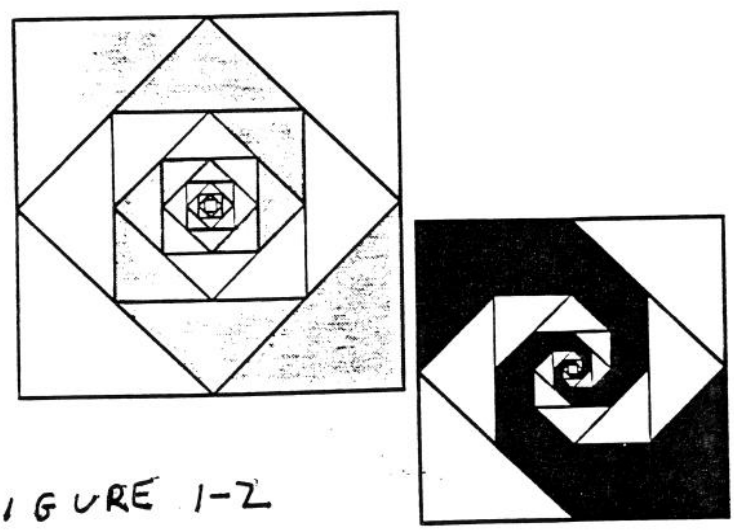


FIGURE 1-2

iven square ABCD

. Locate the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$.

. Label them E, F, G and H, espectively.

. Join them with line segnents to form square EFGH.

. In the same way, join the nidpoints of the sides of :FGH to form another square.

5. Repeat the process until the final square is the desired size.

6. The "spiral" shape becomes visible when the triangles are shaded as illustrated in the diagram.

Now the resulting figure is a Baravelle Spiral.

Project

1 Using this square as your inspiration, create an allover pattern with the Baravelle Spiral. Or develop a variatior of your own in another regular polygon. Use a personal color structure for your finished work.

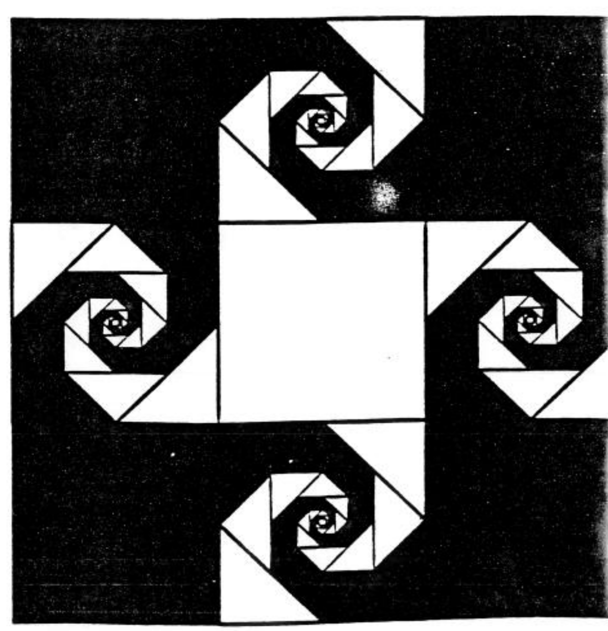


FIGURE 1-3

Activity 2:

Materials:

circle master compass
math ruler
magic markers
tracing paper
triangular grid paper

Group 1 - Infinite Pattern Factory: triangular circle grid

1. Refer to page 1 of: "The Infinite Pattern Factory (IPF)." Create an array of at least 20 circles. Connect the centers of the circles to create a grid of equilateral triangles.

2. Use tracing paper and the circle grid on page 2 or 3 of IPF to draw one of the patterns of Pattern Puzzle A on page 3. Try drawing other patterns found on page 6 or try to invent one of your own.

3. Use tracing paper and the underlying triangular grid to draw one of the patterns shown in Pattern Puzzle B. Other patterns are found on page 6 or invent one of your own on a blank sheet of triangular grid paper.

4. Refer to the instructions on page 7 to create a six pointed star with a triangular grid within it (steps 1-4). Try to find one of the patterns shown on page 8.

5. Create the four varieties of star patterns described in steps 5-8. Use magic markers to color one of the stars.

Group 2 - Infinite Pattern Factory: square circle grid

1. Refer to the instructions on page 9 of: "The Infinite Pattern Factory (IPF)." Carry out steps 1-5 to create at least 20 circles of the square network of circles. Draw the circles lightly with a pencil. Draw them on Worksheet W2-1.

2. Use tracing paper and continue with steps 6-9 to create the pattern in step 9.

3. Use tracing paper to find two of the patterns on page 12 hidden in the circle grids of pages 10 or 11.

4. Create an 8 and 16 pointed star by following steps 1-8 on page 13. Use magic markers to color on of these stars.

Activity 3: Three Dimensional Structures

Lesson 1: Construction of polyhedra with marshmallows and toothpicks.

Materials: 1 bag of miniature marshmallows (Kraft brand) and one box of round toothpicls for every 3 students.

Exercises:

1. Permit the students a period of free play to get used to building structures with the marshmallows and toothpicks. (15 minutes are recommended).

2. Have students find as many planar patterns that they can such that the same number of toothicks are incident to each marshmallow with the exception of the marshmallows on the boundary, i.e., imagine that the pattern that the student constructs would continue on indefinitely with this constraint. This can be done in only three ways if all faces are congruent. Can you find them on a peice of triangular graph paper (see Figure 3-1)? There are also 8 additional tilingls that use more than one kind of face. These are known as semiregular tilings of the plane. How many of them can you find? They are shown in Figure 3-2.

3. Make a connected cycle of about seven marshmallows and toothpicks. See what kind of designs you can make. You can bend the toothpicks into three dimensional space. Now remove one marshmallow and one toothpick and reconnect the cycle to see what kind of designs you can make with the new 6-cycle. Successively remove toothpicks and marshmallows to form 5,4, and finally 3-cycles, each time creating patterns. Note that when you get to a 3-cycle (triangle) there are no degrees of freedom and we arrive at the general principle that triangles are the basic building blocks of "rigid" structures.

4. Have students construct two triangles from six marshmallows and six toothpicks. Now challenge the students to rearrange the six toothpicks so as to form four equilateral triangles. The only way this can be done is to construct a tetrahedron, one of the Platonic soldis, shown in Figure 3-3. This exercise shows that the geometry of 3-dimensional space differes from 2- dimensional space in that it is more "roomy." The extra triangles materialize due to a kind of synergetic interaction between the toothpicks and the space. Have the students verify that the tetrahedron is rigid. It stands up firmly without collapsing. The rigidity is due to its faces being all triangles.

5. Next have the students construct a square and observe that it is not rigid. Ask them to make the square rigid by adding the fewest number of additional toothpicks and marshmallows. The result will be an octahedron, shown in Figure 3-3, another one of

the Platonic solids.

6. Now ask the students to construct a cube. Before doing this, have them predict whether or not it will be rigid. Since it is not rigid, have the students attempt to make it rigid by placing a toothpick along a diagonal of each face. If this is done properly, the octet truss shown in Figure 3-4 results. The octet truss is the basis of space frames.

7. Now lead the students in an exercise to construct a dome made completely out of triangles. To do this have them place five toothpicks in one marshmallows and then ring them by toothpicks so as to form five equilateral triangles surrounding the central marshmallow as shown in Figure 3-5. Note that these triangles will not lie in a plane (why?). Next construct five additional triangles on the outer edges as shown in Figure 3-5. If a belt of toothpicks are used to connect the vertices or these outer triangles, and the resulting figure is capped by another pentagon of equilateral triangles identical to the original, a fourth Platonic solid known as the icosahedron (see Figure 3-3) formed. This polyhedron is the basis of the geodesic dome of Buckminster Fuller.

8. After a short discussion of the Platonic solids the students can try their hands at creating more permanent models by folding up the paterns shown in Figure 3-3, from the plane. See also the enclosed handout. Octahedra and tetrahedra with identical edge lengths are particularly versatile. If you construct several of these in a ratio of 2 tetrahedra for each octahedron, you can use these as building blocks to create more complex structures. Attaching vel-cro to the faces will enable you to stick them together.

9. The Platonic polyhedra are "perfectly symmetric" in other words they look identical when viewing them face-on, vertex-on, or edge-on. It is a good exercise to take your marshmallow and toothpick Platonic soldis and draw their face-on, vertex-on, and edge-on views.

