Nat's Knot Notes:
An Introduction to Knots and Soap Film Minimal Surfaces

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Introduction

The study of knots is a branch of topology. This introduction will be intuitive, visual, and hands-on. One generally ties a knot with a piece of string. When the ends are loose, the knot is referred to as an open curve in space, or open knot. When the ends are joined, the knot is referred to as a closed curve in space, or closed knot. Henceforth a knot will refer to a closed knot. Thus to form a knot, we tie an open knot and then join the ends.

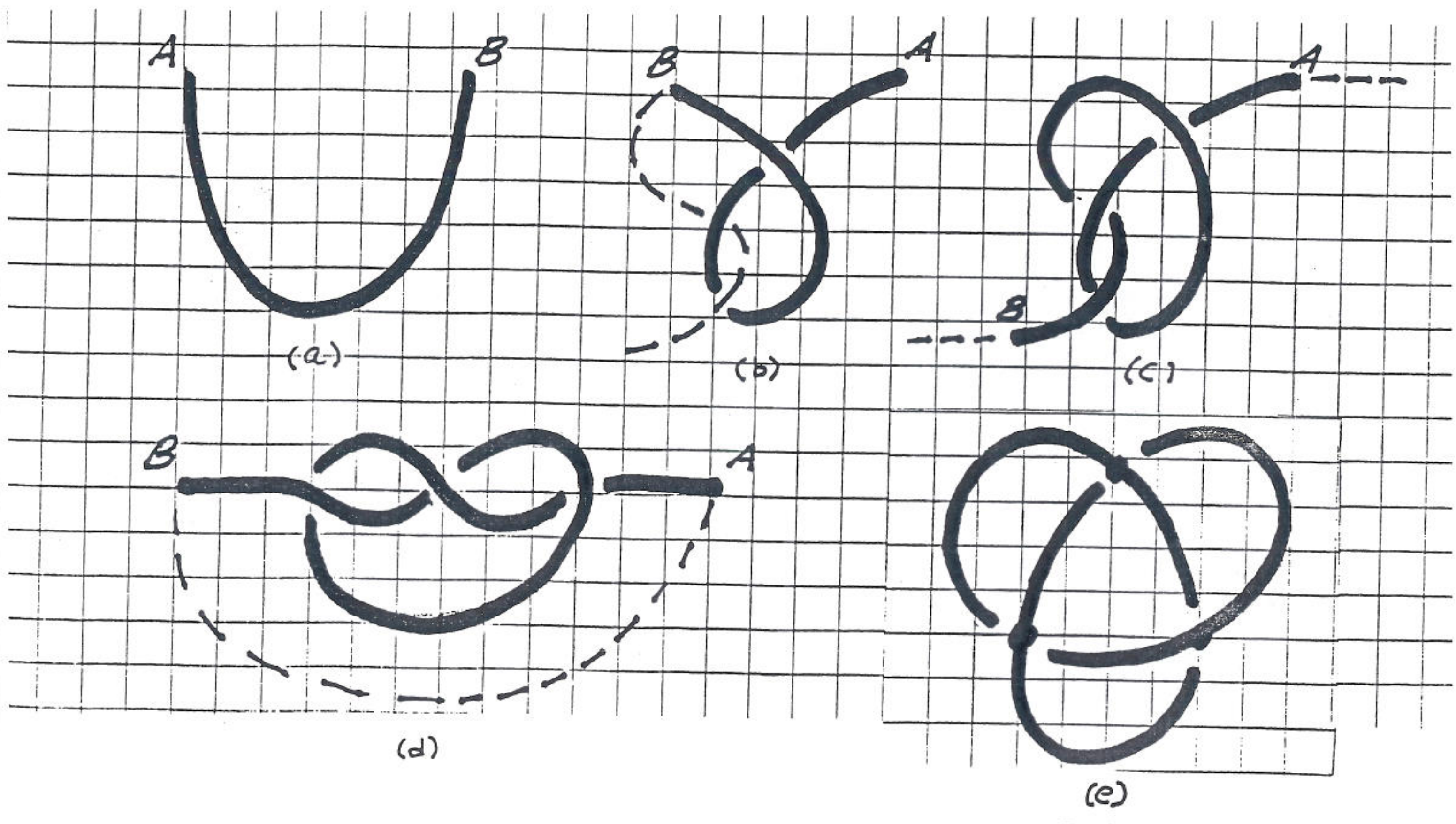
One can form a wire knot and then dip the wire knot in a solution of water, liquid soap, and glycerin. A soap film surface forms with the wire knot as boundary. The surface tension of the soap solution causes the surface to take a shape with minimum surface area for surfaces with the wire as boundary. Hence the surface is referred to as a soap film minimal surface (sfms). These surfaces are a very interesting subject for presentation in grades 4 and above. There is the question of whether the sfms is one-sided or two-sided. Generally the sfms is one-sided. There is then the question of how to deform the knot to obtain a new configuration of the knot that has a two-sided sfms. This motivates the basic topic of knot deformation.

We consider a knot to be made of a flexible material. Allowable knot deformations are bending, twisting, stretching, or shrinking. We say two knots are *equivalent* when one can be deformed into the other. A basic question is when are two knots equivalent? We will discuss this question for some basic examples.

The Trefoil Knot.

We will first tie a so-called trefoil knot. We can use a piece of string, plastic tubing, or wire. Let us suppose we have a piece of string with ends A and B as in Figure 1(a). We first cross B over A as in (b). Note that we leave little gaps on each side of the B-part to denote the B-part is the over-crossing and the A-part is the under-crossing. We now bring B through the center as indicated by the dashed lines in (b) to obtain (c). We can now pull the two ends as indicated by the dashed lies in (c) to obtain (d). In (d) we have an open overhand knot. We now join the ends A and B to obtain the closed knot in (e). Note that we have left little gaps to indicate the over/under crossings. We refer to the picture in (e) as a *knot diagram*. This particular diagram is referred to as a *trefoil knot*. The name refers to the three outer "leaves". In Italian three is tre and leaf is foglia and the plural is foglii so in Italian the name would be trefoglii. In English the name is trefoil.

To join the ends A and B, we can use a piece of scotch tape for string. For plastic tubing, we can use a short piece of insulated wiring that fits inside the tubing. For a wire knot, we can twist the ends together.



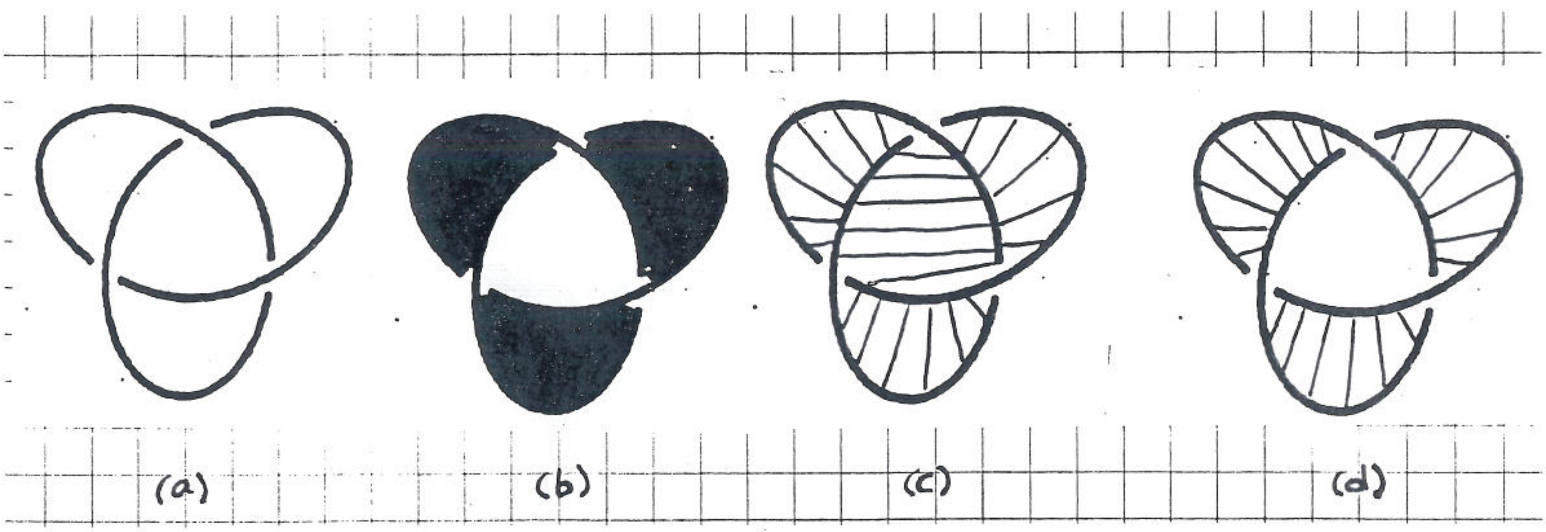
Trefoil knot

Tying A Trefoil Knot

Figure 1.

One can consider the trefoil diagram in Figure 2(a) as a map where each region is a country. The map can be two-colored using shaded and white to distinguish regions that share a boundary. This is called *checker boarding the diagram*. The checker boarding is shown in (b). One can first color the upper left region shaded. Since this outer region shares a boundary with the center region, the center region will be white. Therefore the remaining two outer regions will be shaded. Thus regions that share a boundary have different colors shaded and white.

The checker boarding can be used to predict the corresponding soap film minimal surface (sfms) for a wire model of the knot. Namely, the shaded regions will correspond to surface and the white regions will correspond to windows or spaces. In (b) there is one white central region so the corresponding sfms will have one central window. When a wire model is dipped and removed, the initial sfms will have a central disk as in (c). We represent a transparent sfms by lines. The central disk can be punctured to obtain the final sfms in (d), as predicted by the checker boarding in (b).



Checker Boarding the Diagram to Predict the SFMS.

Figure 2.

We will now consider two more examples. In Figure 3(a) we have a knot diagram. The checker boarding is shown in (b). In this case there are upper and lower shaded regions corresponding to predicted surface and two horizontal white regions corresponding to predicted windows. We can now dip a wire knot corresponding to the diagram in Figure 3(a) to obtain the sfms in (c). There are two horizontal disks that can be punctured to obtain the sfms in (d), which has upper and lower surface regions and two horizontal windows as predicted from the checker boarding in (b).

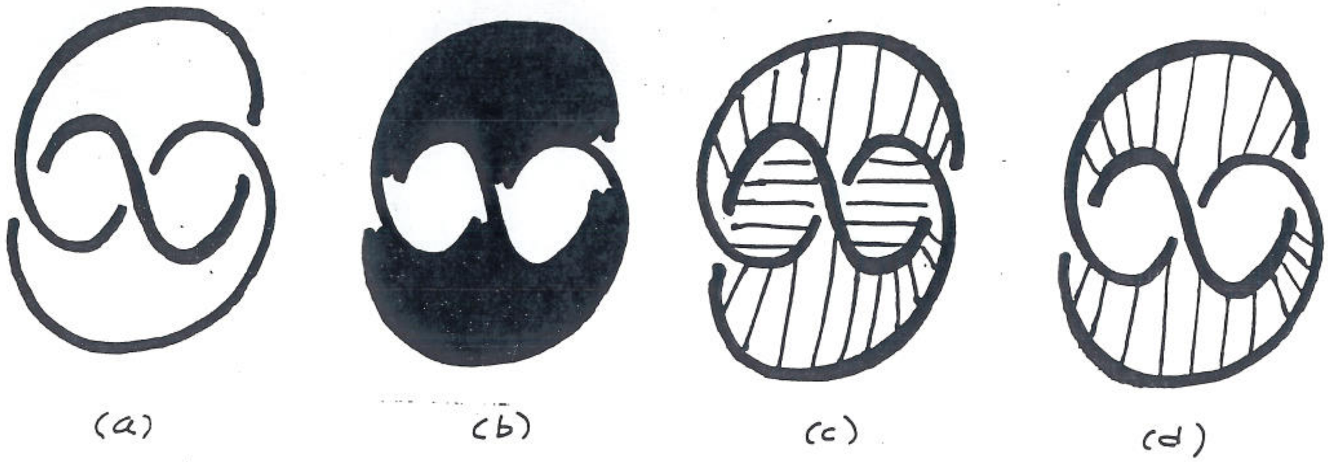


Figure 3.

We have another knot diagram in Figure 4(a) and the checker boarding is shown in (b). In this case there are three shaded regions corresponding to surface and two vertical white regions corresponding to windows. We can now dip a wire knot corresponding to the diagram in (a) to obtain the sfms in (c). There are two vertical disks that can be punctured to obtain the sfms in (d), which has three surface regions and two vertical windows as predicted by the checker boarding in (b).

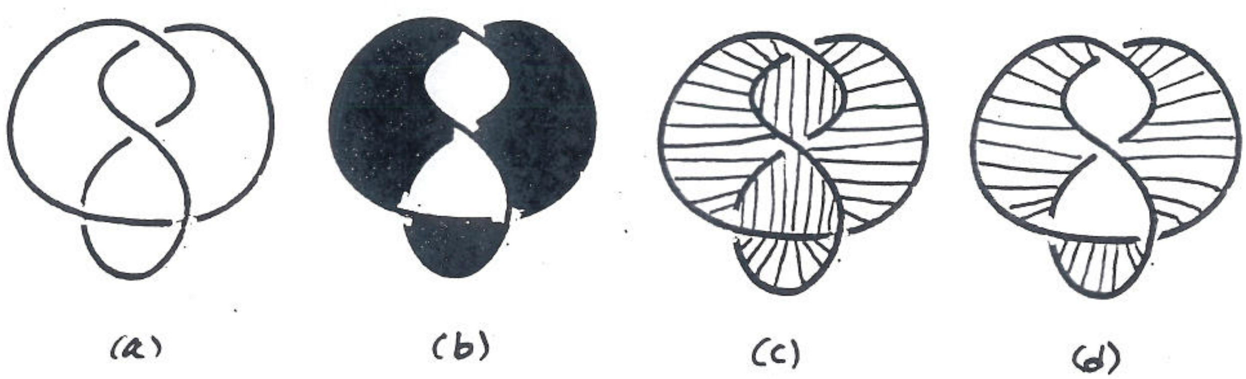


Figure 4.

We will discuss knot deformation below. It will then be shown that the diagrams in Figures 3(a) and 4(a) can be obtained by deforming the trefoil diagram in Figure 1(e). Thus the diagrams in Figures 3(a) and 4(a) are also diagrams of a trefoil knot. We will first discuss one-sided and two-sided surfaces.

One-sided and Two-Sided Surfaces.

We will start with a paper circular band as shown in Figure 5. The circular band has two boundary edges: an inner circular edge C_1 and an outer circular edge C_2 which are outlined in black. The circular band also has two sides: a front side and a back side. In order for a bug on the front side to walk onto the back side, the bug would have to cross over either C_1 or C_2.

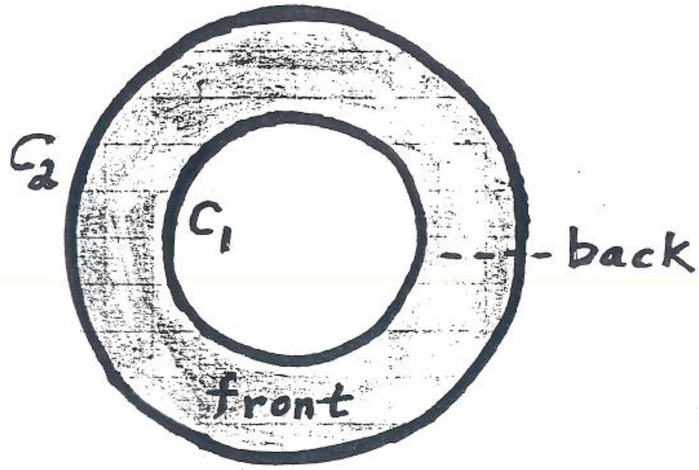


Figure 5.

