Module 1: Projective Geometry Constructions

Construction 1: Projective Transformations

Projective Geometry is the mathematical foundation of the theory of perspective which is of great importance to architects and designers. Fig. 1.1 shows a projective transformation of a series of points A,B,C,D from line x to points A',B',C',D' on line x' with the point of projection being point O. In Fig. 1.2 point O is where the eye of an artist is located as she renders a scene on the horizontal plane by projecting it onto the artist's canvas. The line at infinity in the horizontal plane is projected to the horizon line H on the plane of the canvas, and parallel lines receding to infinity meet on the horizon line in the plane of the painting. Fig. 1.3 shows the rendering of a three dimensional scene. In this picture, all pairs of parallel lines meet on the horizon line. In fact it is the ability to represent infinity in finite terms that is so unique and valuable to projective geometry.

In Euclidean geometry two figures are congruent (identical) if you can translate one and superimpose the translation onto the other figure. In Euclidean geometry certain properties of the original figure such as the angle between lines and edge lengths remain unchanged (invariant) under this transformation. In projective geometry two figures are considered to be congruent if you can obtain one from the other by a series of projective transformations through a series of points O, O',O''...similar to the transformations on the artist's canvas. As a result of these transformations angles and lengths generally change although there is one complex geometrical property that does not change called the *cross-ratio* λ. In Fig. 1.1 the cross-ratio is defined as,

$$\lambda = \frac{AB}{BC} \div \frac{AD}{DC} \qquad (1)$$

This is easy to remember; in the first fraction we take a journey from A to C stopping at B to rest, and in the second fraction we journey from A to C, overshooting and stopping at D. If A,B,C,D are chosen in order as in Fig. 1.1, then AB, BC, and AD will be positive numbers with DC negative making λ a negative number. Since λ is preserved under projective transformations, in Fig. 1.1,

$$\lambda = \frac{AB}{BC} \div \frac{AD}{DC} = \frac{A'B'}{B'C'} \div \frac{A'D'}{D'C'} \qquad (2)$$

Under projective transformations we are free to choose three points, e.g., A,B,C on line x and their transformations, A',B',C' on line x'. Points D and D' must then be positioned so that cross-ratio is preserved; we cannot choose them arbitrarily.

It turns out that all conic sections (circles, ellipses, parabolas, and hyperbolas) are congruent since they can all be obtained by slicing a double cone as shown in Fig. 1.4. Each slice represents a projection of the double cone onto a plane from point O at the vertex of the cone. Depending on the orientation of the plane the projected image is either a circle, ellipse, parabola, or hyperbola. Fig. 1.5 shows four circles juxtaposed next to a line. If the line is transformed by a projective transformation to the line at infinity, the first circle remains unchanged since its line is already at infinity. The second circle transforms to an ellipse, the third circle with a tangent circle opens up to a parabola, while the last circle with the line intersecting the circle breaks open to a hyperbola.

We will describe a series of constructions based on projective geometry without going deeply into the underlying theory. These constructions will show that space has certain intrinsic properties and constraints and is not completely freewheeling.

Let us begin with Construction 1.

Construction 1: Choose two lines x and x' as in Fig. 1.1 and point O of projection. Place four points A,B,C,D on x and find the images: A',B',C',D' with respect to O. Use a ruler to measure the lengths AB, BC, AD, DC and use Eq. 1 to compute the cross-ratio, λ. The verify that the transformation preserves the cross-ratio by comuting the cross-ratio of A',B',C',D' and showing that Eq. 2 holds.

Try to answer the following two questions:

1) Where does the point at infinity on x transform to on x'?
2) Which point on x transforms to the point at infinity on x'?