10. In addition to the tetrahedron, octahedron, and icosahedron, there are exactly five additional polyhedra that can be made with triangle faces. These are called deltahedra. Try to create some of them with your marshmallows and toothpicks. They are shown in Figure 3-6 in case you have trouble finding them all.

Activity 3:

As you construct these polyhedra in the workshop record the following information:

THE PLATONIC POLYHEDRA

NAME	F	V	E	p	q	RIDGID?	F+V-E

The spherical deviation δ of the vertex of a polyhedron is defined as,

δ = 360 - (sum of the angles θ around a vertex)
where θ is a typical face angle.

After you have constructed a Platonic polyhedron complete this table:

NAME	q	V	θ	SUM OF θ AROUND A VERTEX	δ	δ x V
TETRAHEDRON	3	4	60°	180°	180°	720°
CUBE						
OCTAHEDRON						
DODECAHEDRON						
ICOSAHEDRON						

WHAT DO YOU NOTICE?

F = Number of faces

V = Number of vertices

E = Number of edges

P = Number of edges per face

q = Number of faces surrounding each vertex or edges per vertex

TABLE 1. THE PLATONIC SOLIDS

Name	F	V	E	p	q	F + V - E
Tetrahedron	4	4	6	3	3	2
Cube	6	8	12	4	3	2
Octahedron	8	6	12	3	4	2
Dodecahedron	12	20	30	5	3	2
Icosahedron	20	12	30	3	5	2

FIG. 3-1

(a)

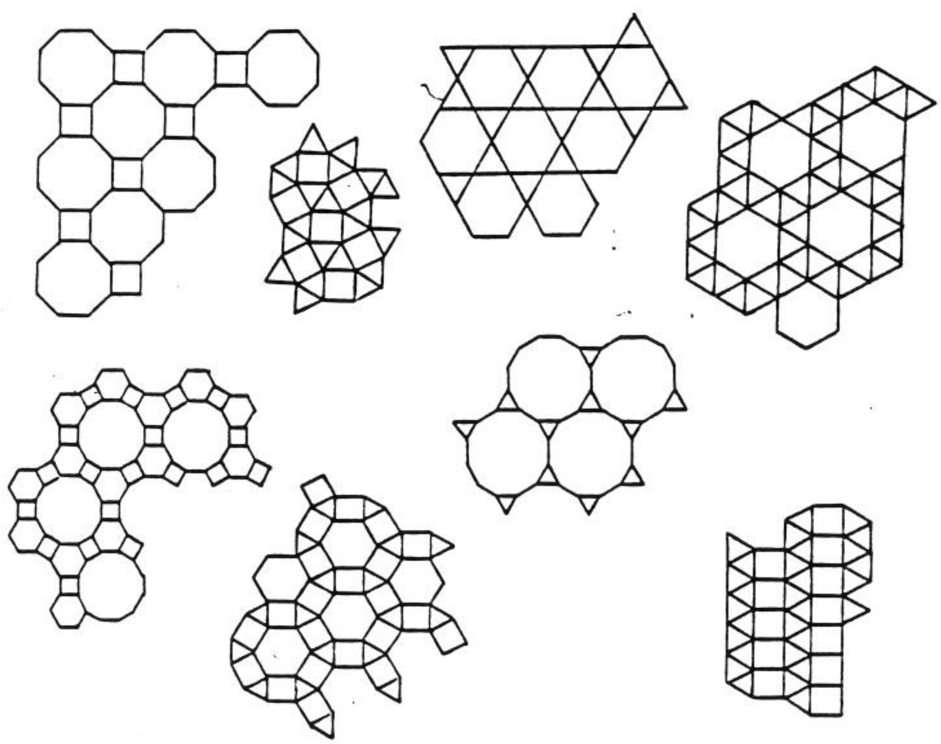
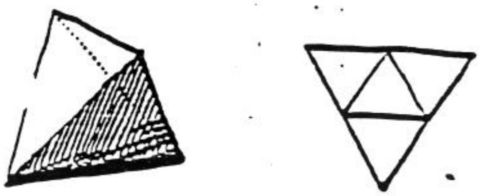


FIG. 3-2

(b)

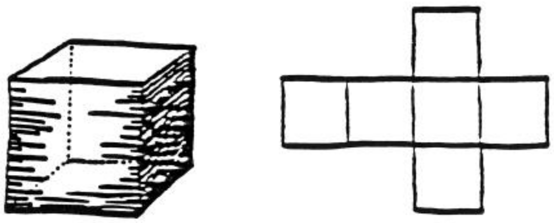
The three regular tilings and eight semiregular tilings of the plane. The tiling 3^4 6 exists in two mirror-symmetric (enantiomorphic) forms.

Tetrahedron {3,3}



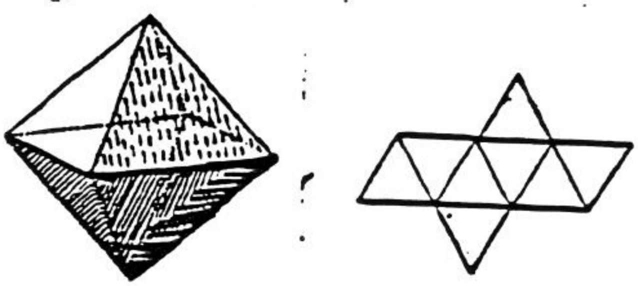
Faces	Vertices	Edges
F_3	V_3	E_{3-3}
4	4	6

Cube (Hexahedron) {4,3}

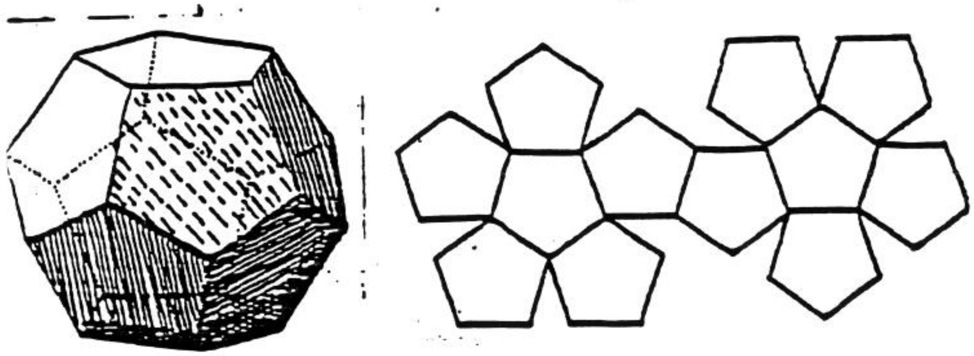


Faces	Vertices	Edges
F_4	V_3	E_{4-4}
6	8	12

)ctahedron {3,4}

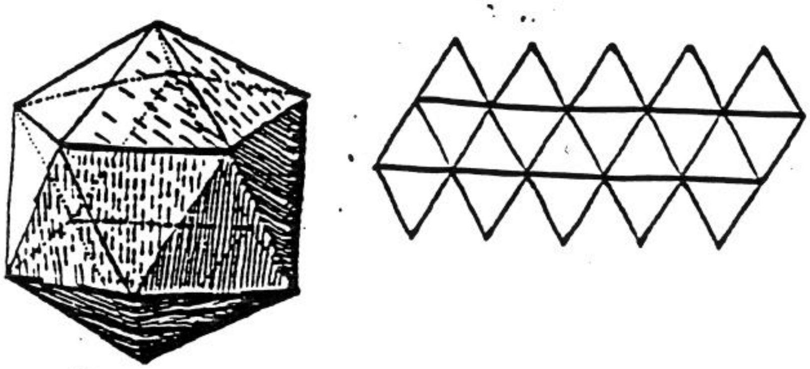


Faces	Vertices	Edges
F_3	V_4	E_{3-3}
8	6	12



Faces	Vertices	Edge
F_5	V_3	E_{5-}
12	20	30

Icosahedron {3,5}



Faces	Vertices	Edges
F_3	V_5	E_{3-3}
20	12	30

FIG. 3-3

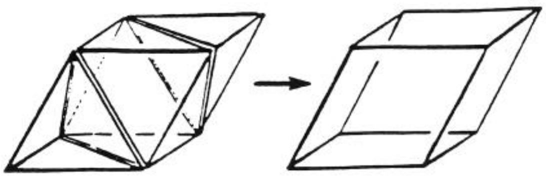
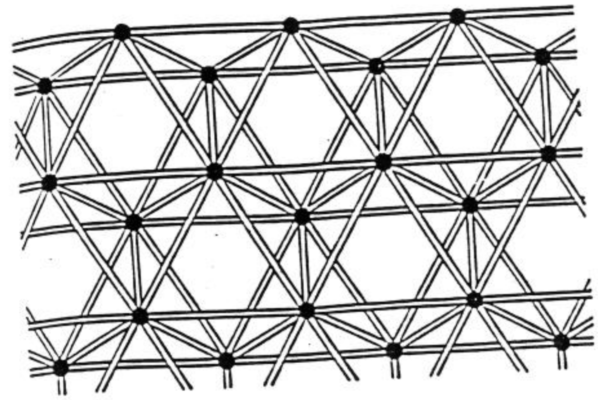