We will now cut the circular band as indicated by the dashed line in Figure 6(a). We give the bottom right end a half-twist, as indicated by the arrow, and then join the two ends to obtain the surface in (b). Note that the half-twist results in joining the front side to the back side so that the surface in (b) has only one side. A bug could walk from any point on the surface to any other point without crossing over the boundary edge. The surface also has only one edge C consisting of C_1 joined to C_2. This surface is referred to as a Möbius band, named after the German mathematician August Möbius (1790-1868).

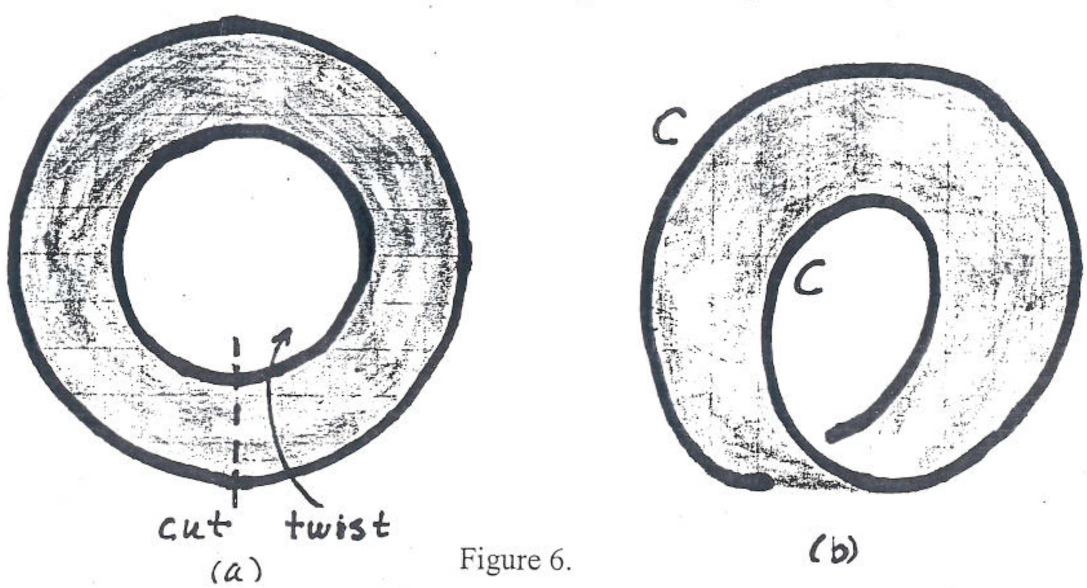
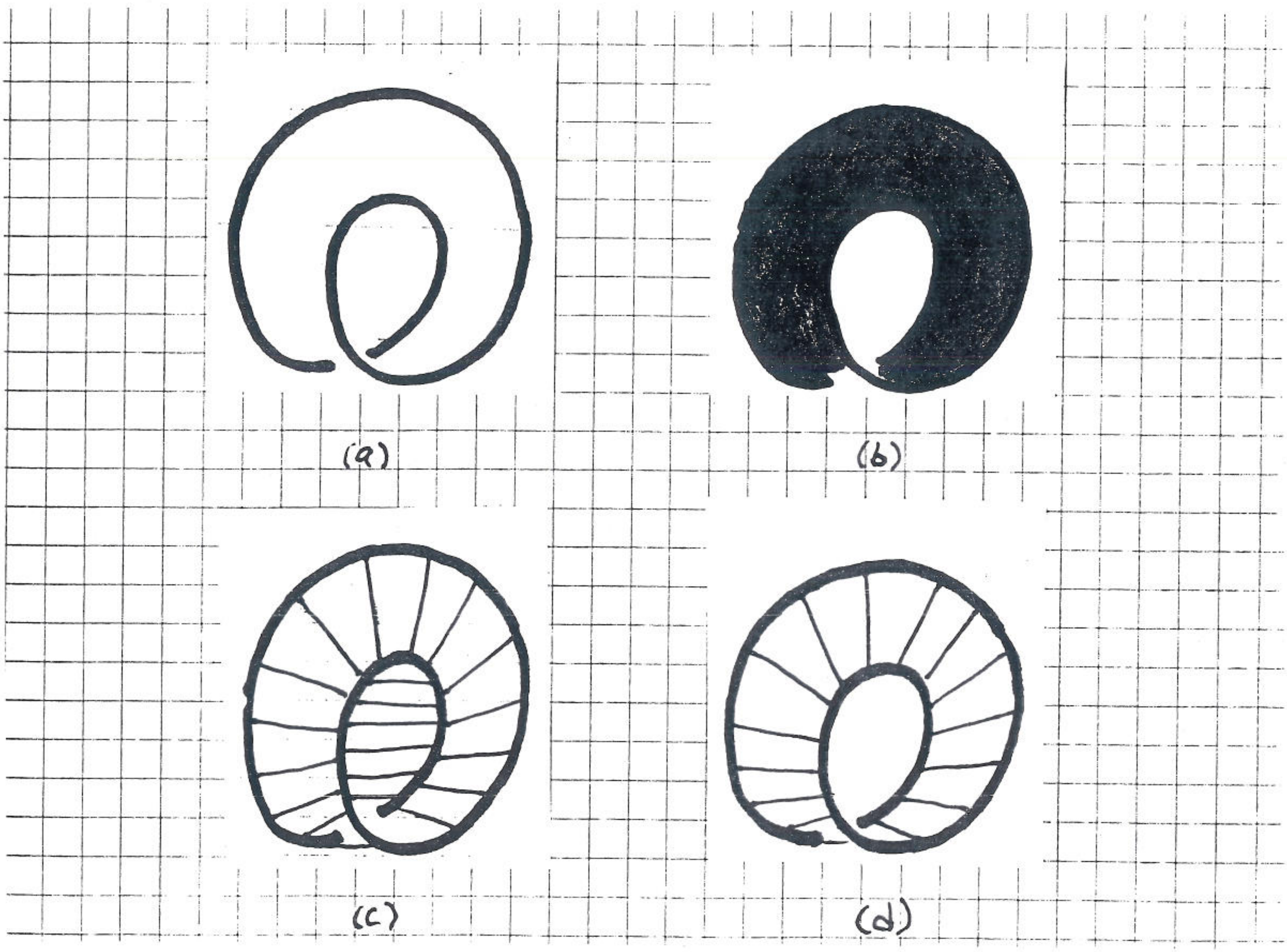


Figure 6.

The edge of the paper Möbius band in Figure 6(b) is outlined in black and is shown isolated in Figure 7(a) as a diagram of a closed curve in space. It is really not knotted and is referred to as a *loop*. A loop is also referred to as an *unknot* or *trivial knot*. The diagram can be checker boarded as in (b). There is just one shaded outer region and one white inner region. A wire model of the edge can be dipped to obtain the initial sfms in (c) with an inner disk. This disk can be punctured to obtain the sfms in (d) predicted by the checker boarding in (b). The sfms in (d) corresponds to the paper Möbius band in Figure 6(b). Thus a Möbius band can occur naturally as a sfms as in (d).

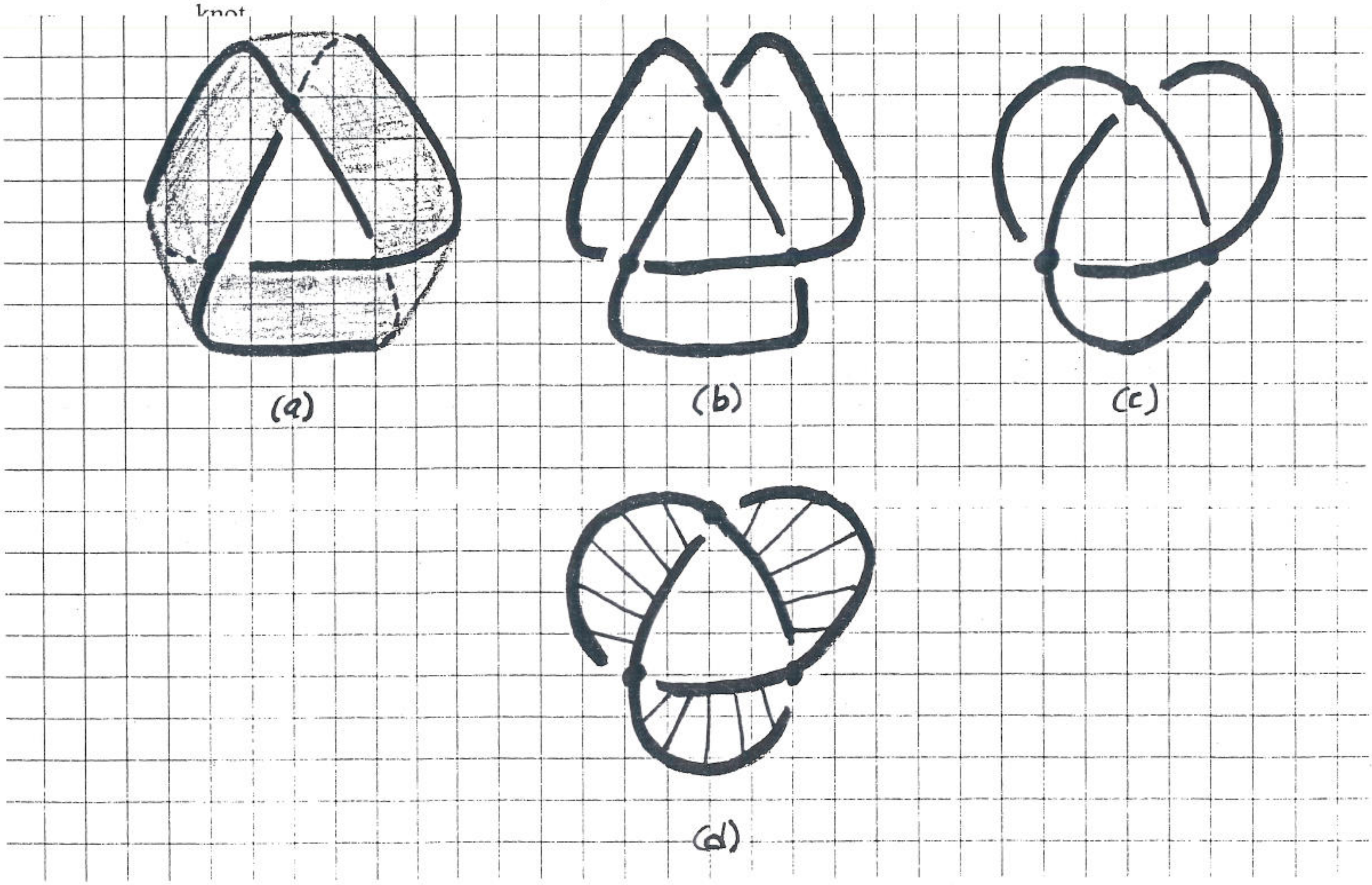


A Möbius Band as a SFMS.

Figure 7.

Triple Twist Möbius Band.

We will now form a triple twist Möbius band. In this case we start with a long paper rectangle, which works out better then a long circular band for this example. A ratio of length to width of 10 works-say 2 inches wide and 20 inches long. When we give one end three half-twists, we obtain the surface in Figure 8(a). This surface will also have only one side and one edge. The edge is outlined in black. The dashed lines indicate the continuation of the edge behind the surface. A bug travelling on the surface can move from any point to any other point without crossing the edge. The single isolated edge is shown in (b). Note that the diagram in (b) is a trefoil knot. If we round out the straight parts of the edge, we obtain our previous diagram of a trefoil knot as in (c). Thus the edge of a triple twist Möbius band is a trefoil knot. The sfms for a wire trefoil knot in Figure 2(d) is shown again in Figure 8(d). This sfms corresponds to the paper triple twist Möbius band in (a). Thus a triple twist Möbius band can occur naturally as the sfms on a trefoil knot.



A Triple-Twist Möbius Band as a SFMS.

Figure 8.

Since a Möbius band is a one-sided surface, we can color a Möbius band with one color red. The single- twist and triple- twist Möbius bands are shown in red in Figures 9(a) and (b), respectively.

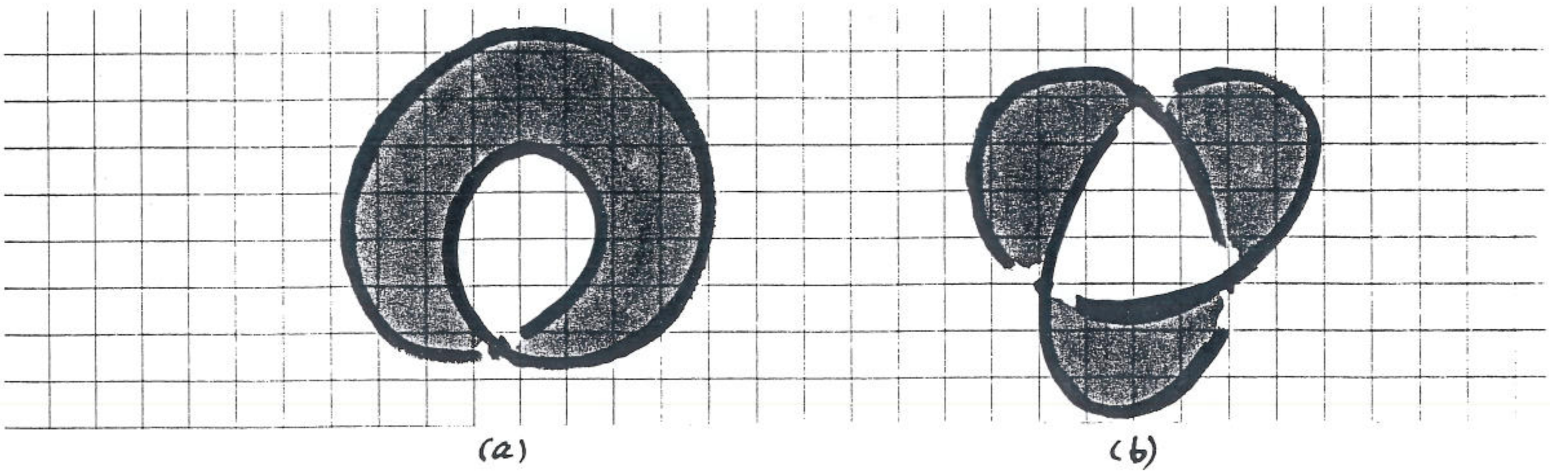


Figure 9.

The Möbius bands in Figure 9 are basic one-sided surfaces. We will now discuss basic two-sided surfaces. We begin with a rectangular piece of paper, which is red on the front and blue on the back, as in Figure 10(a). The side edges are outlined in black. If we give the bottom a half-twist to the right, we obtain the two-sided twisted strip in (b), where the edges cross in the middle as shown. If we give the bottom a half-twist to the left, we obtain the two-sided twisted strip in (c), where the edges cross in the middle as shown. The main point is that the colors switch at the crossing from red at the top to blue at the bottom. If we looked at the rear view, the colors would switch at the crossing from blue at the top to red at the bottom. Thus in either view, the colors switch at the crossing in the two-sided case.

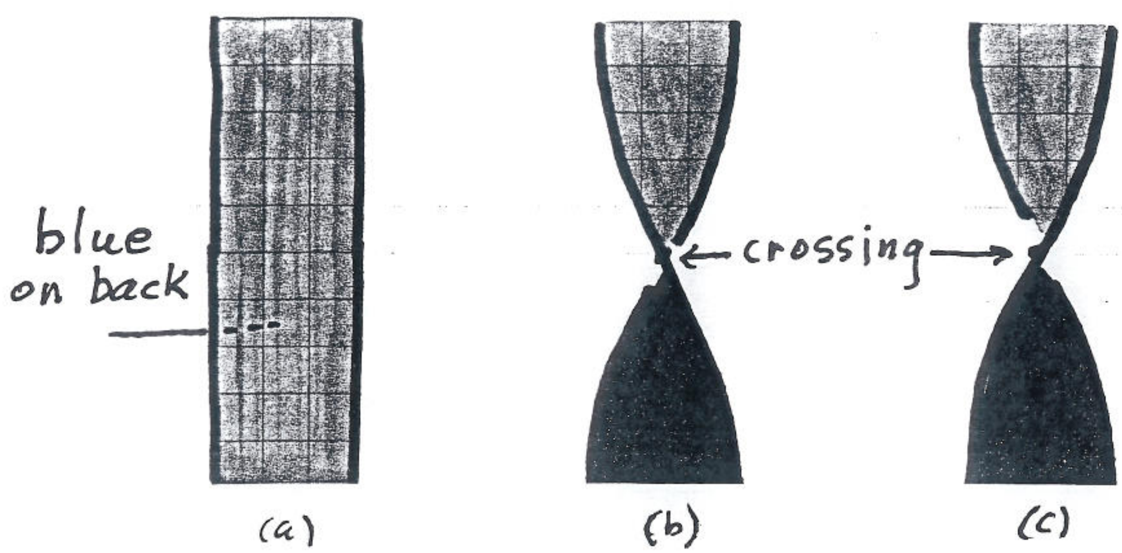


Figure 10.