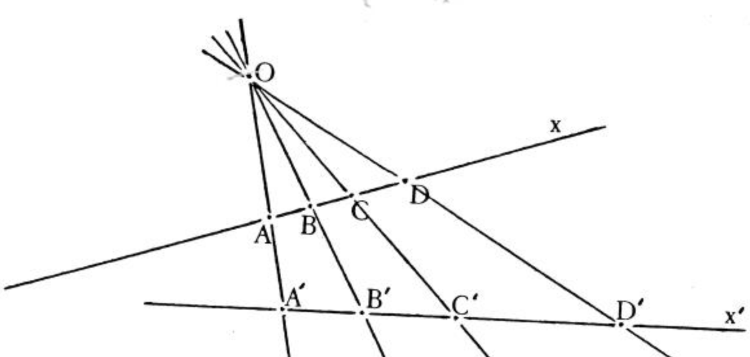

FIG. 1.1

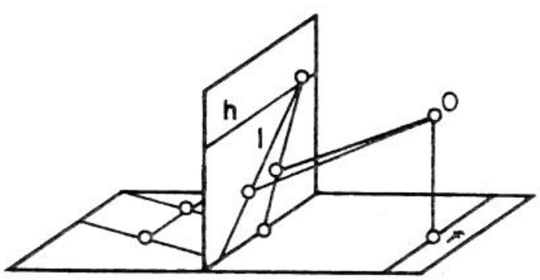

FIG. 1.2

A road l receding to infinity depicted as converging to a point on the horizon lin h of an artist's canvas.