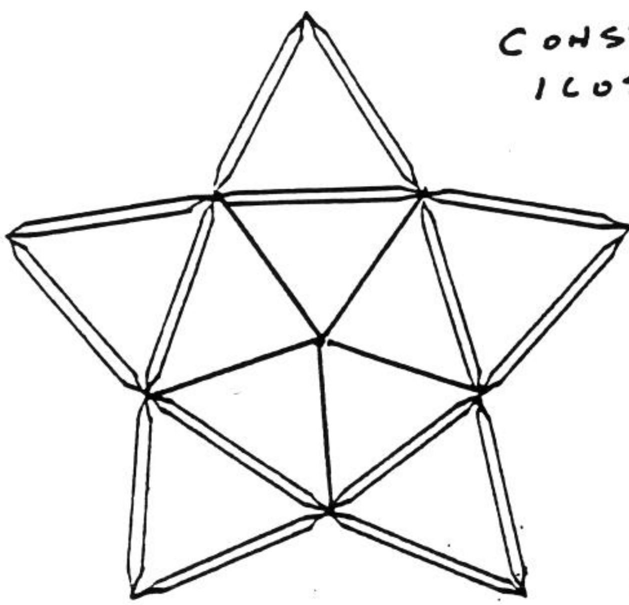
Figure 7.18 Two tetrahedra and one octahedron form a parallelopiped that fills space.



An octet truss.

FIG. 3-4

CONSTRUCTION OF AN ICOSAHEDRON



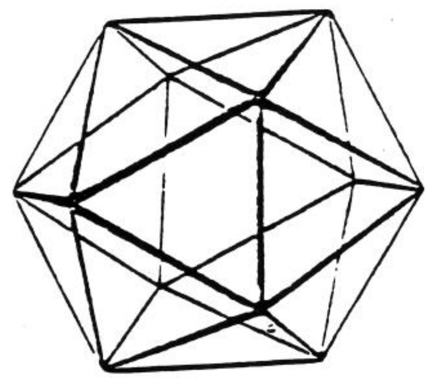


FIG 3-5 Icosahedron

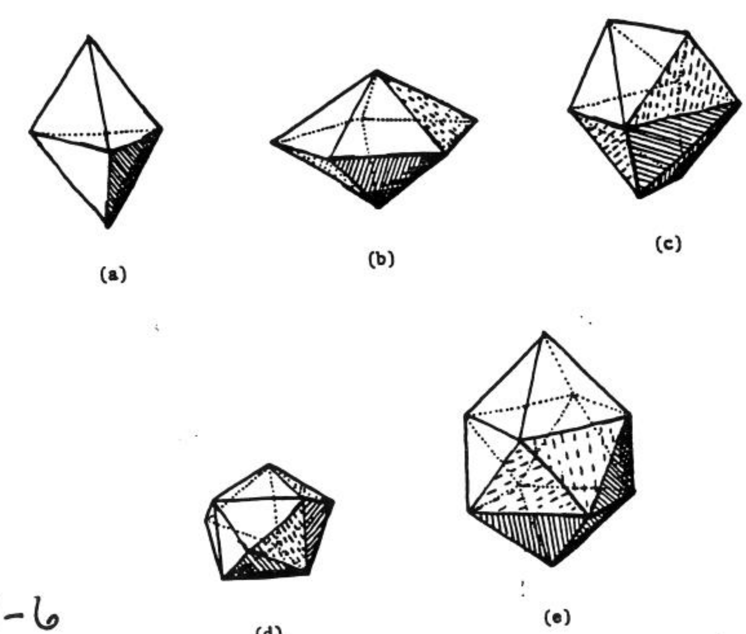


FIG. 3-6

~~Figure 3-44~~. *Deltahedra: (a) six faces, (b) ten faces, (c)* … *faces, and (e) sixteen faces.*

Lesson 2: The triangle is the only polygon that is rigid.

Materials: 10 cardboard rectangular pieces 11 inches by two inches
20 paper fasteners

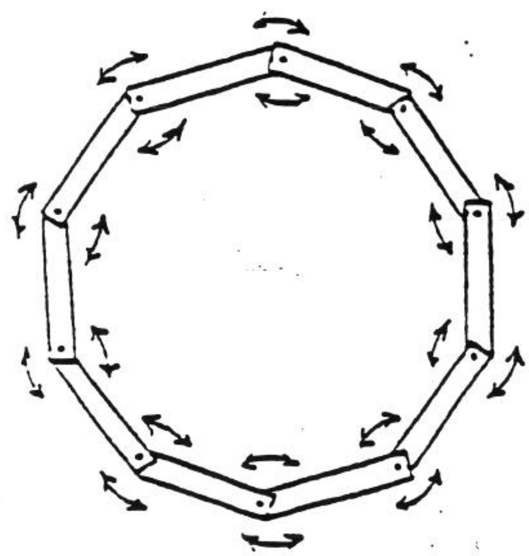
Estimated Time of Accomplishing Lesson: One 45 minute period

Underlying Rationale: Project Synergy is concerned with concepts that are reflected in three-dimensional real space. Most geometry courses begin with the Euclidean axiomatic assumption of the point, line and plane. Since these axioms, by definition, cannot exist in real space, we begin with the primary object that does exist: the finite edge or line. From this finite edge, we experiment with interacting combinations to find the primary structure of real space: the triangle.

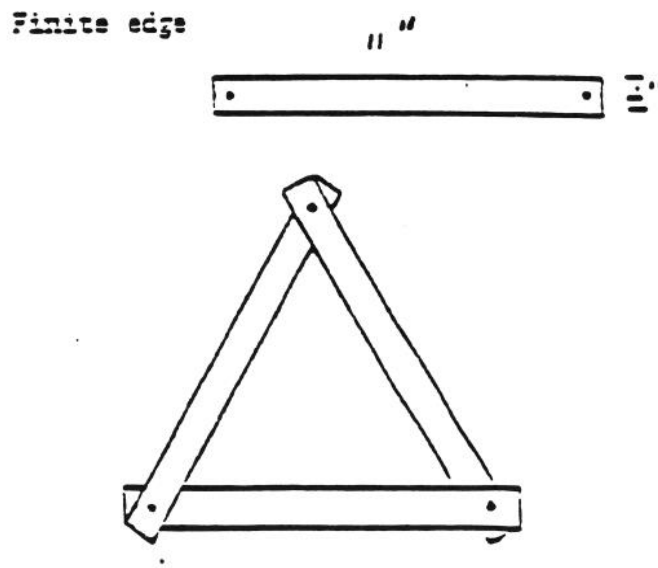
Procedure: Ten cardboard rectangular pieces 11" by 2" are connected end to end by paper fasteners. Remove one piece and reconnect the fasteners to create a closed nine-sided polygon. Test to see if the pieces cannot move freely about the paper fasteners. Proceed to repeat the operation until the pieces cannot move about the fasteners.

How many sides will the figure have?
Will a smaller sided figure be rigid?
Will a larger sided figure be rigid?

Vocabulary: vector edge structure finite polygon



Movable Ten-sided Polygon



Triangle is rigid

→ NOTES ON TRIANGLES AND DOMES by GERRY SEGAL.
SYNERGY: AN INTERDISCIPLIN-
ARY ~~MATHEMATICS~~ EXPERIENCE
FOR THE MIDDLE SCHOOL.
N.Y. BOARD OF EDUCATION
CENTER FOR CURRICULUM DEVELOPMENT

Lesson 3: Building a structure with the triangle as the basic element

Materials: 96 cardboard rectangular pieces 11 inches by two inches, 36 paper fasteners

Estimated Time of Accomplishing Lesson: Three 45 minutes periods

Underlying Rationale: Having explored the idea of the triangle as a rigid structure, we are now going to explore the function of the form.

Procedure: Step 1 - Connect 4 pieces of cardboard in order to make a quadrilateral.

Step 2 - On each vertex place three cardboard pieces. Label each piece a-b-c. (see diagram)

Step 3 - Connect slat a of each vertex with slat c of its neighboring vertex. Four triangles should have been completed.