Predicting Whether the SFMS is One-Sided or Two-Sided.

For the one-sided surfaces in Figure 9, we see that the color does not switch at a crossing. For the two-sided case in Figure 10, the colors do switch at a crossing. Let us now consider a sfms for a diagram. At each crossing as in Figure 11(a), a sfms will have a half-twist as in (b). If we consider the surface colored, then in the one-sided case, the color does not switch as in (c). In the two-sided case, the colors do switch as in (d).

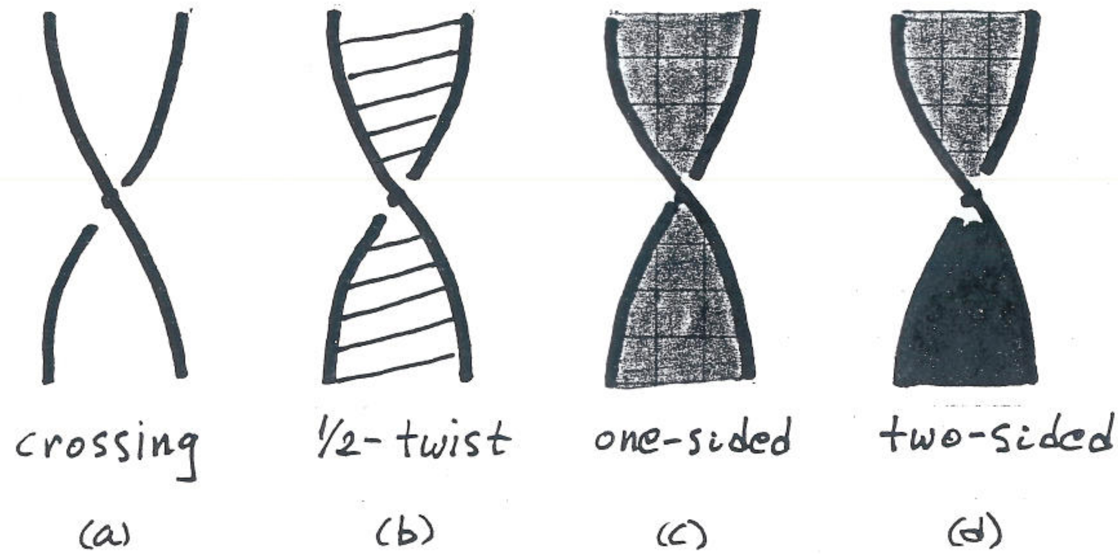


Figure 11.

Now given the checker boarding of a knot diagram, the surface corresponds to the shaded regions. The surface will be two-sided if we can color the shaded regions red and blue so that we can satisfy the switching condition that the colors switch at each crossing, as in Figure 11(d). If we cannot satisfy the switching condition that the colors switch at each crossing, then the surface is one-sided, as in Figure 11(c).

We will first discuss an example where the switching condition can be satisfied. Consider the sfms in Figure 3(d), as shown below in Figure 12(a). The sfms has an upper region and a lower region. We can color the upper region red, as indicated by an R in (b). In order that the colors switch at the crossings, we color the lower region blue, as indicated by a B in (b). We now see that the colors switch at each of the three crossings. The colored sfms is shown in (c).

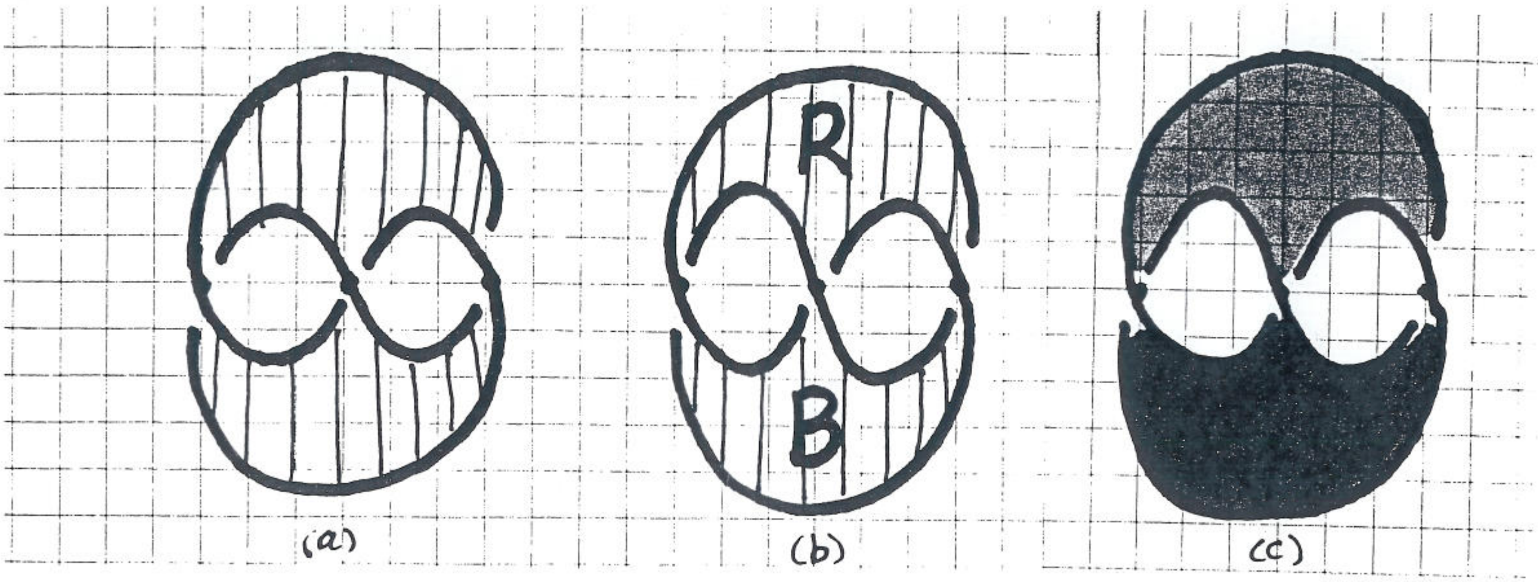


Figure 12.

We will next consider the sfms in Figure 4(d), as shown below in Figure 13(a). The sfms has three regions. We can color the left region red as indicated. In order to satisfy the switching condition at the center crossing, we would have to color the right region blue. In order to satisfy the switching condition at the lower left crossing, we would have to color the lower region blue. But then the switching condition would not be satisfied at the lower right crossing since the right region and lower region are both blue. Thus the switching condition cannot be satisfied at each crossing so the surface is one–sided, as indicated by all red in (b). The colored sfms is shown in (c).

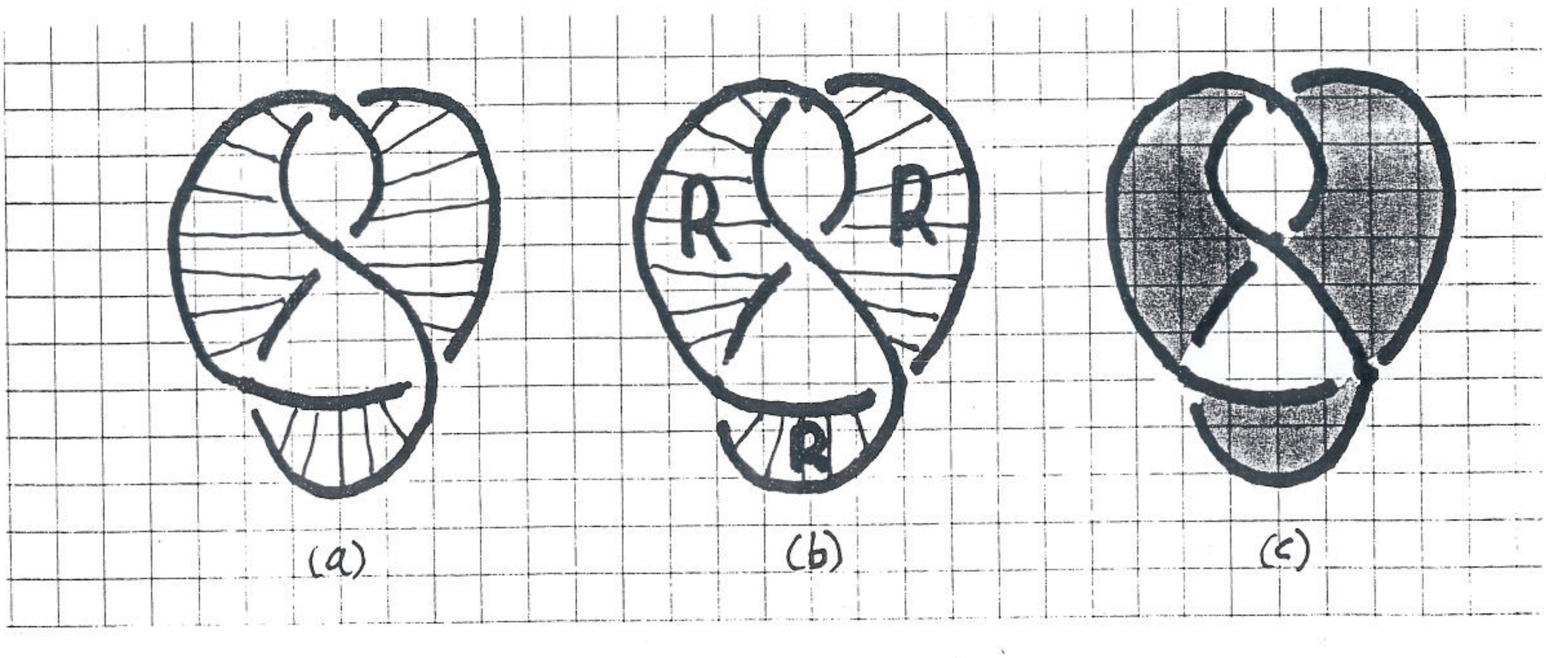


Figure 13.

For another example, consider the sfms for the trefoil knot in Figure 2(d). This sfms has three regions arranged left, right, and below as in Figure 13(a). Thus the sfms is one-sided. We had previously deduced this since the sfms corresponded to a one-sided triple twist Möbius band as in Figure 9(b). Here we deduced the sfms is one-sided because the switching condition could not be satisfied. We will consider additional examples of one-sided and two-sided surfaces after we discuss knot deformations.

Knot Deformations.

As mentioned in the introduction, we consider knots to be made of flexible material. Allowable knot deformations are bending, twisting, stretching, and shrinking. However, we are not allowed to cut the knot, deform, and then rejoin. Two knots are said to be *equivalent* when one can be deformed into the other.

We will describe knot deformations by using knot diagrams. For our first example, consider the trefoil diagram in Figure 14(a). We deform by lifting the point P up as indicated by the dashed lines in Figure 14(a). The deformed diagram is shown in (b) where we have numbered the three crossing points 1, 2, and 3. We can now reshape the diagram in (b) by moving crossings 1 and 3 up and 2 down, as shown in (c). Note that the diagram in (c) is the same as in Figure 3(a). Thus this diagram is also a diagram for a trefoil knot. We will refer to the diagram in Figure 14(a) as *trefoil 1*. We have shown trefoil 1 has a one-sided sfms that is a triple –twist Möbius band. We will refer to the diagram in Figure 14(c) as *trefoil 2*. We have shown above that trefoil 2 has a two-sided sfms. Thus in Figure 14 we see how to deform trefoil 1 with a one-sided sfms into trefoil 2 with a two-sided sfms. We could also deform trefoil 2 back to trefoil 1 by reversing the steps in Figure 14 by lowering the point P. Thus trefoils 1 and 2 are equivalent.

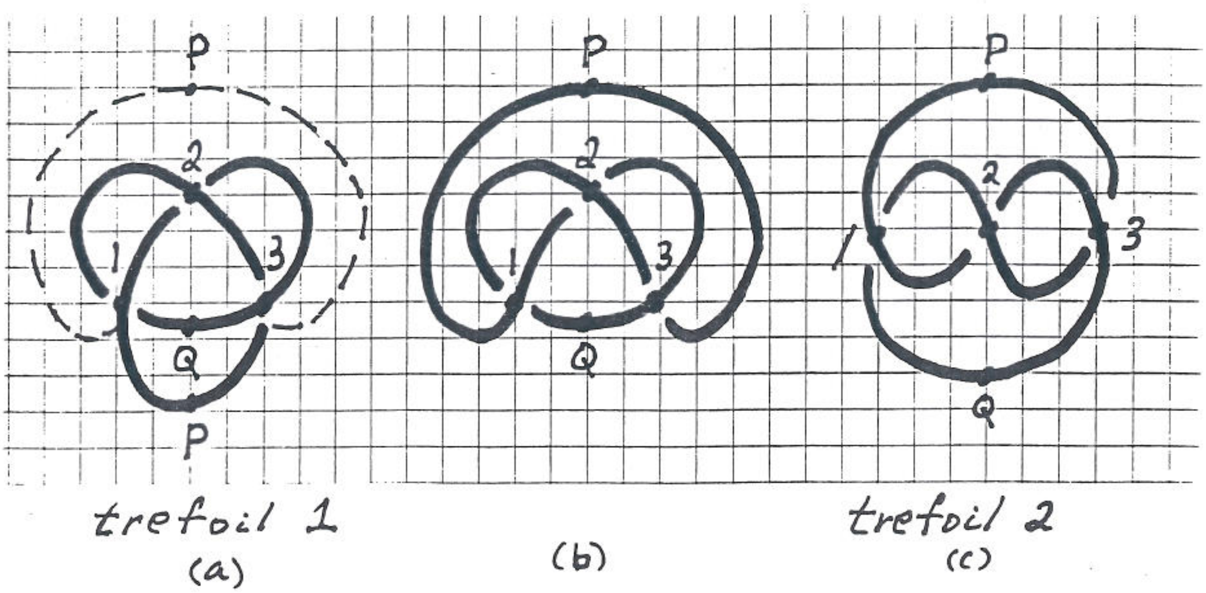


Figure 14.

For our next example, we again start with trefoil 1 in Figure 15(a).The diagram of trefoil 1 has three crossings. We will now obtain an equivalent diagram with four crossings. We deform trefoil 1 by giving the bottom a half-twist to the right as indicated by the arrow in Figure 15(a). The deformation is indicated by the dashed lines. The resulting diagram is shown in (b). This diagram has four crossings and is referred to as *trefoil 3*. Note that we can deform trefoil 3 back to trefoil 1 by untwisting the bottom. Thus trefoils 1 and 3 are equivalent. Note that we previously discussed trefoil 3 in Figure 4 and Figure 13 where we saw trefoil 3 has a one-sided sfms.