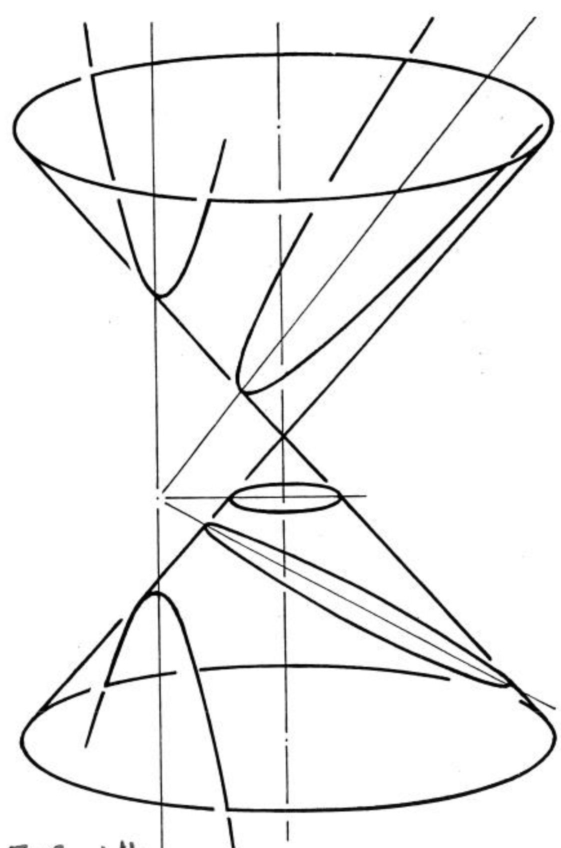

FIG. 1.4

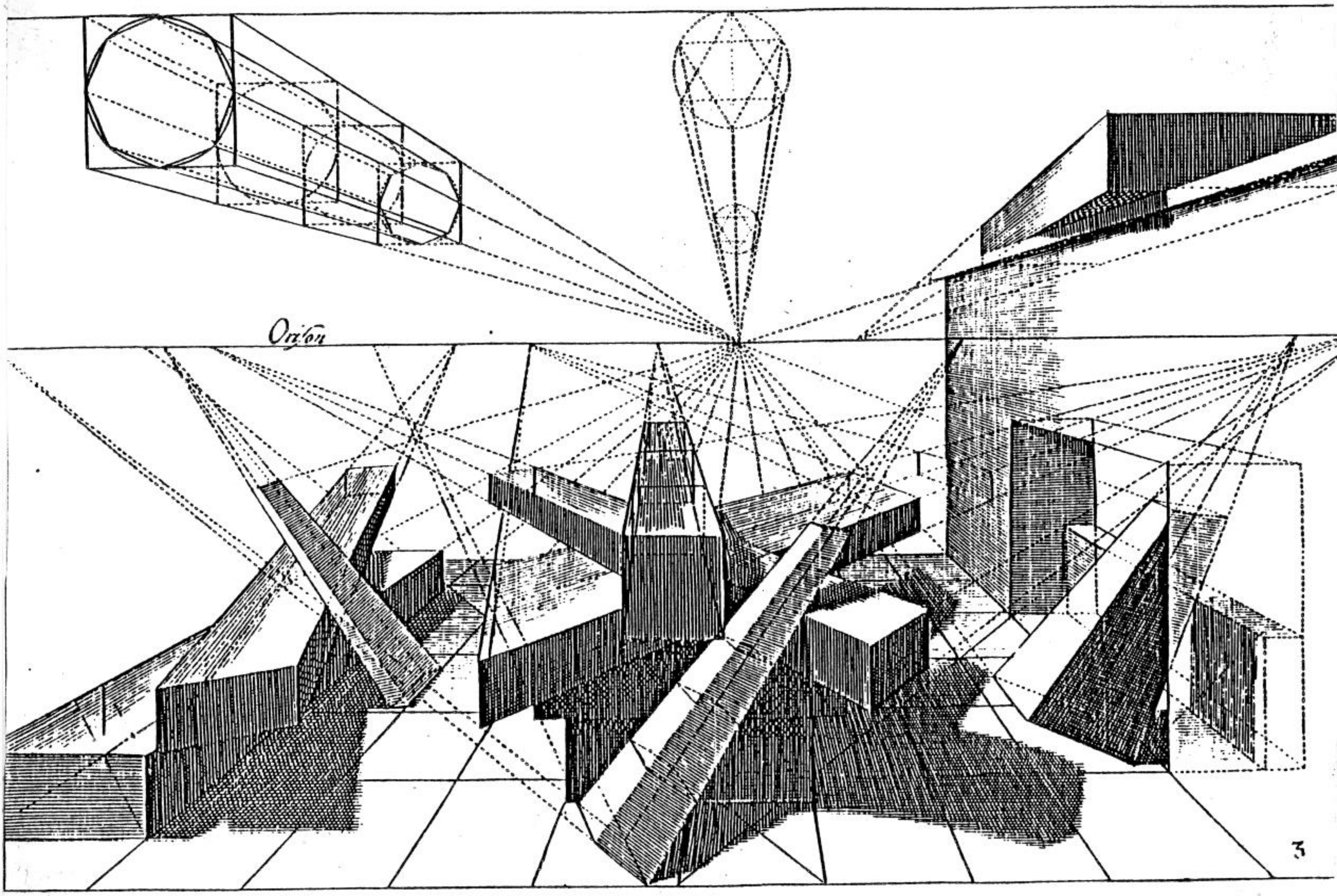

FIG. 1.2

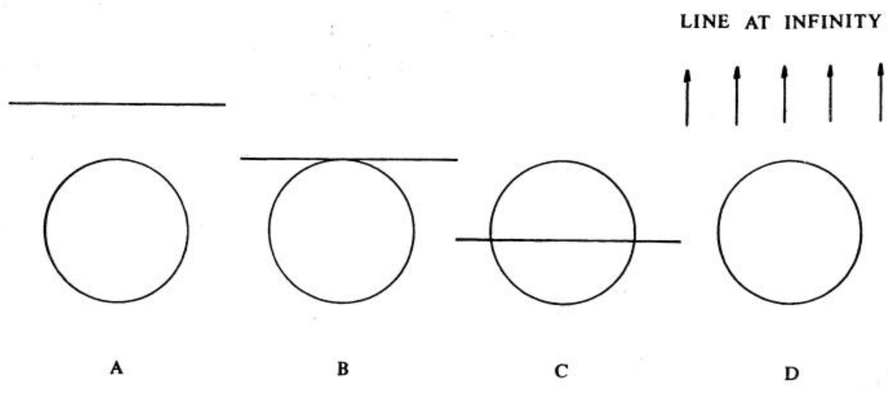

Figure 2.3 When the special line is projectively mapped to infinity, the circles become (a) an ellipse, (b) a parabola, (c) an hyperbola, (d) a circle.

FIG. 1.5

Construction 2:

Pappos' Theorem (see Fig. 2.1):

We describe this theorem by using an analogy to the color wheel. Beginning with two arbitrary lines and two sets of three points, a third line mysteriously materializes

1. Consider an arbitrary pair of lines labeled a and b. Place three arbitrary points labeled R_1, B_1, Y_1 on line 'a' and three other arbitrary points labeled R_2, B_2, Y_2 on line 'b' where R,B, and Y can be thought of as the colors red, blue, and yellow.
2. Connect R_1 with B_2 and B_1 with R_2. The intersection of these two lines are labeled P for purple (mixture of red and blue).
3. Connect R_1 with Y_2 and Y_1 with R_2. Label the intersection of these two lines, O for orange.
4. Connect B_1 with Y_2 and Y_1 with B_2. Label the intersection G for green.
5. The points P, O, G lie on a third line c which mysteriously materializes..

Remark 1: Note that the sequence of six points : $R_1 B_2 Y_1 R_2 B_1 Y_2 R_1$ define six lines which form a six-sided closed figure or star-hexagon where P,O,G lie at the intersection points of opposite edges of the hexagon.