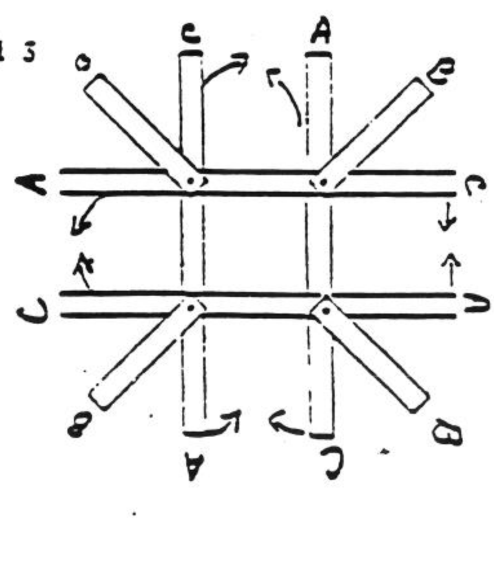
Step 4 - Between each triangle there is a single slat b. Connect the outer most vertex of each triangle (where a and c meet) to slat b with a new piece of cardboard. Since each slat b is between two triangles, a belt of 8 cardboard pieces will be needed to connect the triangles to slat b. At this point the size of the edges are not large enough to maintain a two-dimensional design. Each slat should curve by the tension and an object resembling a basket will be formed. (see diagram)

Step 5 - Place two slats on every joint of the octagonal belt. Connect neighboring slats to create eight new triangles. Then connect the outer vertices of neighboring triangles by adding eight more cardboard pieces.

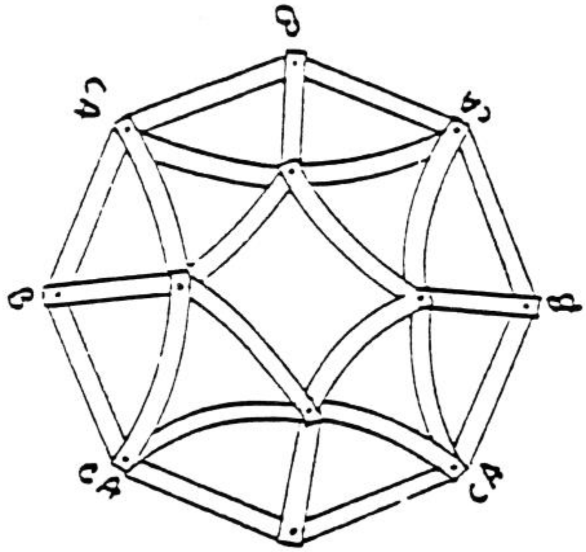
Step 6 - Repeat Step 5 until all 96 slats are utilized.

Step 1

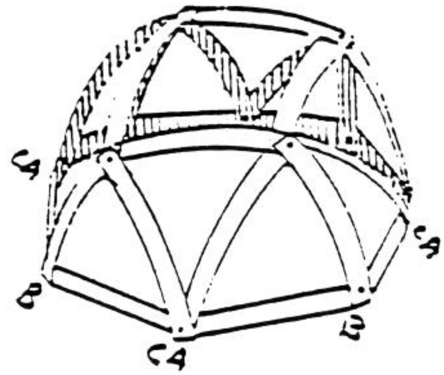
Steps 2 and 3



Step 4

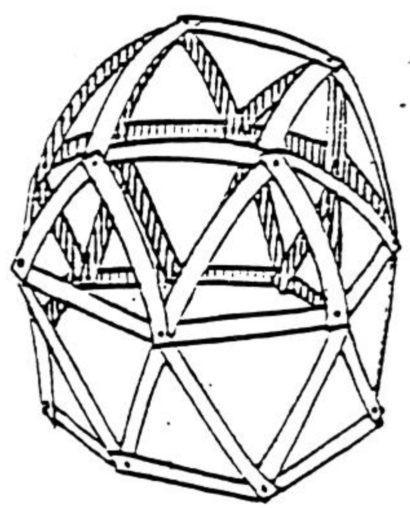


Schematic



Side view

Step 5



Side view

Questions: Where do we see triangles being used as basic structures?
What would happen if we made the tower out of quadrilaterals?

Vocabulary: Quadrilateral

Activity 4

Materials

Construction paper
Magic markers
scissors
file folders
graph paper

Group 1 - Tangrams and Amish Quilt Patterns

Tangrams is an ancient Chinese puzzle made up of pieces that come from a dissection of square into seven polygons: five 45-degree right triangles, a square, and a parallelogram as shown on Worksheet W4-1. The earliest reference to tangrams was in a Chinese book dated 1813. It was the Rubik's cube of the nineteenth century due to its popularity in those days.

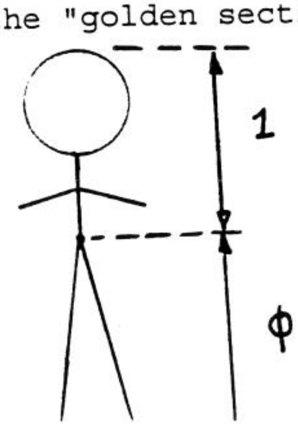
1. Cut out the tangram pieces found on Worksheet W4-1. Assemble them to reproduce the pictograms shown in Figure 4-1. You must use all of the tangram pieces once only with no overlaps in each pictogram.

2. Try to create an octagon from three tangram squares and two tangram diamonds. Take it as a challenge to recreate the tangram square. Can you construct a triangle using the tangram pieces?

3. Create an "Amish quilt" pattern using the tangram pieces found on Worksheet W4-2. You may use any of these shapes that you wish, and you may repeat a shape as many times as you wish to create a pattern reminiscent of an Amish quilt such as the one shown in Fig. 4-2. It may be helpful to first cutout the shapes from the worksheet and then trace them onto a piece of stiff cardboard such as a file folder. The cardboard template can then be used to recreate these polygon shapes on your construction paper.

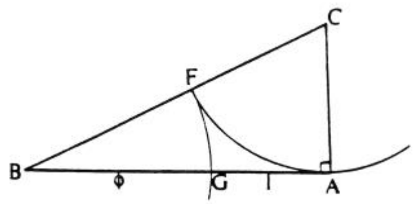
Group 2 - Golden Mean and Golden Triangle Patterns

1. The golden mean proportion comes up frequently in art, architecture, music, and science. The golden mean is symbolized by ϕ and equals $\phi = (1 + \sqrt{5})/2 = 1.618...$ In a large population, it is found that the ratio of your bellybutton to the ground divided by the the top of your head to your bellybutton is approximately the golden mean. In other words your bellybutton divides your length by the "golden section" (try this!) the ratio : 1.



2. The following procedure can be used to divide a line into the golden section (the ratio ϕ :1)

   a. Begin with a line segment AB drawn on a piece of graph paper
   b. Draw AC = 1/2 AB perpendicular to AB
   c. Circular arc CA intersects CB at F.
   d. Circular arc BF intersects AB at G, breaking AB into the golden section.



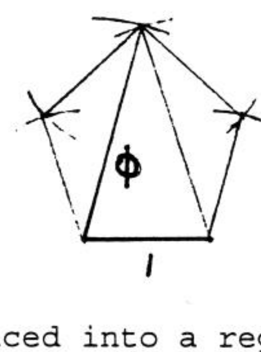
[redacted] Dividing a line into its golden section with compass and straightedge.

3. After you have created the two lengths of the golden section, you can use these to create a regular pentagon since the ratio of diagonal to side of a regular pentagon is ϕ : 1. Follow this procedure:

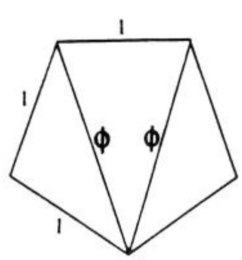
   a. Begin with a line segment divided into the golden section (see 2 above). Mark off the shorter of the two lengths (length = 1 unit) of the golden section on your graph paper.

   b. Create an isoceles triangle with base 1 unit and side equal to ϕ units (the longer length of the golden section) as shown below.

   c. With your compass set to 1 unit, recreate the other four sides of the pentagon.



4. Two diagonals placed into a regular pentagon divides it into two species of "golden triangles" as shown in 3 above.



[redacted] A pentagon subdivides into one type 1 and two type 2 golden triangles.

formed at different scales. In this figure the base angle of triangle I is successively bisected to produce triangles I and II at a smaller scale. Worksheet W4-3 presents a template from which to cut out species of triangles I and II. Create a pattern with construction paper using golden triangles I and II at more than one scale. It may be helpful to cut the triangles out of the template and reconstruct them on a piece of stiff cardboard such as a file folder, and then use the template to trace onto construction paper.