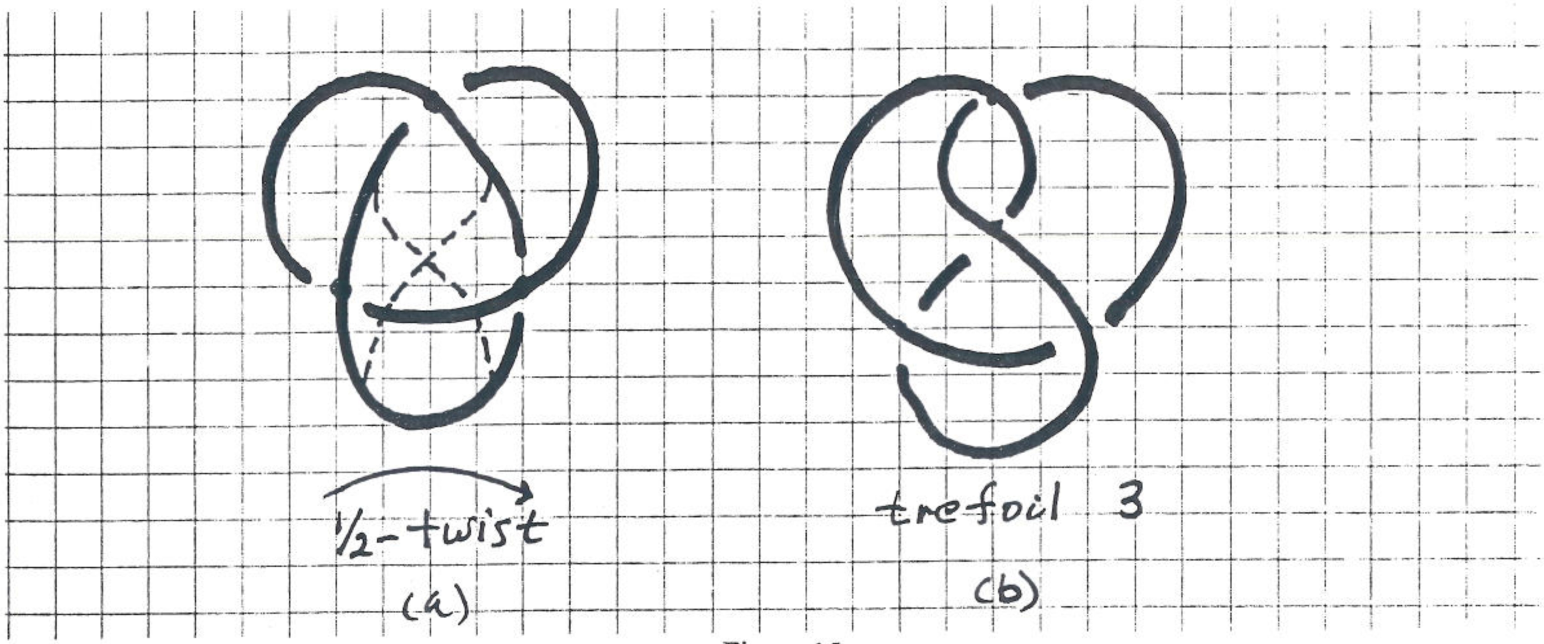


Figure 15.

We have now shown that trefoils 1 and 2 are equivalent and trefoils 1 and 3 are equivalent. Note that we could deform trefoil 2 into trefoil 1 by a first deformation and then deform trefoil 1 into trefoil 3 by a second deformation. We can start with trefoil 2 and apply the first deformation followed by the second deformation to deform trefoil 2 into trefoil 3. By reversing the deformations, we can deform trefoil 3 back to trefoil 2. Thus trefoils 2 and 3 are equivalent. Therefore trefoils 1, 2, and 3 are all equivalent. They are equivalent diagrams of the trefoil knot.

Alternating Diagrams.

The diagram for trefoil 1 is shown in Figure 16(a). A direction for moving around the knot is indicated by the arrows. The crossings are labeled 1, 2, and 3. Suppose we start at point P and move in the direction of the arrows. We first cross over at 1; then cross under at 2; then cross over at 3. Returning to 1, we cross under at 1;then cross over at 2; and cross under at 3 before returning to P. Thus the crossings alternate over, under, over, under, over, under. In this case we say the diagram is an *alternating diagram.* We could also reverse the direction of the arrows and still obtain an alternating diagram. That is, whether the diagram is alternating does not depend on the choice of arrows. The diagram for trefoil 2 is shown in Figure 16(b). Starting at P, one can also check that this is an alternating diagram. The diagram of trefoil 3 is shown in Figure 16(c). The diagram of trefoil 3 is not alternating. Starting at P, we cross over at 1, cross over at 2, cross under at 3, return to 1 and cross under, cross over at 4, etc. In this case the crossings are over, over, under, under, etc.

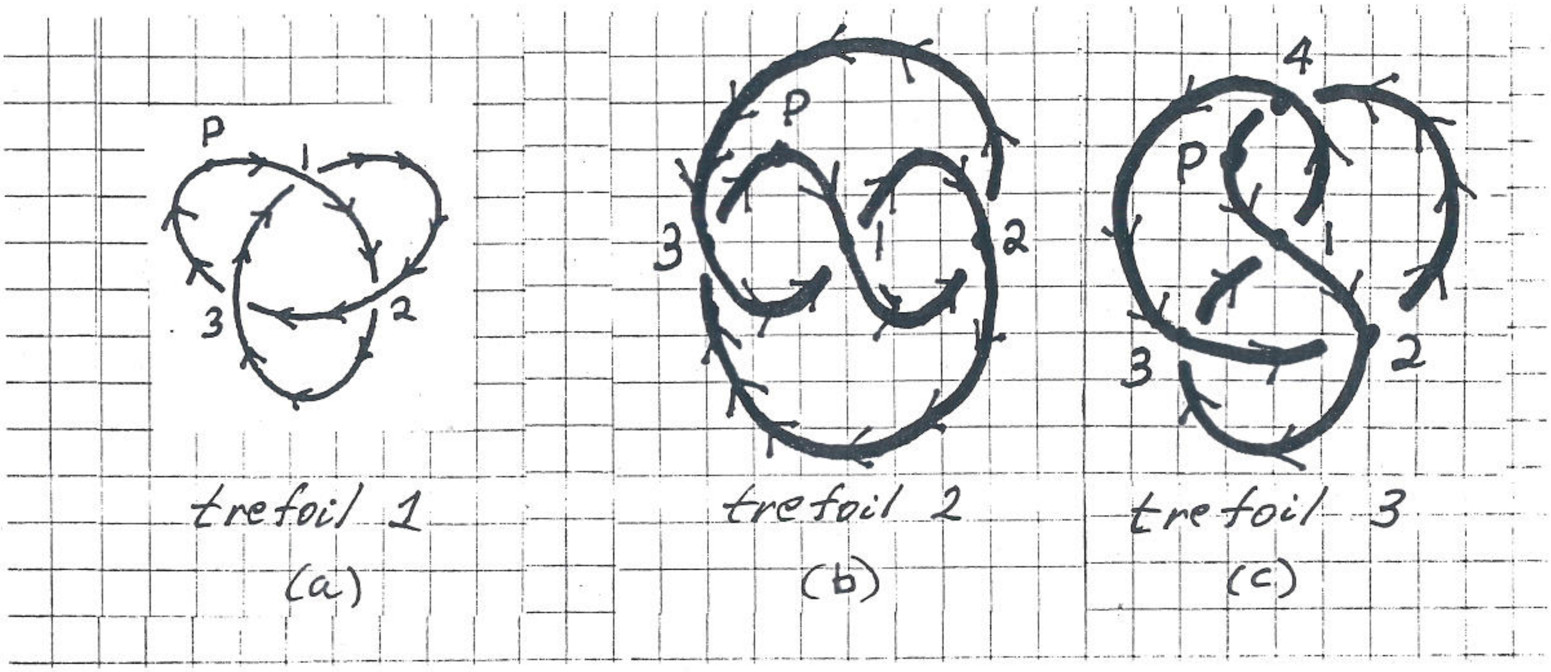


Figure 16.

We have now seen that trefoils 1 and 2 are both alternating diagrams with 3 crossings. Trefoil 3 is not alternating with 4 crossings. Thus equivalent diagrams may or may not be alternating and may have different numbers of crossings. The crossing number of a knot is the least number of crossings in a diagram of the knot. The trefoil knot has diagram trefoil 1 with 3 crossings and it can be shown that there is no diagram for the trefoil knot with less then 3 crossings. Thus the crossing number of the trefoil knot is 3. A standard Knot Table is shown in Figure 17. Here knots are listed according to their crossing number.

There is one knot 3_1 with crossing number 3, which is the trefoil knot with diagram trefoil 1. There is one knot 4_1 with crossing number 4 which is referred to as the figure-eight knot and will be discussed in detail later. There are two knots with crossing number 5, three knots with crossing number 6, seven knots with crossing number 7, and twenty-one knots with crossing number 8. In Italian five is chinque and 5_1 is referred to as a *chinquefoil.* Seven in Italian is sette and 7_1 is referred to as a *settefoil.*

In general, the knot denoted by N_j has crossing number N and the diagram has N crossings. Since a diagram with crossing number N cannot be equivalent to a diagram with a smaller number of crossings, knots in the Knot Table with different crossing numbers cannot be equivalent. Furthermore it can be shown that the knots with the same number of crossings in the Knot Table are not equivalent. Knots that are not equivalent are said to be *distinct.* Thus all the knots in the Knot Table are distinct.

One can check that all the diagrams in the Knot Table are alternating except for the diagrams of the last three knots 8_{19}, 8_{20}, and 8_{21}. In general, a knot is said to be an *alternating knot* if it has an alternating diagram. Thus all the knots preceding the last three knots are alternating knots. A knot is *non-alternating* if it does not have an alternating diagram. It can be shown that the last three knots are non-alternating. Thus no matter how one of these knots is deformed, one never sees an alternating diagram.

We will say that a *diagram is two-sided* if the switching condition is satisfied. Otherwise the *diagram is one-sided.* The only two-sided diagrams in the Knot Table are 5_2, 7_2, and 7_4. The red-blue coloring for these diagrams is shown in Figure 18 after the Knot Table. One can check that all the other diagrams are one-sided. In particular, all the diagrams with 8 crossings are one-sided. However, just as the one-sided diagram trefoil 1 can be deformed into the two-sided diagram trefoil 2, every one-sided diagram in the Knot Table can be deformed into a two-sided diagram by some deformation. In particular, this will be shown for the figure-eight diagram 4_1 below.

Knot Table

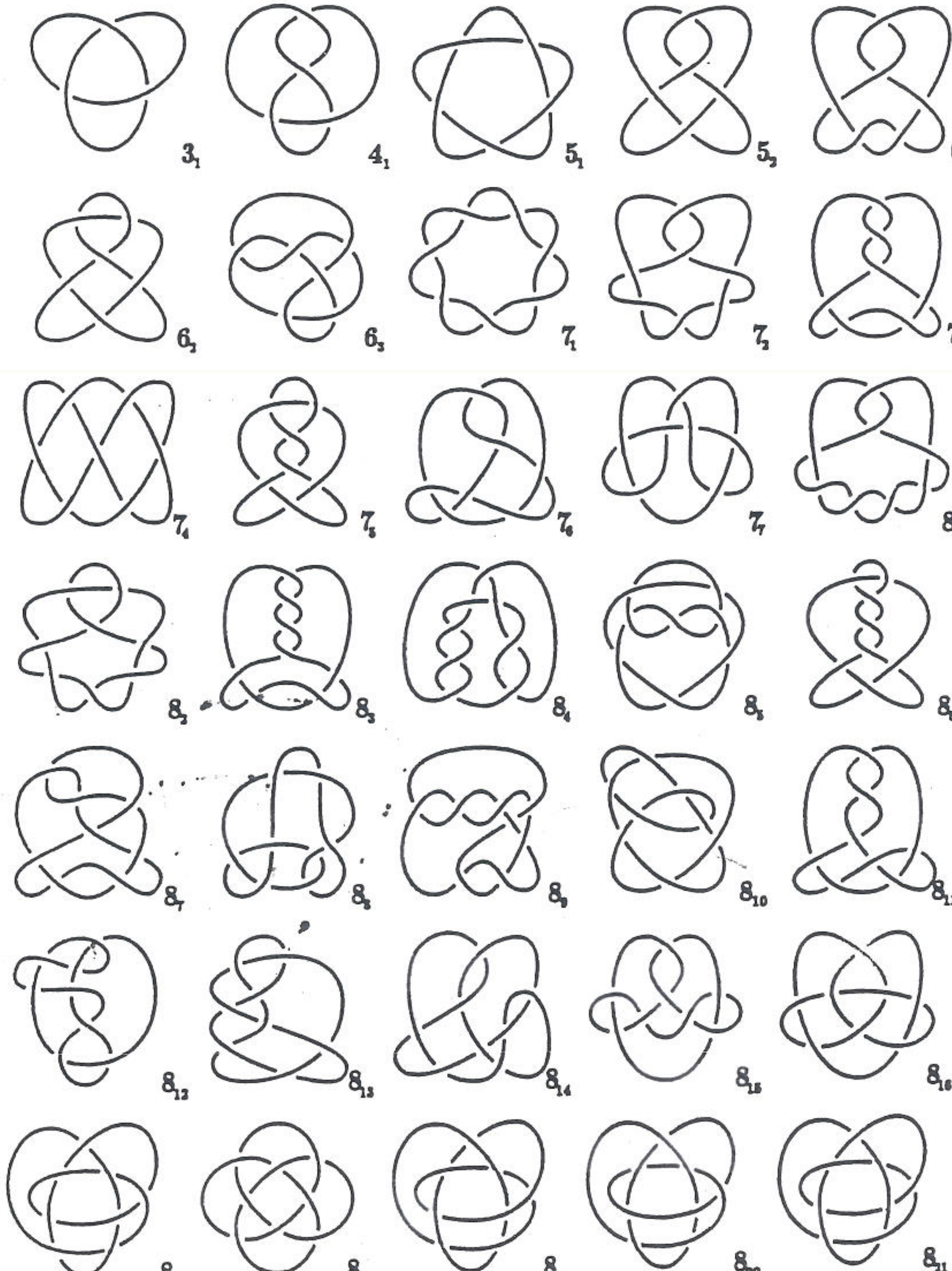
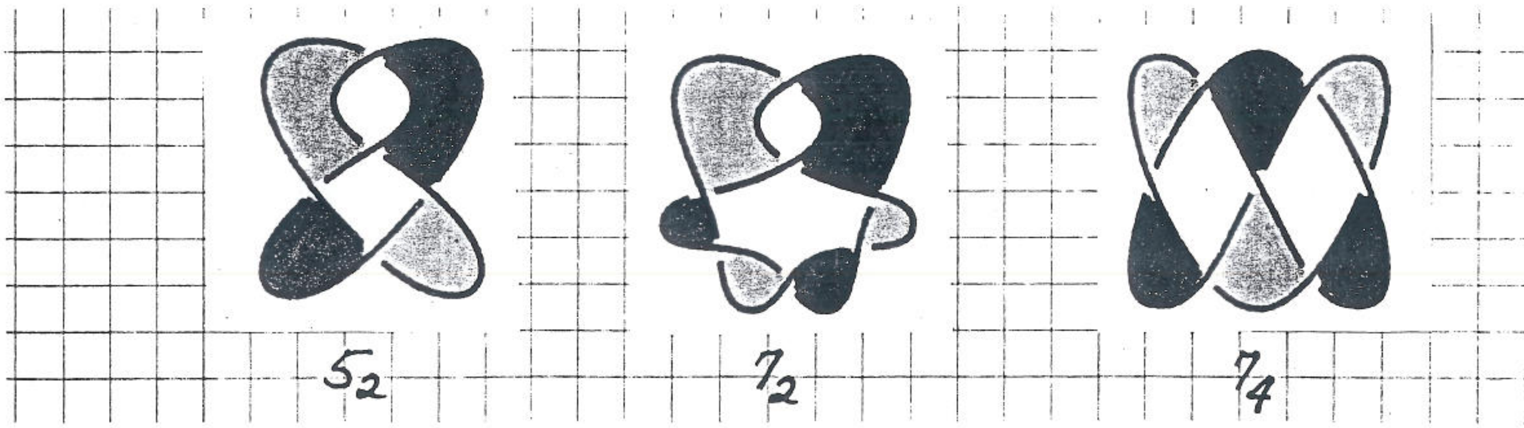


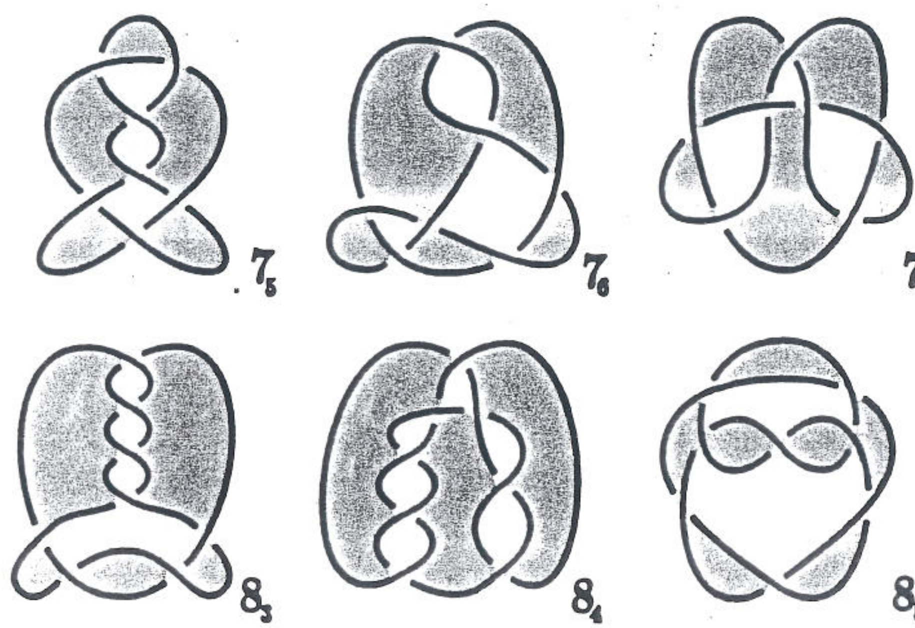
Figure 17.