Remark 2: We are using upper case letters to represent points and lower case letters to represent lines.

Remark 3: The assignment of R,B,Y to the three points on each line is arbitrary. If the labeling was different the new intersections would still lie on a line although the line could be different.

Construction 2: Demonstrate Pappos' Theorem by choosing a pair of lines a and b and three points on each line and show that your construction leads to a third line c.

Warning: Depending on where you place the points and in which order you name them, the third line might be off of the page so you may have to hunt for points which avoid this problem. The same hold for other projective geometry constructions.

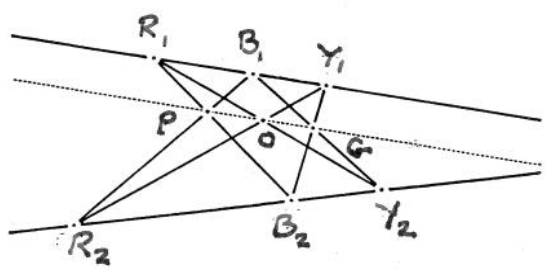

FIG. 2.1

Construction 4:

Pascal's Theorem (Pappos' Theorem on an ellipse or a circle)

In Fig. 4.1 two sets of three points, R_1, B_1, Y_1 and R_2, B_2, Y_2 are placed on an ellipse and the Pappos' construction is carried out to show that once again P,O,G lie on a line, the so-called *Pascal line*, and a hexagon is again inscribed in the ellipse.

This construction goes by the name, *Pascal's Theorem*. You will notice that once a hexagon is inscribed in the ellipse, opposite sides meet at the three points on the Pascal line. In Fig. 4.2, we show another example of Pascal's Theorem, but this time the three pair of opposite edges, labeled 1,2,3, meet at the three points on the Pascal line.

Construction 1: Depending on how one numbers the six points on an ellipse which are the vertices of the hexagon, a different Pascal line results. In Fig. 4.3, six equidistant points are placed on a circle, and the hexagons are illustrated that are the result of the 12 possible orderings of the points. Draw the Pascal line defined by the intersection of opposite edges for three of these hexagons.

Construction 2: Place six points anywhere on a circle and draw the Pascal line. You may also use the ellipse FIG. 4.4.