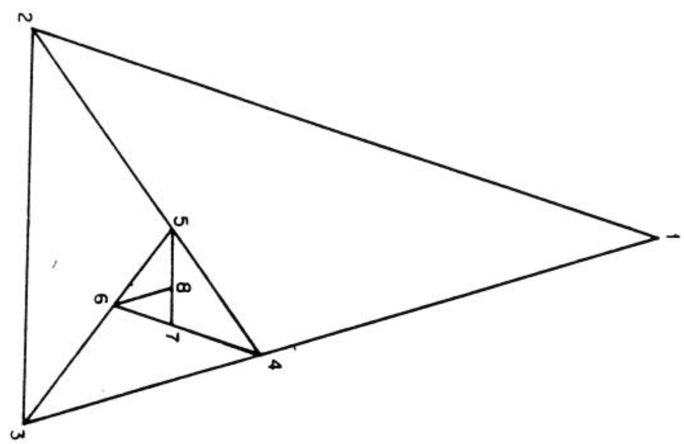


Figure 3.13 Whirling golden triangles.

Group 3 - The Brunes Star

1. The remarkable Brunes Star is shown in figure 4-3. A square divided into four half-squares is shown in Worksheet W4-4. By placing two diagonals into each half-square you can recreate the Brunes star. Create an interesting design by using magic markers to color this star.

2. This star is made up entirely of 3,4,5-right triangles or fragments of 3,4,5-right triangles. Examine the demonstration corkboard to see how it is created from four large 3,4,5-right triangles. Figure 4-4 shows some of the measurements. 3,4,5-right triangles are present at four different scales. Can you identify some of them.

3. At different levels, the Brunes star divides the width of the star naturally into either 3,4,5,6, or 8 sublengths. Can you find these dissections? If you place the point of your compass at the lower left-hand vertex of the square and sweep an arc of a circle from the bottom edge to the left side of the square a line drawn from left to right on the square is approximately divided into 7 equal segments. Try this. The result is shown in Figure 4-5.

4. Worksheet W4-5 illustrates a square divided into four subsquares. With a pencil, lightly draw four Brunes stars, one inside of each square. By judiciously removing lines and coloring the remaining spaces, create an interesting composite pattern.

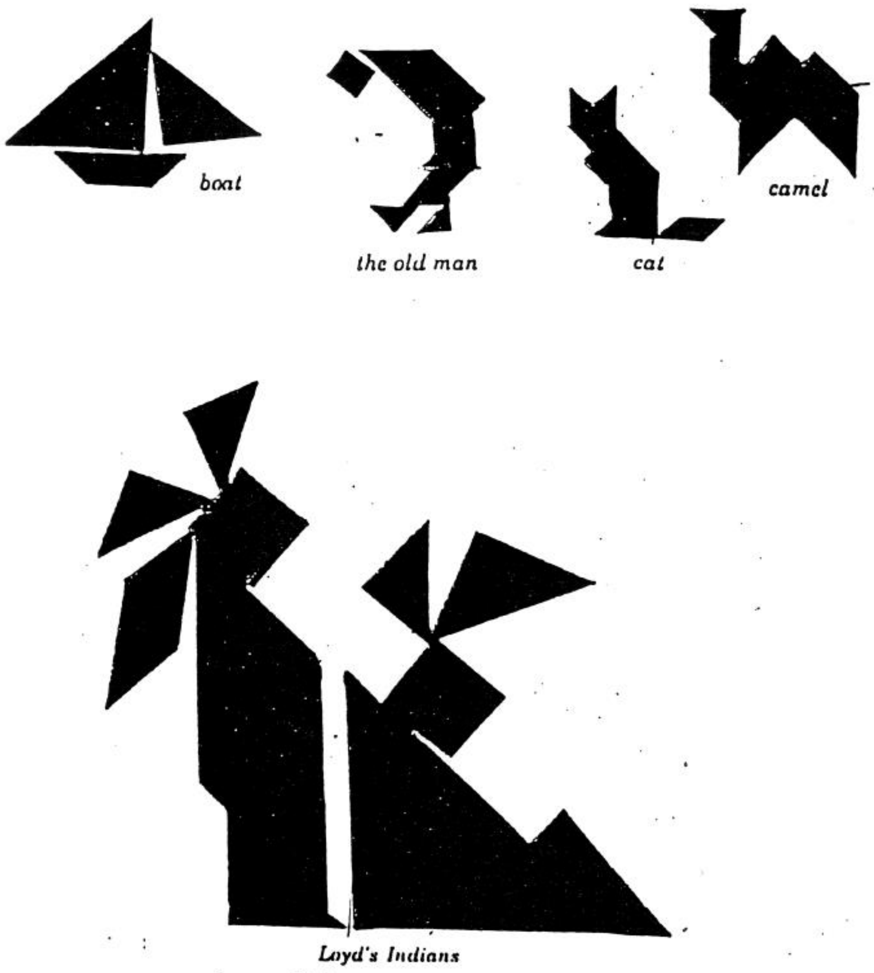
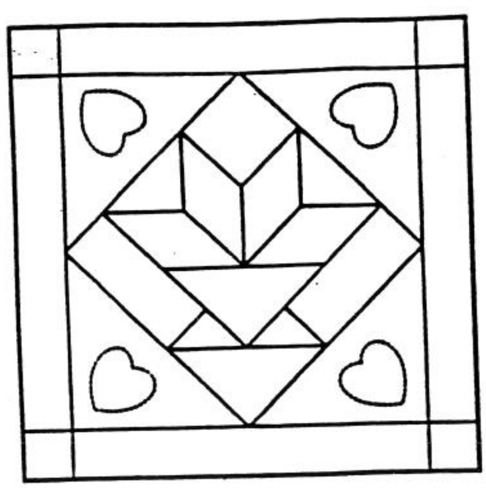


FIG 4-1



~~Figure 1a (left)~~. A quilt with tangram pieces.