As stated above, the only two-sided diagrams in the Knot Table are 5_2, 7_2, and 7_4. The red-blue colorings of the sfms are shown in Figure 18. All of the other diagrams in the Knot Table are one-sided. A selection of six one-sided diagrams are shown in Figure 19. As mentioned above, each one-sided diagram can be deformed to obtain an equivalent two-sided diagram. In particular, an equivalent two-sided diagram will be given below for the figure-eight knot 4_1.



Two-Sided Diagrams

Figure 18.

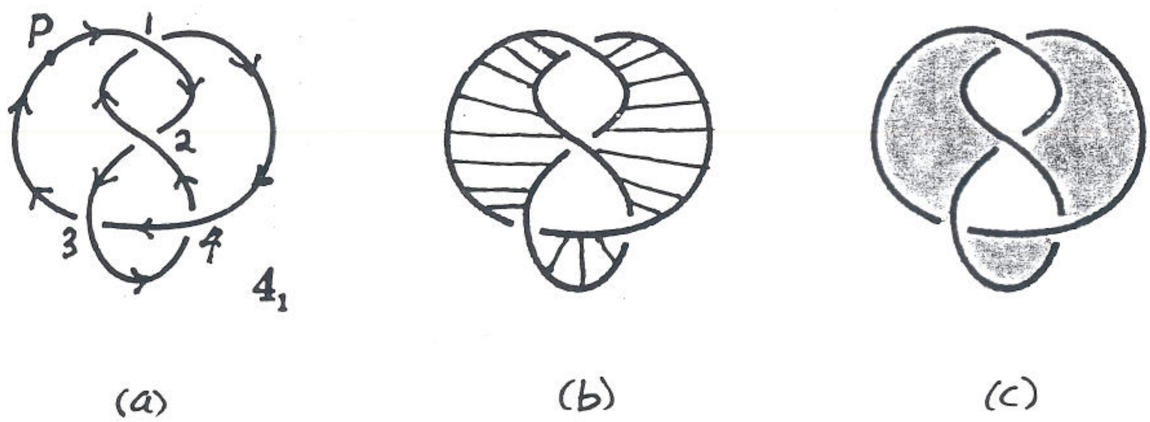


One-Sided Diagrams

Figure 19.

The Figure–Eight Knot 4_1.

We will now discuss the figure-eight knot 4_1 in detail. The diagram is shown in Figure 20(a). The diagram of 4_1 is alternating as can be seen by starting at P and following the direction of the arrows. Note that the diagram of 4_1 at the crossings 1 and 2 looks like the diagram of trefoil 3 in Figure 15(b). However, the crossings at 3 and 4 are reversed from over/under to under/over. In this sense, 4_1 is related to trefoil 3. The sfms of 4_1 in (b) is similar to the sfms of trefoil 3 in Figure 4(d). In particular, the sfms of 4_1 is one-sided as shown in (c).

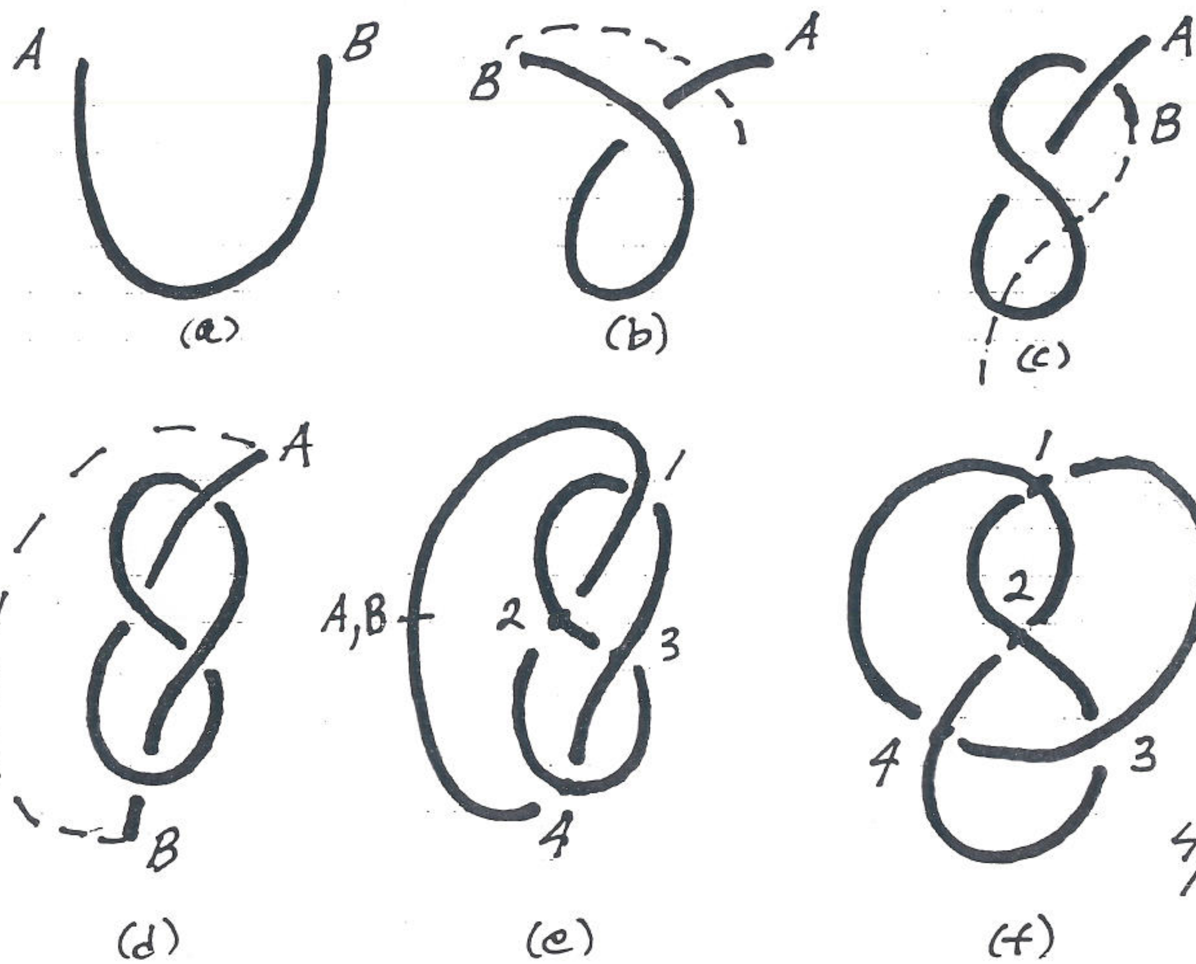


The Figure-Eight Knot 4_1.

Figure 20.

On the following two pages we will show how to tie the figure-eight knot and how to deform the one-sided diagram into a two-sided diagram.

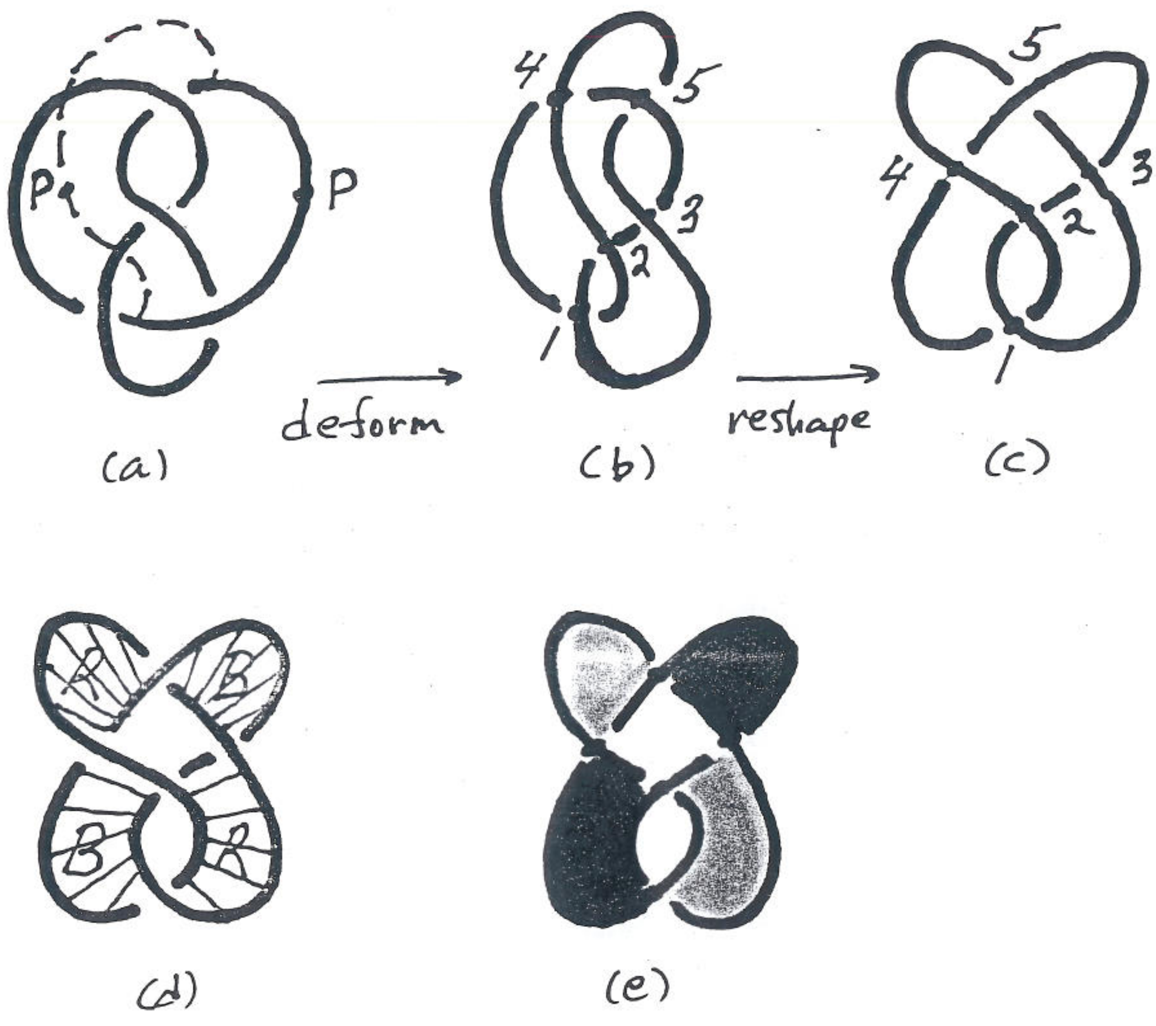
We have seen that the diagram of the figure-eight knot can be obtained from the diagram of trefoil 3 by reversing the two lower crossings. We will now show how to tie the figure-eight knot directly. In Figure 21(a) we begin with two loose ends A and B and cross B over A, as in (b). We now bring B behind A, as indicated by the dashed lies in (b) to obtain (c). Next we move B over and through the loop as indicated by the dashed lines in (c) to obtain (d). We now join A and B to obtain (e). The diagram in (e) is now reshaped by rearranging the crossings 1,2,3,4 to obtain the diagram for 4_1 in (f).



Tying The Figure –Eight Knot

Figure 21.

We will now deform the one-sided diagram in order to obtain a two-sided diagram, as shown in Figure 22. The point P on the right is lifted over to the left. The deformation is indicated by the dashed line in (a). The deformed diagram is shown in (b) with the crossings labeled 1,2,3,4,5. We now reshape this diagram as in (c) by moving 1 down so 2 is above 1. Crossing 4 will move down and 5 will move up at the top. Crossing 3 will move over across from 4. We now have the equivalent diagram in (c). The checker boarding of the diagram in (c) is shown in (d). The corresponding two-sided sfms is shown in (e).



Deforming The One-Sided Diagram of 4_1 Into A Two-Sided Diagram.

Figure 22.

Part Four

Visualizing Knots and Soap Film Minimal Surfaces: Mirror Image Writhe and Knot Equivalence Figure Eight Knot Complementary Two-Sided Diagrams

Preface

We will first introduce the mirror image of a knot. It is then shown that the figure eight knot is equivalent to its mirror image. In order to decide whether a knot can be equivalent to its mirror image, we will introduce the writhe of a diagram. The writhe is a number that is easy to compute and it can be shown that the writhe must be zero if a reduced alternating knot is equivalent to its mirror image. Thus if the writhe is not zero, then the knot cannot be equivalent to its mirror image. In particular, this will imply that a reduced alternating knot with an odd number of crossings cannot be equivalent to its mirror image.

We will introduce the projection of a knot in preparation for a detailed discussion of the figure eight knot. In particular, we will see that the figure eight knot can be obtained from the projection of trefoil 3. The standard diagram for the figure eight knot has a one-sided sfms. It will be shown how to deform this diagram to obtain a diagram with a two-sided sfms.

We will also discuss coloring the white regions of the checker boarding of a knot diagram. If the white regions can be colored red and blue so that the colors switch at each crossing, then a simple deformation called a flip will yield a two-sided diagram. Since the white regions are the complement of the shaded regions, we say that the diagram is complementary two-sided in this case.

Mirror Image

Consider placing a crossing in front of a mirror, as shown in Figure 1. For the crossing on the left, the vertical line is the over-crossing and the horizontal line is the under-crossing. However, in the mirror image on the right, the horizontal line is the over-crossing and the vertical line is the under-crossing. This is because the horizontal line is closer to the mirror so its mirror image will appear in front of the vertical line. Thus the crossings switch in the mirror image: the vertical over-crossing becomes an under-crossing and the horizontal under-crossing becomes an over-crossing.