FIG 4-2

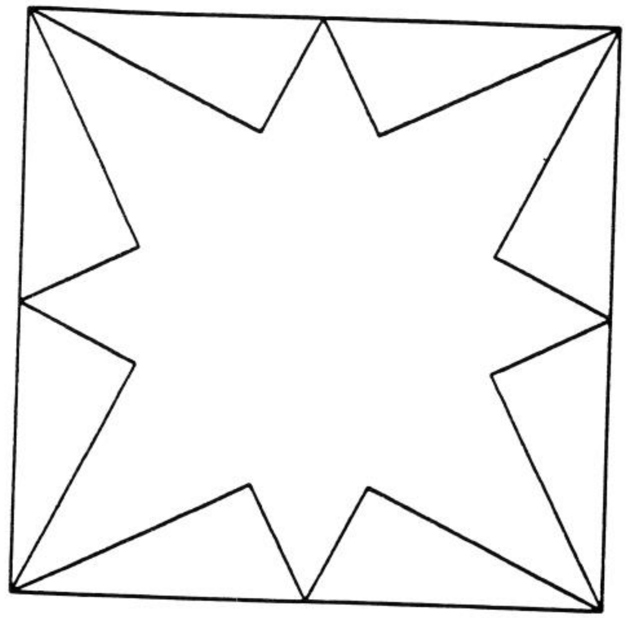


FIG. 4-3

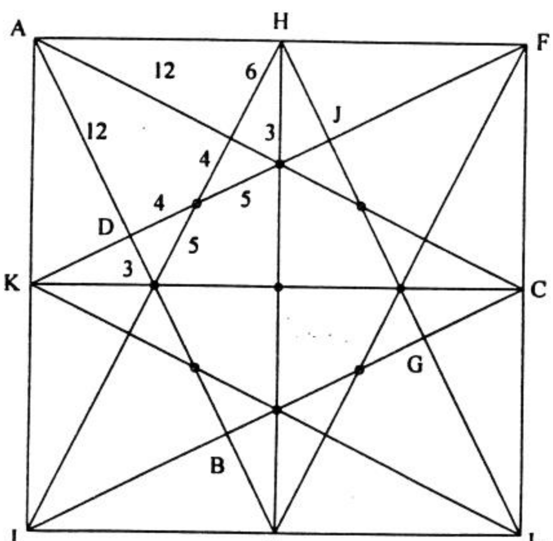


FIG 4-4

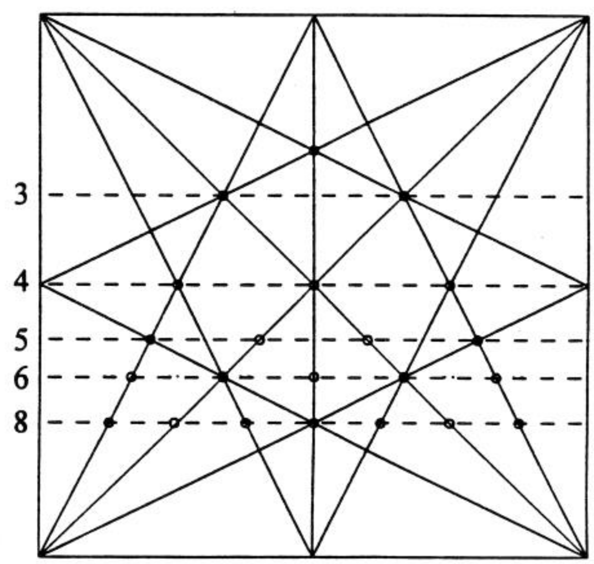
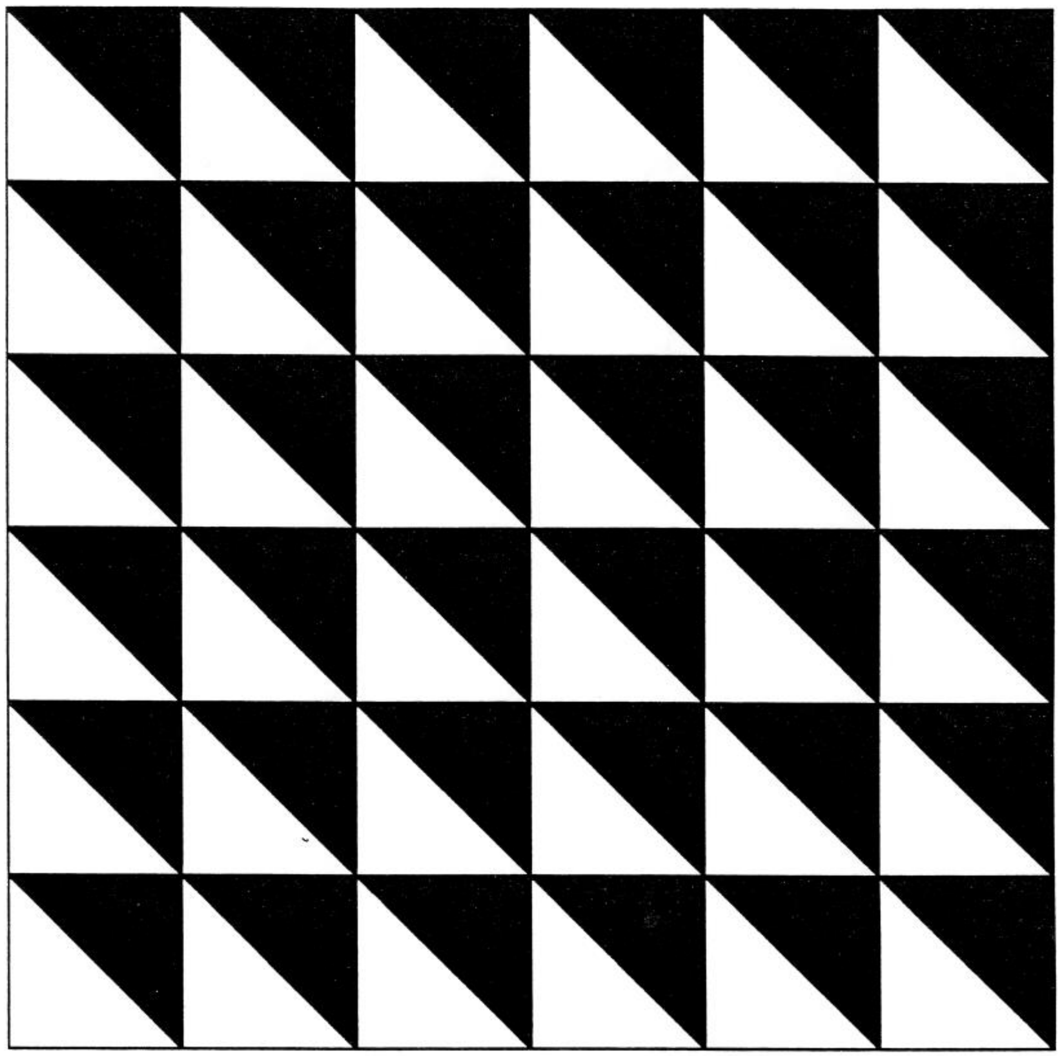


FIG. 4-5

HALF SQUARES

W1-1



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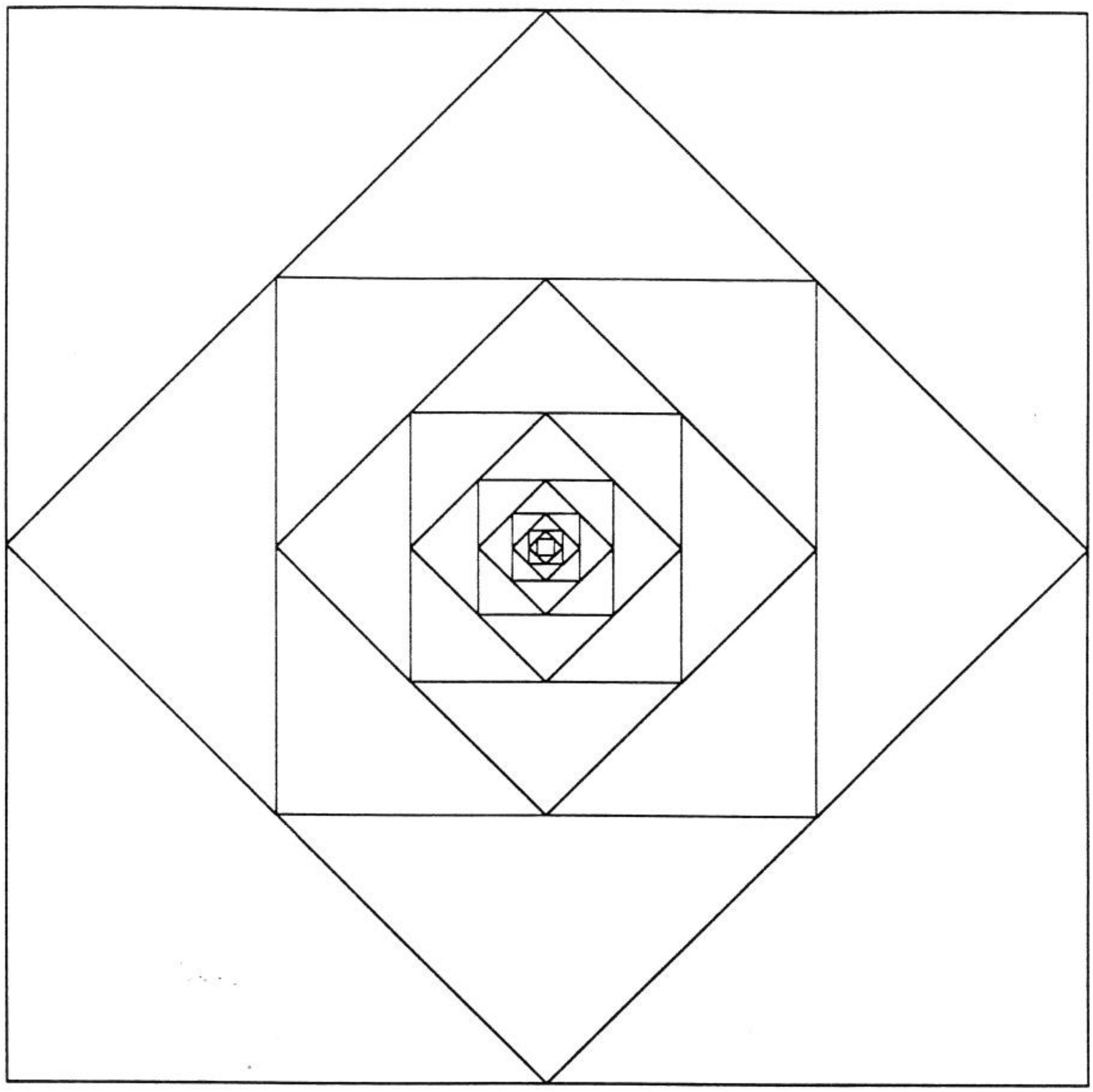
W1-2