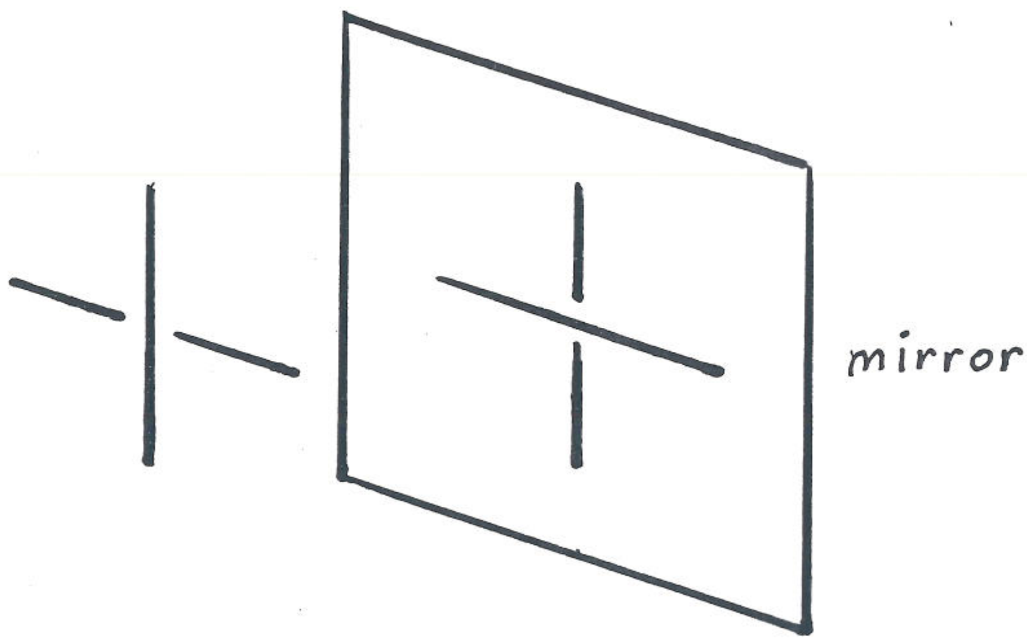


Figure 1.

In Figure 2 we have a trefoil knot shown on the left and its mirror image is on the right. Note that at each crossing point 1,2,3 the over/under crossings switch to under/over in the mirror image. Thus to obtain the mirror image of a knot, one switches the crossings at each crossing point.

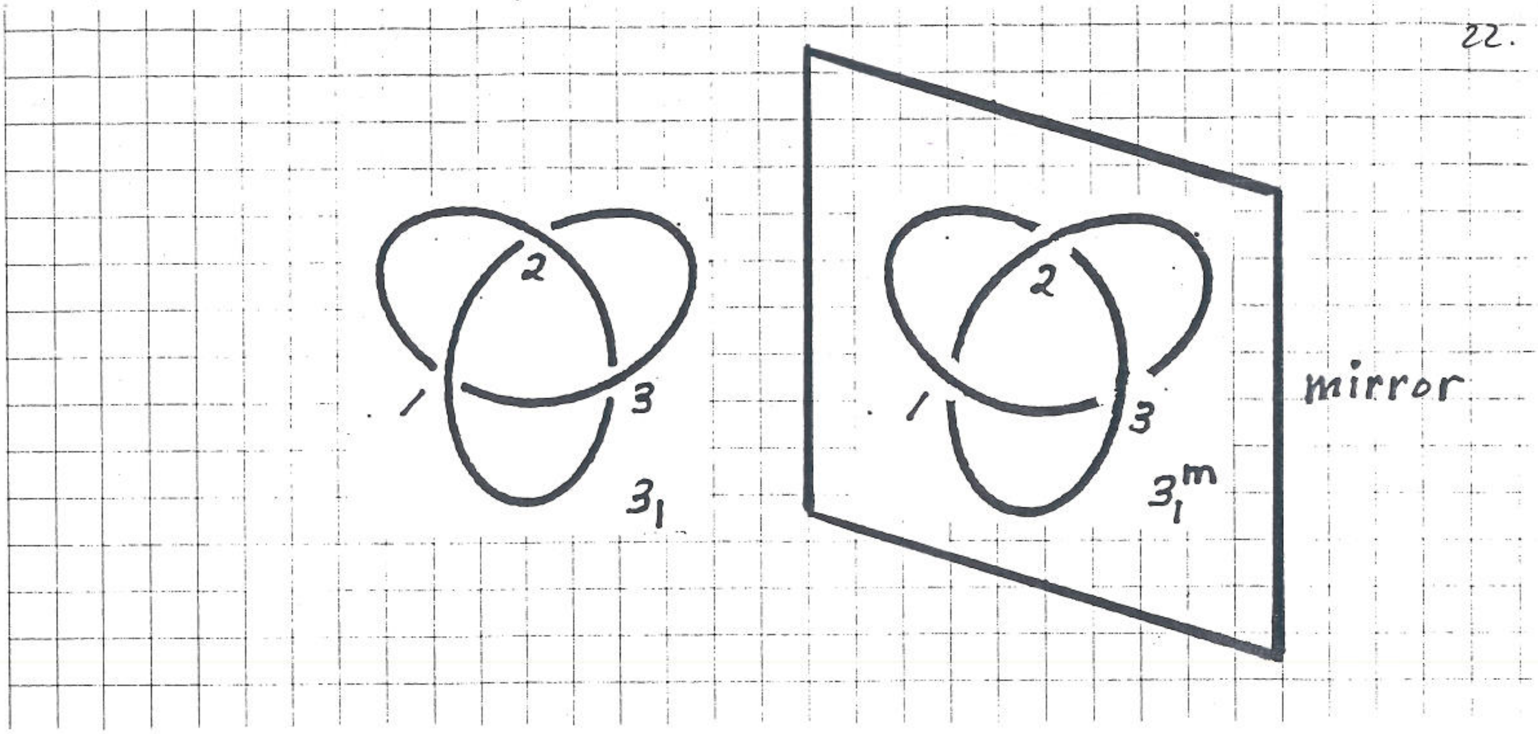


Figure 2.

The mirror image of a knot diagram D is denoted by D^m. In Figure 2, 3_1 is shown on the left and $3_1{}^m$ is shown on the right. In Figure 3 we have 4_1, 6_3 , and their mirror images.

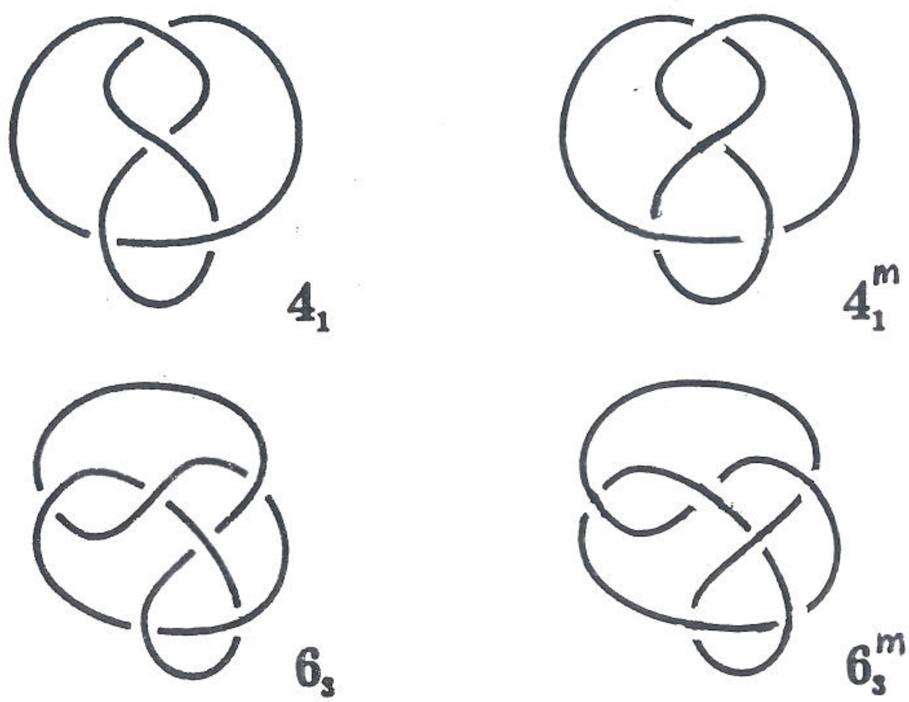


Figure 3.

It is natural to ask whether a knot is equivalent to its mirror image? The answer depends on the knot. It will be shown below that if a reduced alternating knot has an odd number of crossings, then the knot is not equivalent to its mirror image. In particular, 3_1, 5_1, and 5_2 are not equivalent to their mirror images. However, the figure-eight knot 4_1 is equivalent to its mirror image. The deformation of 4_1 into 4_1^m is shown in Figure 4.

We begin with 4_1 in Figure 4(a). We have numbered the four crossing points 1, 2, 3, 4 in (a). The point P on the left moves to the right when the knot is deformed, as indicated by the dashed lines in (a). The deformed knot is shown in (b). We now reshape the deformed knot in (b) by moving crossing 1 to the right and moving crossing 2 to the left of 1, as in (c). We also move 3 to the center and move 4 below 3, as in (c). The diagram in (c) is actually 4_1^m upside down. If we turn the diagram in (c) right-side up, we obtain 4_1^m as shown in (d). Thus 4_1 is equivalent to its mirror image.

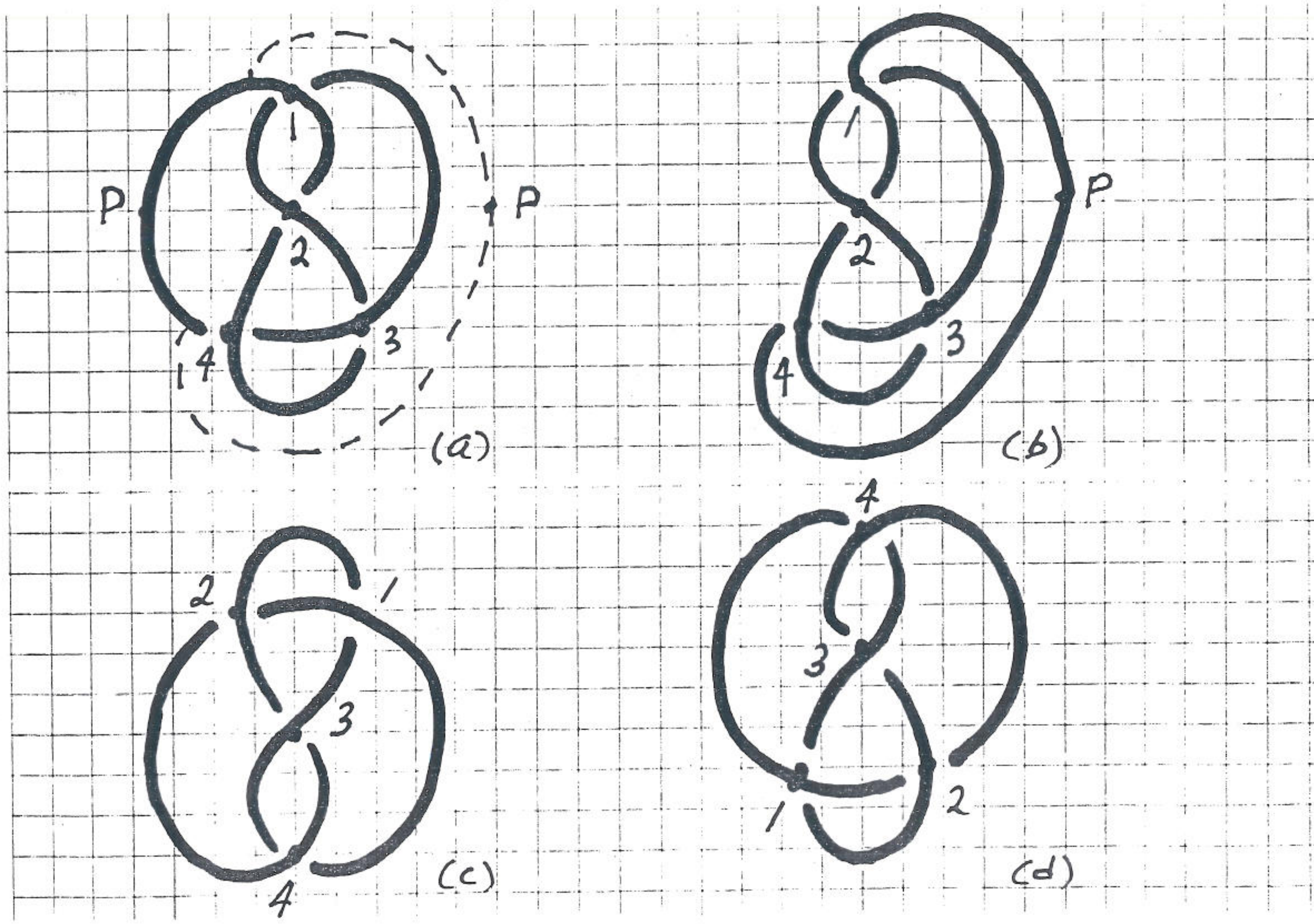


Figure 4.

Writhe and Knot Equivalence

The writhe of a knot diagram will now be defined. The writhe is very useful for deciding whether a knot is equivalent to its mirror image. In order to define the writhe, we choose a direction on the diagram, as shown in Figure 5 for the diagram of 4_1.

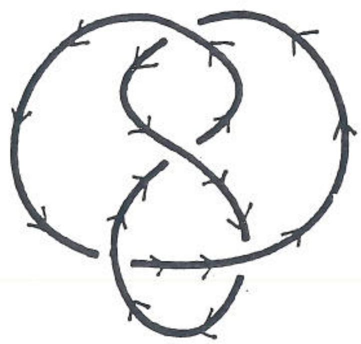


Figure 5.

There are two possibilities at a crossing, as shown in Figure 6. We consider that the over-crossing is a bridge that we are walking on in the direction of the arrows and the under-crossing is a river below the bridge. If the river is coming from the right as in Figure 6(a), then the crossing is a *right-hand crossing* and we assign +1 to the crossing. If the river is coming from the left, as in Figure 6 (b), then the crossing is a left-hand crossing and we assign −1 to the crossing.

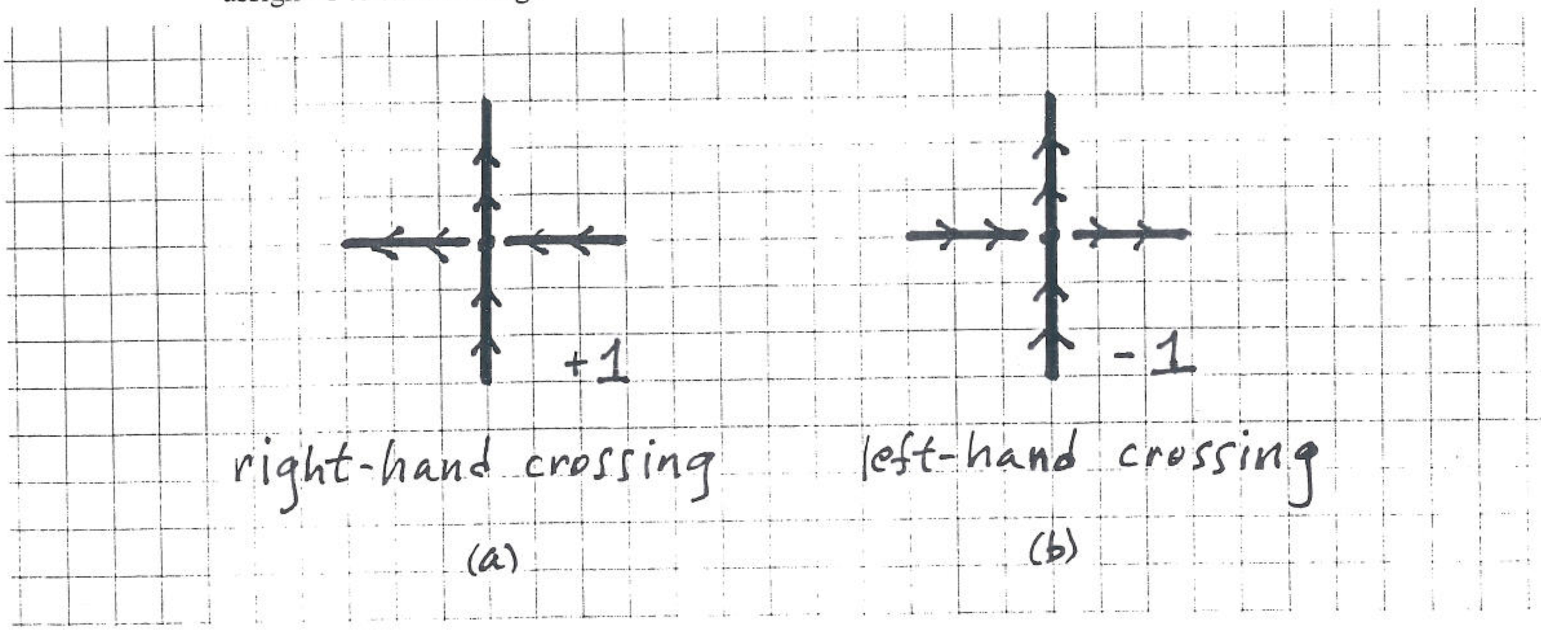
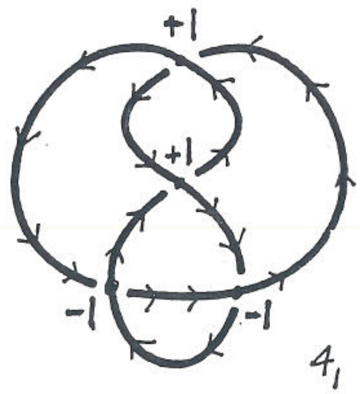


Figure 6.

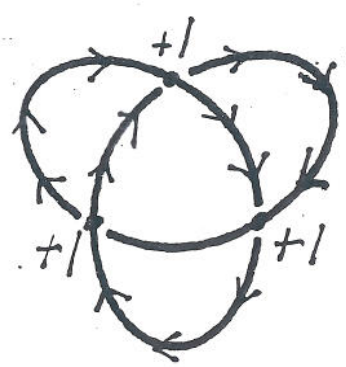
For the figure-eight knot 4_1 in Figure 5, there will be two right-hand crossings and two left-hand crossings, as indicated in Figure 7. Remember to walk in the direction of the arrows on the over-crossing in order to see if the direction on the under-crossing is coming from the right or left. The *writhe* w is defined as the sum of the +1and –1 values for all the crossings. In this case the sum is 1+1-1-1=0 so the writhe is w = 0.
In general, given a diagram D, we denote the writhe by w(D). Thus w(D) is the sum of the +1 and –1 values for all the crossings in D. For the case in Figure 7, we found that $w(4_1) = 0$. One can also check that $w(4_1^m) = 0$; hence $w(4_1) = w(4_1^m) = 0$.



$w(4_1) = 0$

Figure 7.

In Figure 8(a) we see that 3_1 has three right-hand crossings so $w(3_1) = 3$. We refer to 3_1 as the *right trefoil.* In Figure 8(b) we see that 3_1^m has three left-hand crossings so $w(3_1^m) =$ - 3. We refer to 3_1^m as the *left trefoil.*



right-trefoil = 3_1, $w(3_1) = 3$.

left-trefoil = 3_1^m, $w(3_1^m) = -3$

Figure 8.

In Figure 8, for $D = 3_1$, we have shown that $w(D^m) = -3 = -w(D)$. We will now verify that $w(D^m) = -w(D)$ for any diagram D. In Figure 9 we have a right-hand crossing for D on the upper left. Note that when we switch the crossings for the mirror image, we obtain a left-hand crossing on the upper right.

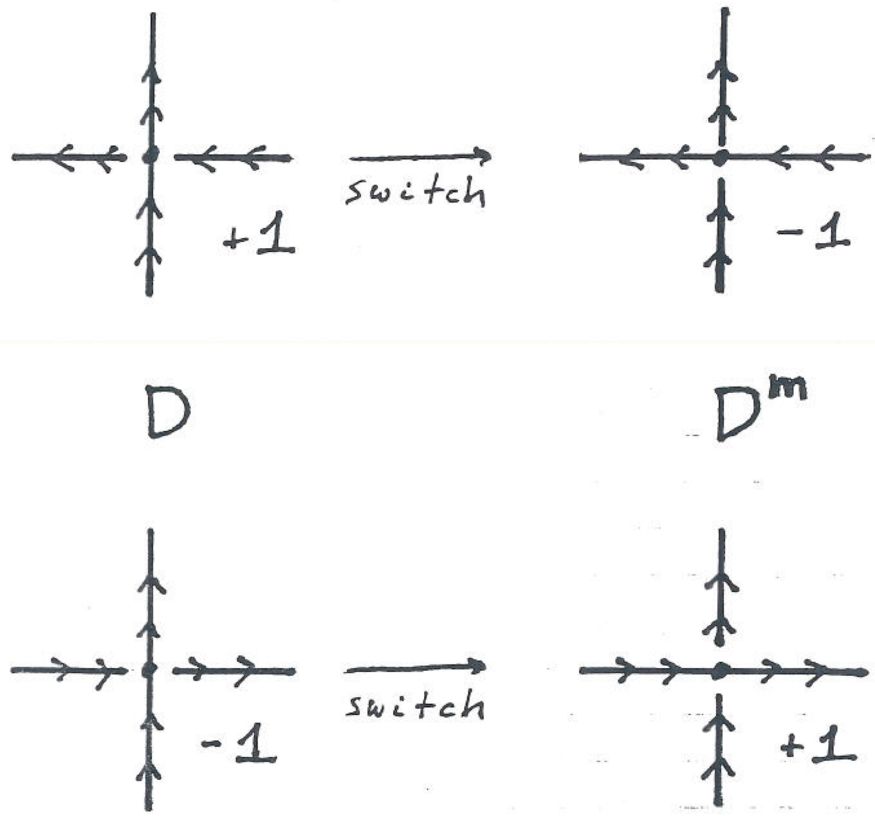


Figure 9.

Thus the mirror image of a right-hand crossing is a left-hand crossing. Hence a +1 switches to a –1. Similarly, for a left-hand crossing for D on the lower left, the mirror image on the lower right is a right-hand crossing. Thus the mirror image of a left-hand crossing is a right-hand crossing. Hence a –1 switches to a +1. Since $+1 = -1 \times -1$, we can say that for either a right-hand crossing or a left-hand crossing, the value at the crossing for the mirror image D^m is –1 times the value at the crossing for D. Thus to obtain the writhe of D^m, we multiply crossing values for D by –1 and add. This is the same as first adding to obtain w(D) and then multiplying by –1. Hence we have the following result.

Theorem 1. For any diagram D, $w(D^m) = -w(D)$.

For example, $w(3_1^m) = -w(3_1) = -3$ and $w(4_1^m) = -w(4_1) = -0 = 0$.

Reduced Alternating Diagrams

In Figure 10(b) we have trefoil 1 with writhe w = 3. In (a) we have added an extra half-twist or curl at the upper left. The diagram is still alternating but the extra curl is a right hand crossing and adds a +1 to the writhe so the writhe is w = 4 in (a). Note that the diagrams in (a) and (b) are equivalent since we can simply untwist the curl to deform (a) into (b). Thus the diagrams are both alternating and equivalent but the writhe values w = 4 and w = 3 are different. When we deform the alternating diagram (a) into (b) by removing the curl, we say the diagram in (b) is *reduced alternating* because the diagram in (b) has no extra curls.

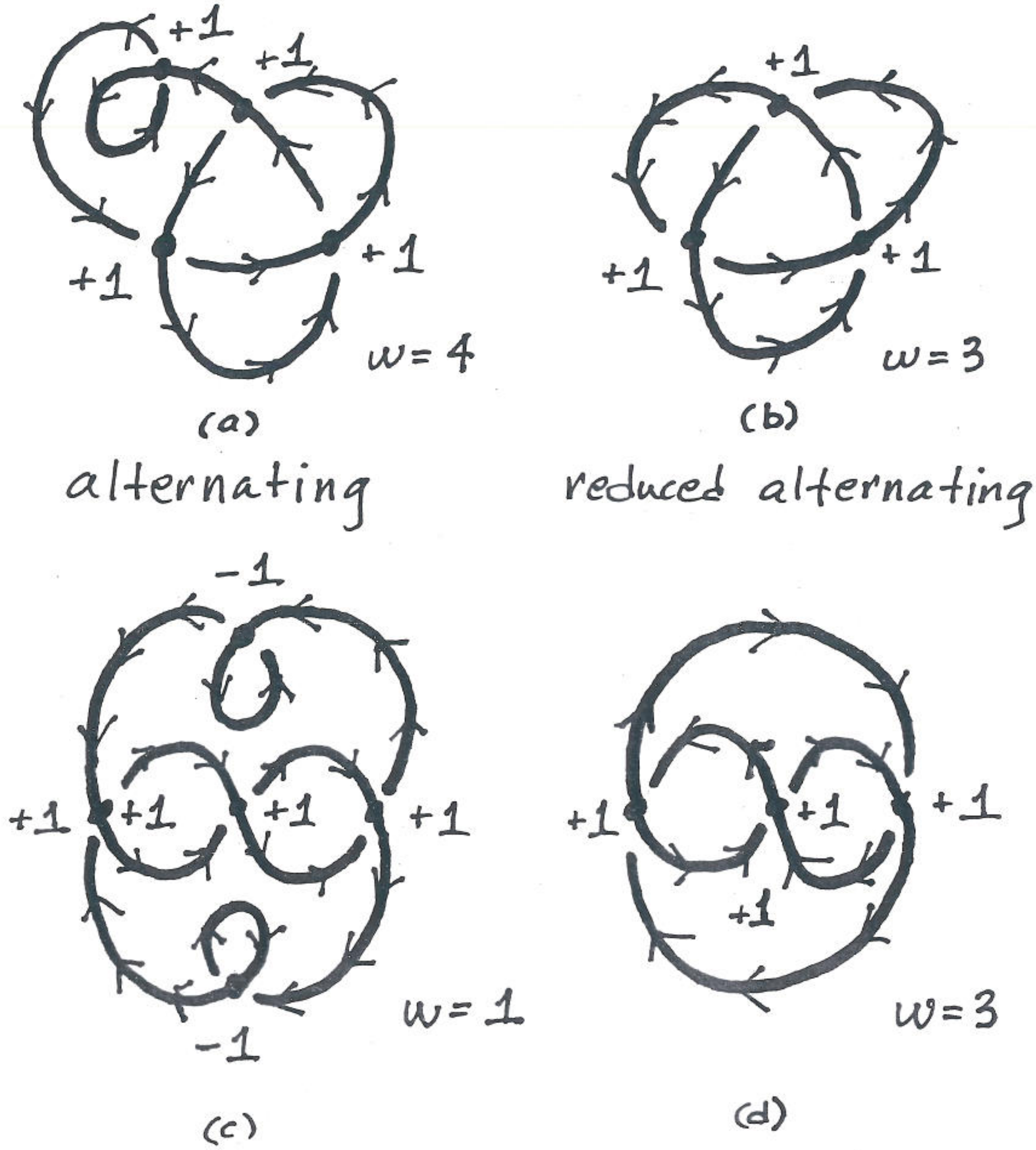


Figure 10.

Similarly in Figure 10(d) we have trefoil 2 with writhe w = 3. In (c) we have added two extra curls. The diagram is still alternating but the two curls are left-hand crossings and both −1. Therefore the writhe is w = 3 − 2 = 1 in (c). The two diagrams in (c) and (d) are equivalent since we can simply untwist the two curls to deform (c) into (d). Thus the diagrams are both alternating and equivalent but the writhe values w = 1 and w = 3 are different. When we deform the alternating diagram (c) into (d) by removing the two curls, we say the diagram (d) is reduced alternating because the diagram in (d) has no extra curls. Since trefoil 1 in (b) and trefoil 2 in (d) are equivalent, we conclude that all four diagrams in Figure 10 are equivalent but their writhe values are all not equal.

However, in the reduced alternating cases of trefoil 1 in (b) and trefoil 2 in (d), the writhe values are both equal to 3. Thus for these examples the writhe values are equal for equivalent diagrams only in the reduced alternating case. This result is true in general as stated in the following theorem , due to M. B. Thistlethwaite and K. Murasugi.. The proof of this theorem is beyond the scope of these notes. However, it is very useful for our purposes.

Theorem 2. If diagrams D_1 and D_2 are both reduced alternating and equivalent, then $w(D_1) = w(D_2)$.

Thus Theorem 2 says that reduced alternating diagrams that are equivalent have the same writhe. Therefore if two reduced alternating diagrams have different writhes , then they cannot be equivalent. We state this as follows.

Corollary 1. If diagrams D_1 and D_2 are both reduced alternating and $w(D_1) \neq w(D_2)$, then D_1 and D_2 are not equivalent.

Corollary 1 applies to the right and left trefoils. They are both reduced alternating but their writhes are different. Thus the right and left trefoils are not equivalent.

Theorem 1 also implies that if a reduced alternating diagram is equivalent to its mirror image , then its writhe must be zero.

Corollary 2. If D is a reduced alternating diagram that is equivalent to D^m, then $w(D) = 0$.

Proof. Let $D_1 = D$ and let $D_2 = D^m$. Since D is reduced alternating, D^m is also reduced alternating since in forming the mirror image we only switch crossings but we do not add any "curls". Thus we can apply Theorem 1 to obtain

(1) $w(D) = w(D^m)$.

However, Theorem 1 states that

(2) $w(D^m) = -\,w(D)$.

From (1) and (2), we obtain w(D) = – w(D), hence 2 w(D) = 0. Therefore dividing by 2 implies w(D) = 0, as required.

Note that an example of Corollary 2 is the figure-eight knot 4_1, which is equivalent to its mirror image and does have writhe 0. On the other hand, we note that there are reduced alternating diagrams that do have writhe zero but they are not equivalent to their mirror images. The first example that occurs in the Knot Table is knot 8_4. One can check that $w(8_4) = 0$ but it can be proved that 8_4 is not equivalent to its mirror image.

The contrapositive of Corollary 2 states that if a reduced alternating diagram D has non-zero writhe, then D cannot be equivalent to its mirror image. We state this result as Corollary 3.

Corollary 3. If D is a reduced alternating diagram with $w(D) \neq 0$, then D cannot be equivalent to D^m.

Corollary 3 gives us another proof that the right trefoil is not equivalent to its mirror image the left trefoil since the right trefoil has non-zero writhe 3.

In fact, Corollary 3 implies that any reduced alternating diagram with an odd number of crossings cannot be equivalent to its mirror image, which we state as Corollary 4.

Corollary 4. If D is a reduced alternating diagram with an odd number of crossings, then D cannot be equivalent to D^m.

Proof. In order for the sum w(D) = 0, the +1 values and the –1 values must cancel. Thus there must be an equal number of right and left crossings. Let n be the number of right crossings. Hence there are also n left crossings. Therefore the total number of crossings is 2n, which is always even. Thus the writhe cannot be zero for an odd number of crossings. Hence D cannot be equivalent to D^m.

Since the right trefoil has 3 crossings, Corollary 3 immediately implies that the right trefoil cannot be equivalent to its mirror image. It is not necessary to even compute the writhe since we now know it cannot be 0. Furthermore, one can check that the diagrams in the Knot Table with 5 or 7 crossings are all reduced alternating; hence they cannot be equivalent to their mirror images.

Given a reduced alternating diagram with an even number of crossings, the first thing to do is compute the writhe in order to decide if its possible that the diagram is equivalent to its mirror image. If the writhe is not zero, then Corollary 3 implies the diagram cannot be equivalent to its mirror image. This is the case for 6_1 and 6_2, whose writhes are not zero, as seen in the exercises. If the writhe is zero, then it is possible that the diagram is equivalent to its mirror image. This is the case for the figure-eight knot 4_1. It is also true for knot 6_3. One can check that $w(6_3) = 0$ and in Part 5 we will show that 6_3 is equivalent to 6_3^m. On the other hand, we mentioned above that knot 8_4 has zero writhe but is not equivalent to its mirror image. Thus zero writhe is necessary but not sufficient for a diagram to be equivalent to its mirror image.

We will now state two very fundamental results for the reduced alternating case. Firstly, reduced alternating equivalent diagrams have the same number of crossings. Secondly, for a knot with a reduced alternating diagram, the number of crossings is the crossing number. These results were first conjectured in the late 1800's and finally proved by Louis Kauffman, Kunio Murasugi, and Morwen Thistlewaite in 1986. The proof required the Jones polynomial proved by Vaughn Jones in 1985. For a discusssion of the Jones polynomial, see The Knot Book by Colin Adams, Freeman Press.