BARAVELLE SPIRAL



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10/23/96

W2-1

SQUARE GRID OF CIRCLES

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10/23/96

W4-1

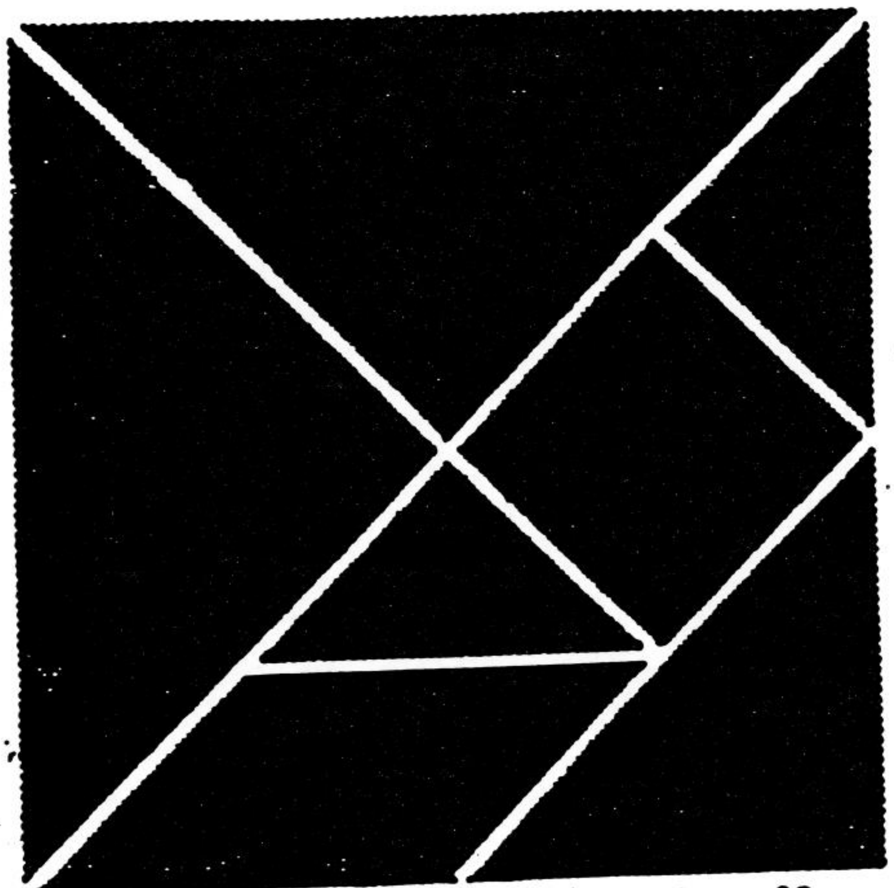
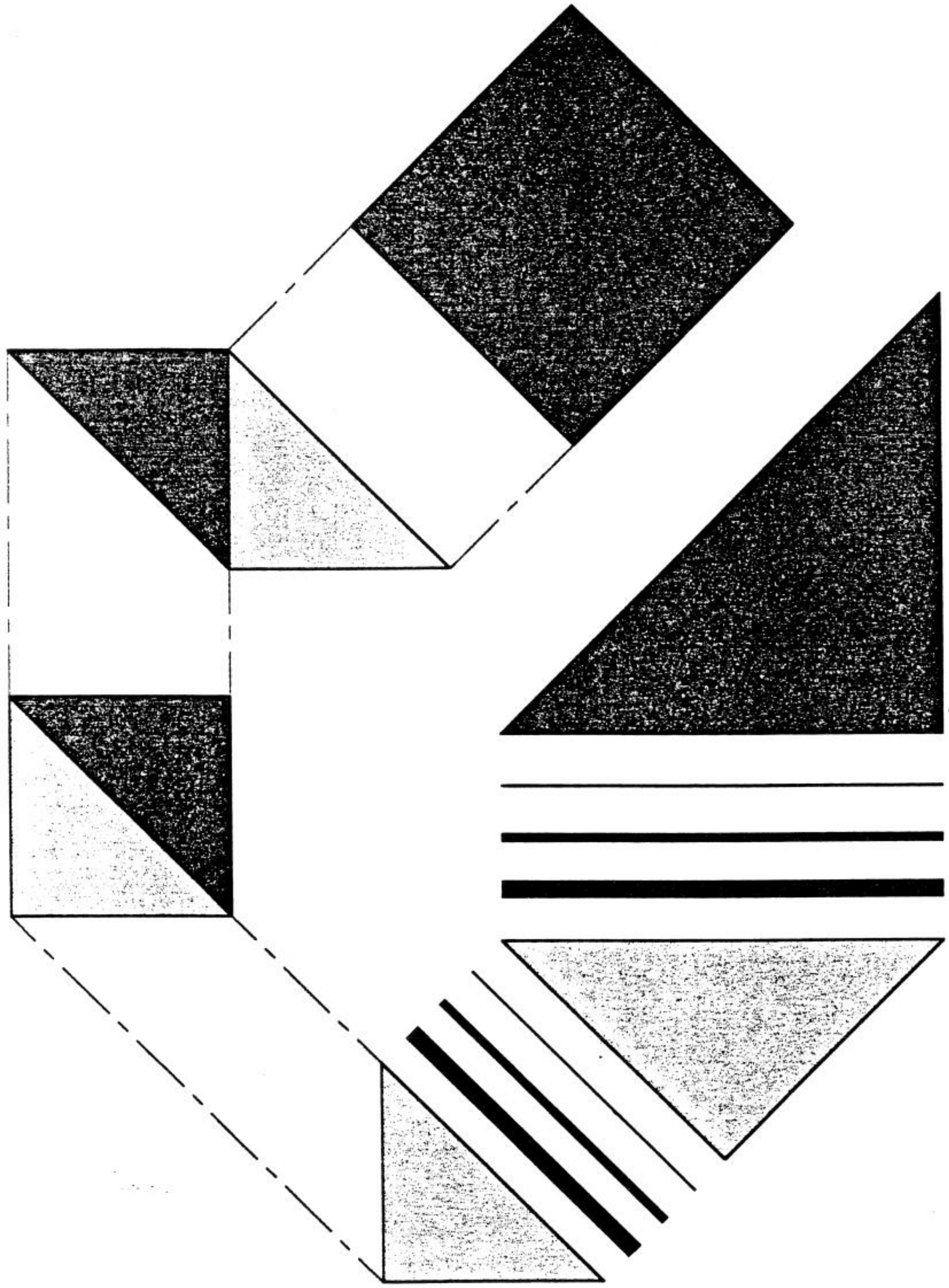


Figure 3a. The seven Tangram pieces formed by dissecting a square.