Theorem 3. If D_1 and D_2 are reduced alternating diagrams and equivalent, then they have the same number of crossings. If a knot K has a reduced alternating diagram with N crossings , then c(K) = N.

Note that all the knots in the Knot Table through seven crossings have reduced alternating diagrams. Therefore Theorem 3 implies $c(N_j) = N$, as was stated in Part Three.

Projection

If we place a knot in front of a white screen and shine a light directly on the knot, we will obtain the knot shadow on the screen. For example , the shadow of a trefoil knot is shown on the right in Figure 11. Note that the shadow is simply a two-dimensional curvo as discussed earlier. The shadow is generally referred to as the knot *projection.*

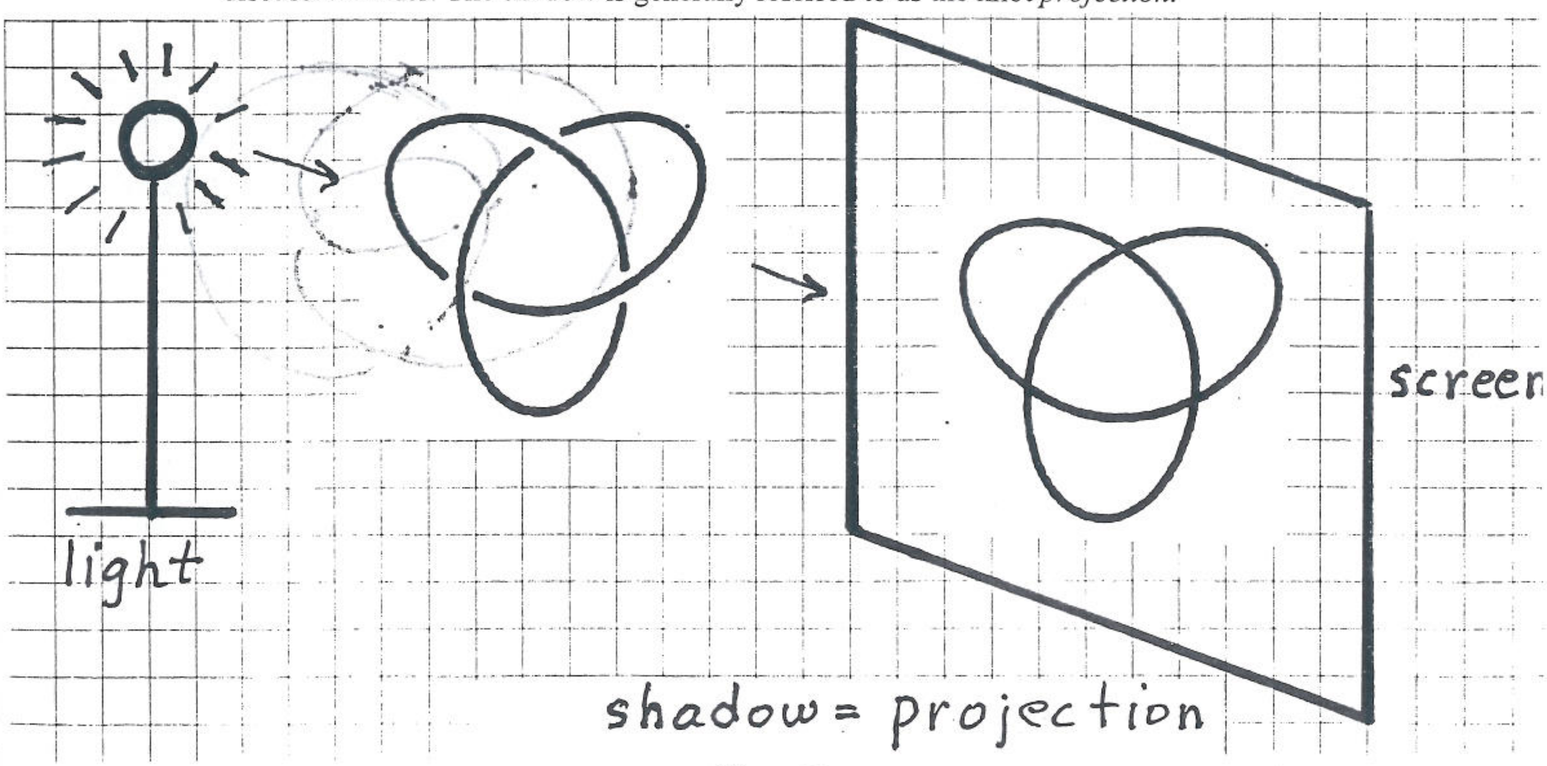


Figure 11.

(See Ernst and Summers, 1987.) Note that in this lower bound, both a given knot and its mirror image are counted if they are not equivalent. Hence, this number can exceed the number of prime knots of n crossings given by Thistlethwaite for $n \leq 13$. We talk more about this result in Section 3.2.

Very recently, Dominic Welsh of Oxford University has proved that the number of distinct prime n-crossing knots is bounded above by an exponential in n.

2.2 The Dowker Notation for Knots

The Dowker notation is an extremely simple way to describe a projection of a knot. First, let's start with an alternating knot. Suppose we have a projection of an alternating knot that we want to describe, like the one in Figure 2.3. Choose an orientation on the knot, given by placing coherently directed arrows along the knot. Pick any crossing and label it 1. Leaving that crossing along the understrand in the direction of the orientation, label the next crossing that you come to with a 2. Continue through that crossing on the same strand of the knot, and label the next crossing with a 3. Continue to label the crossings with the integers in sequence until you have gone all the way around the knot once. When you are done, each crossing will have two labels on it, as the knot passes through each crossing twice (Figure 2.4). Notice that, in fact, each crossing has one even number and one odd number labeling it.

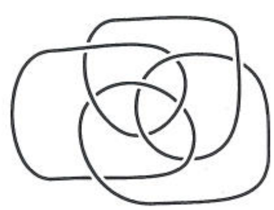


Figure 2.3 An alternating knot.

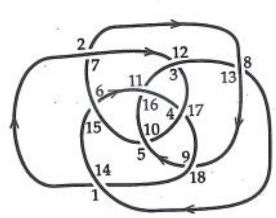


Figure 2.4 Label each crossing of the knot with two numbers.

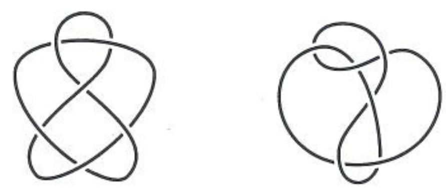
Exercise 2.2 Why does every crossing get one even numbered label and one odd numbered label?

Thus, we can think of this labeling as giving us a pairing between the odd numbers from 1 to 18 and the even numbers from 1 to 18. In this case, we get

1	3	5	7	9	11	13	15	17
14	12	10	2	18	16	8	6	4

As a shorthand, we could just write 14 12 10 2 18 16 8 6 4, and keep in mind that this means 1 is paired with 14, 3 with 12, 5 with 10, and so forth. Thus, from a projection of a knot, we obtain a sequence of even integers, where the number of even integers is exactly the number of crossings in the knot.

Exercise 2.3 Find a sequence of even integers that represents the projection of the knots 6_2 and 6_3 (Figure 2.5). How about a second sequence of even integers that also represents the same projection of 6_3?



6_2 6_3

Figure 2.5 The knots 6_2 and 6_3.

Now, suppose we want to go the other way. Given a sequence of even integers that represents a projection of an alternating knot, how do we draw the projection? Say the sequence is 8 10 12 2 14 6 4. This is shorthand for

1	3	5	7	9	11	13
8	10	12	2	14	6	4

So let's begin drawing the knot. Start by drawing just the first crossing, labeling it with a 1 and an 8. We extend the understrand of the knot and then draw in the next crossing, which corresponds to 2. Since 2 is paired with 7, we label this crossing with a 2 and a 7. Because the knot is

alternating, we know that the strand that we are on goes over this crossing. We continue the overstrand through this crossing to the next crossing where it becomes the understrand, labeling the new crossing with a 3 and the integer that is paired with 3, namely 10 (Figure 2.6). We continue this process until the next integer that should be placed on a crossing already labels an existing crossing. We then know that the knot must now circle around to pass through that crossing. Note that we have two choices as to how to circle around: either circling to the right or to the left in order to pass back through the previously drawn crossing. For the time being, let's ignore this ambiguity and just choose either direction for circling around.

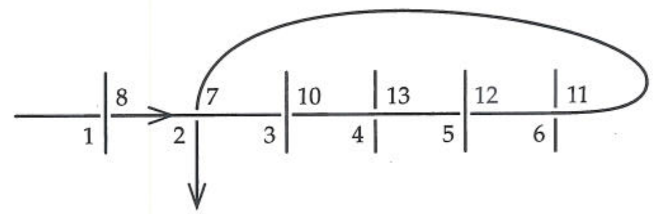


Figure 2.6 Constructing a knot projection from the Dowker notation.

We continue in this manner. If neither of the labels on the next crossing has occurred before, then we make a new crossing. But if one of the labels has occurred before, we circle the knot through that crossing. All the way along, we will be sure that the crossings alternate as we progress along the knot. Finally, we end up with a picture of our knot (Figure 2.7).

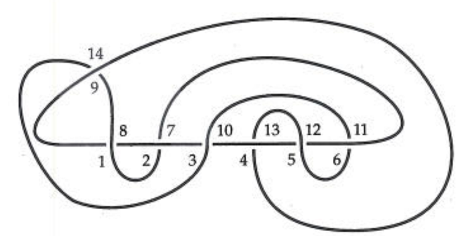


Figure 2.7 The knot that comes from 8 10 12 2 14 6 4.

Exercise 2.4 Which seven-crossing knot from the table at the end of the book is this knot?

Exercise 2.5 Draw a picture of the projection of an alternating knot corresponding to the sequence 10 12 8 14 16 4 2 6.

Now what about that ambiguity in our choice of how the knot circles around? Our choice *can* change the resulting knot. For instance, the sequence 4 6 2 10 12 8 represents two distinct knots, as shown in Figure 2.8. Note that the two knots are composite knots, and that this is reflected in the fact that the sequence 4 6 2 10 12 8 is actually a shuffling of the three numbers 2, 4, 6 and then a shuffling of the three numbers 8, 10, and 12. When the permutation of the even numbers can be broken into two separate subpermutations, the resulting knots are composite (assuming each of the factor knots is nontrivial) and the knot is not completely determined by the Dowker notation. However, if we restrict ourselves to sequences of even numbers that cannot be split into subpermutations, either a particular knot or its mirror image results (Figure 2.9). When the knot is amphicheiral, only one knot can be the result.

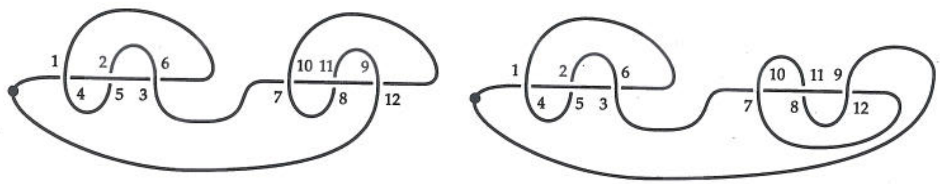


Figure 2.8 Two knots with the same Dowker notation.

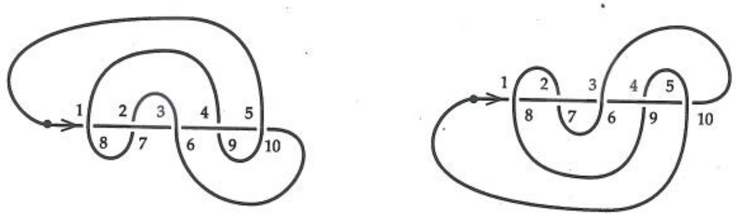


Figure 2.9 A knot and its mirror image are both given by 8 6 10 2 4.

Although the possible projections look different, they will all correspond to the same pair of knots. The best way to see this is to think of projecting the knot onto a sphere (Figure 2.10) rather than onto a plane. (Just as the earth looks planar until you get far enough away from it, so does any sphere.) The advantage to projecting onto a sphere is that there is no special outer region with infinite area as there is in a projection onto the plane. Figure 2.11 contains two projections described by 8 6 10 2 4 that are distinct as projections on the plane but that are equivalent projections on the sphere.

Figure 2.10 Projecting a knot onto a sphere.

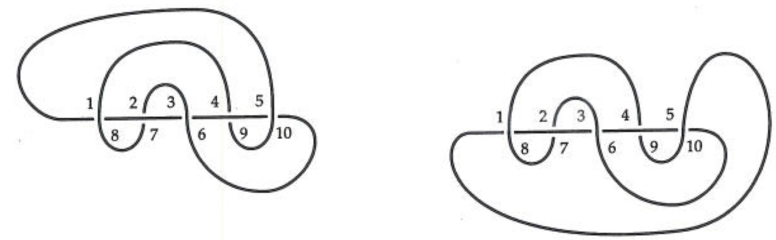


Figure 2.11 Two projections of 8 6 10 2 4.

Exercise 2.6 Draw two projections given by 10 12 2 14 6 4 8, which are inequivalent as projections in the plane but which are equivalent as projections on the sphere.

Exercise 2.7 How many different sequences of the integers 2 4 6 8 10 12 14 are there? (This exercise gives us an upper bound on the number of possible alternating knot projections with seven crossings; however, it's far from accurate.)

The system that we have explained works very well for describing the projection of an alternating knot, but how can we extend it to knots that aren't alternating? We add in + and − signs to our sequence of even numbers. Our rule is as follows: When traversing the knot using the labeling system that we have described; we assign an even integer and an odd integer to each crossing. If the even integer is assigned to the crossing while we are on the overstrand at that crossing, we leave the even integer positive. But if the even integer is assigned to the crossing while we are on the understrand of that crossing, we make the corresponding even number negative. So, for example, in the knot in Figure 2.12, the numbers 14, 12, 4, and 8 become negative.

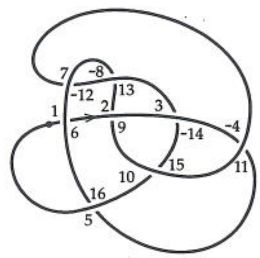
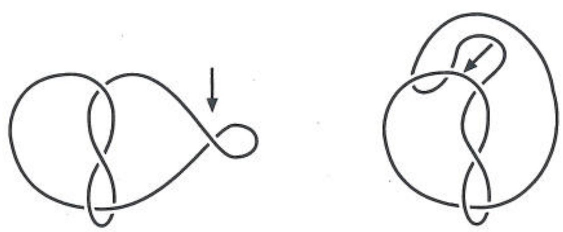


Figure 2.12 A nonalternating knot with sequence 6 −14 16 −12 2 −4 −8 10.

Exercise 2.8 Draw a projection of the knot corresponding to the sequence 14 12 −16 2 18 6 8 10 −4.

Exercise 2.9 How do you recognize from the sequence of numbers that a projection has a trivial crossing in it like this? How about recognizing a Type II Reidemeister move that will reduce the number of crossings by two? (See Figure 2.13.)



a b

Figure 2.13 (a) Trivial crossing. (b) Type II Reidemeister move.

Dowker's notation allows us to feed projections of knots into the computer simply as a sequence of numbers. In particular, suppose we wanted to attempt a classification of 14-crossing knots. The number of sequences of the 14 numbers 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28 is 14!, which is about 87 billion. Then we can put a +1 or −1 in front of each of the even numbers, giving us another factor of 2^{14}. Of course, there aren't this many different knots with 14 crossings. Lots of the sequences represent the same knot. In fact, lots of the sequences represent the same projection of the same knot.

Morwen Thistlethwaite used the Dowker notation to list all of the prime knots of 13 or fewer crossings. Perhaps it will turn out to be the best way to list knots of 14 or fewer crossings.