W4-2



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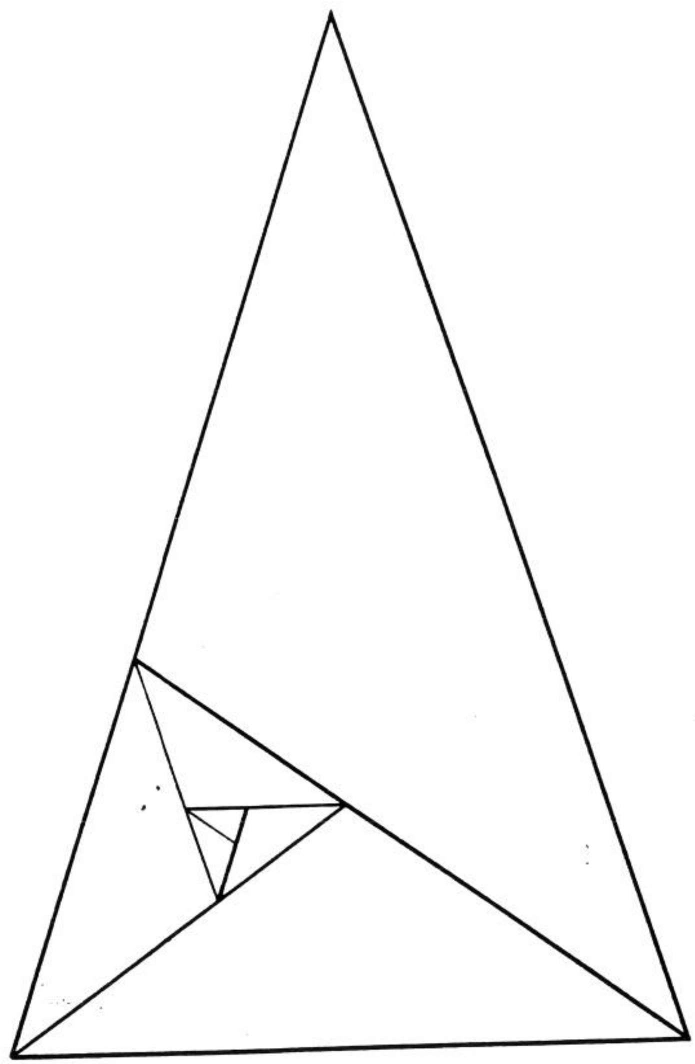
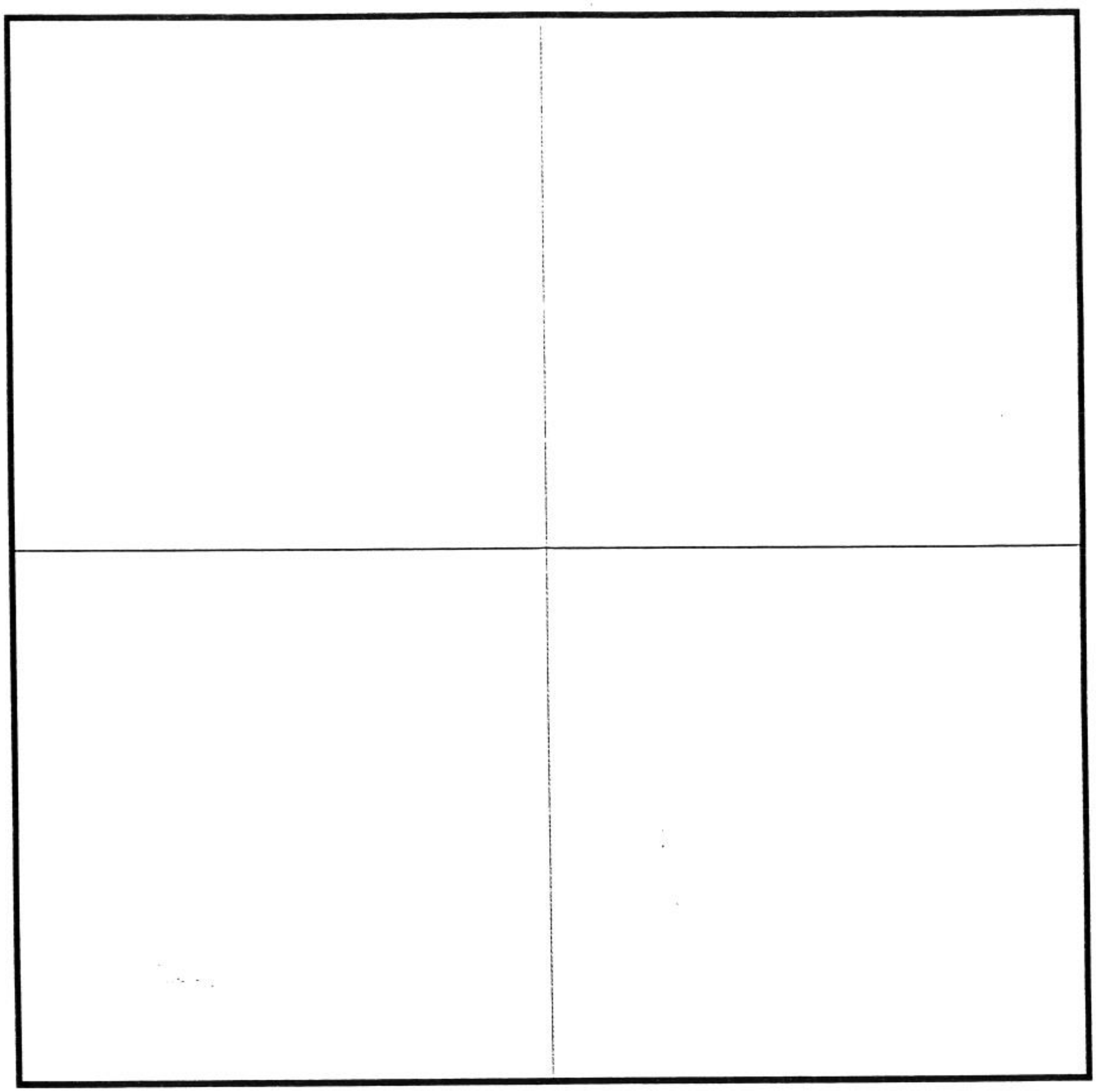
11/25/95
revised 10/23/96



The Golden Triangle

W4-4

BRUNES SQUARE DESIGN



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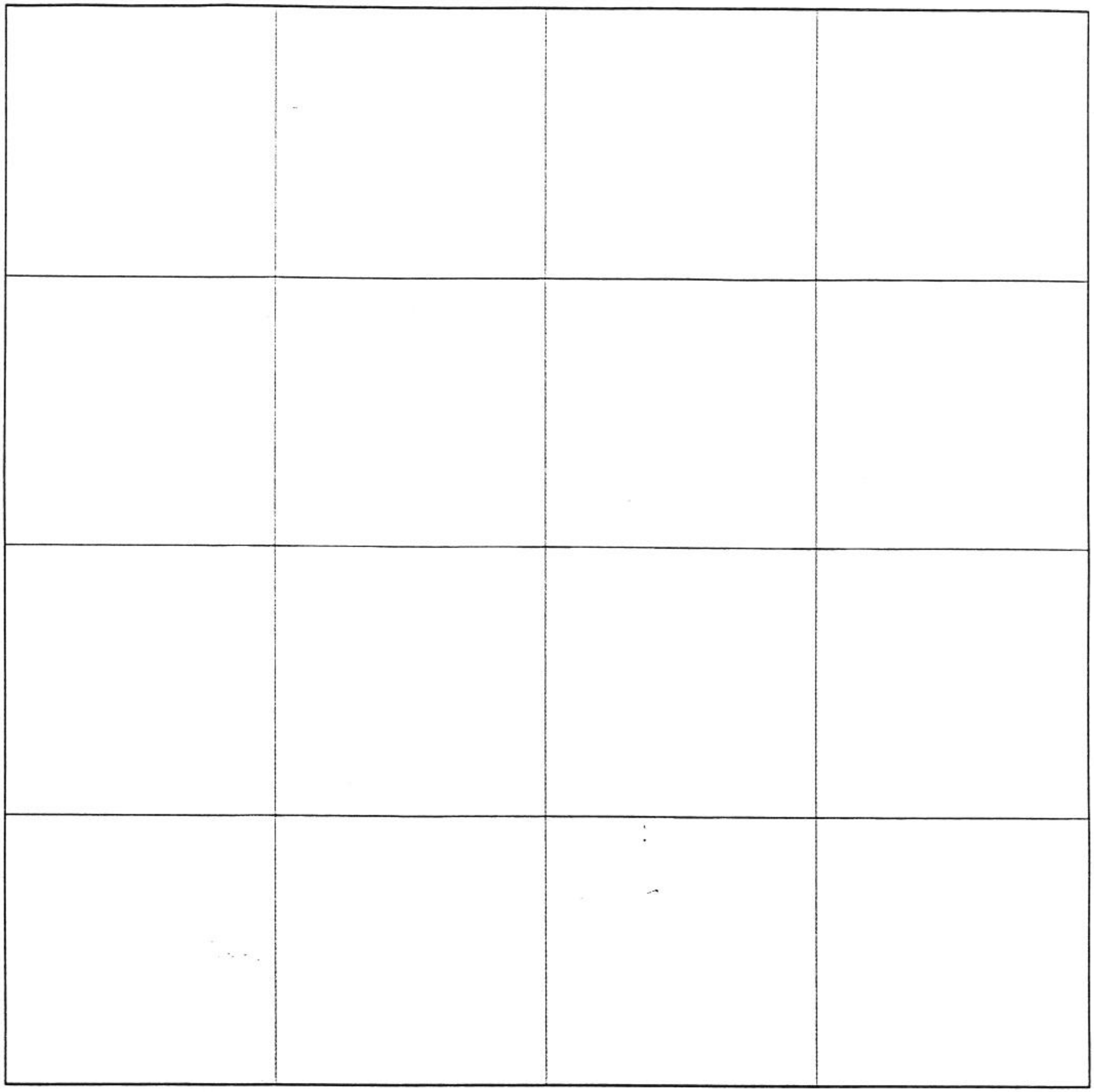
BRUNES SQUARE DESIGN



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