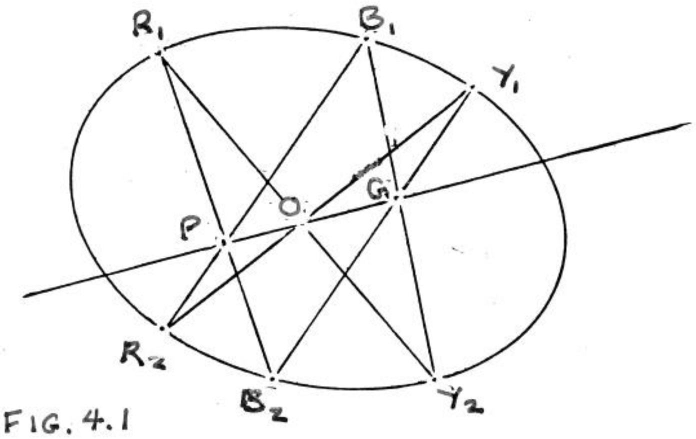

FIG. 4.1

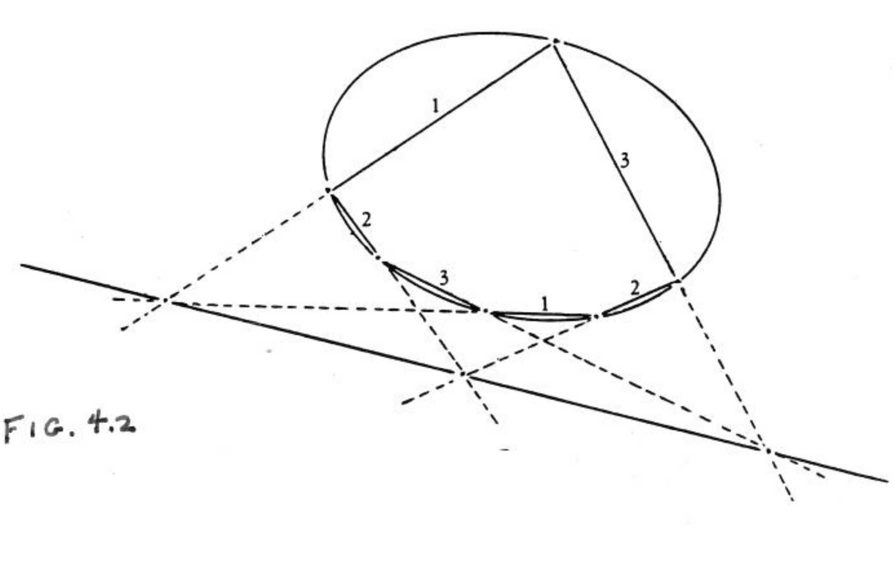

FIG. 4.2

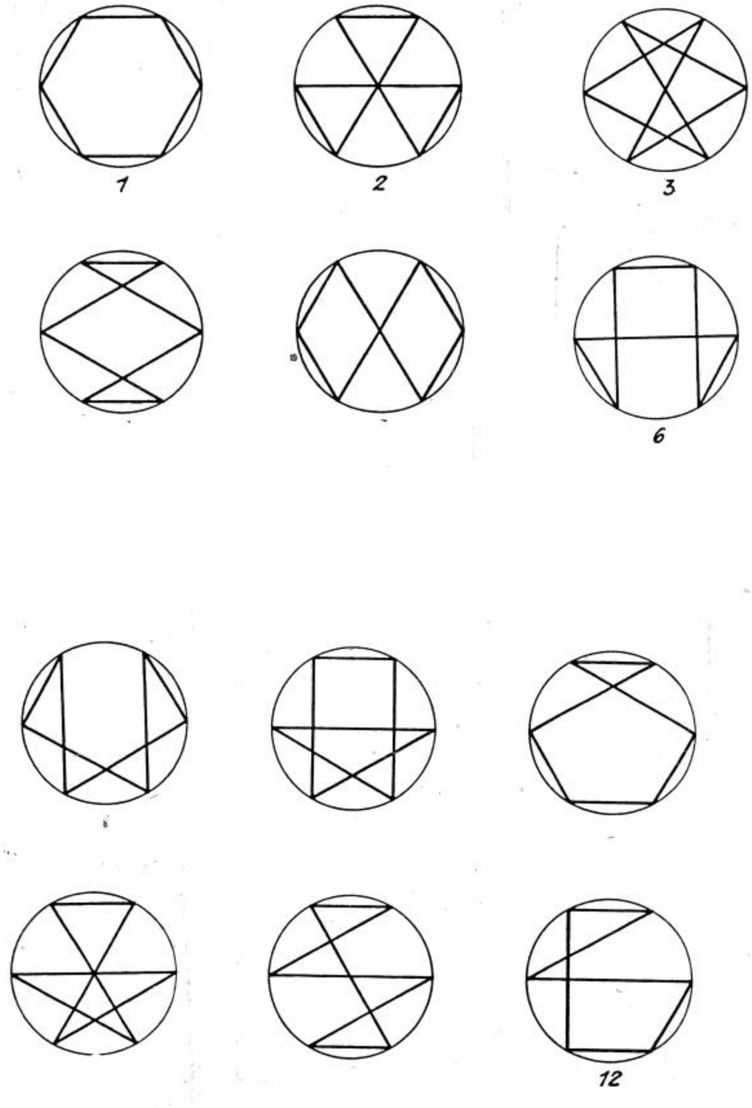

FIG. 4.3

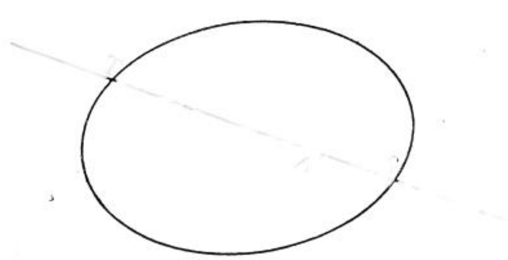

FIG. 4.4

Construction 3:

The Dual of Pappos' Theorem. (Fig. 3.1)

Beginning with two points and two sets of three lines radiating from these two points a mysterious third point materializes..

In projective geometry, for any construction or theorem in the plane there is a dual construction where the word 'line' is replaced by the word 'point' and 'point' is replaced by 'line.'

The dual of Pappos' theorem states that beginning with an arbitrary pair of points labeled A and B, consider a pencil of three lines labeled r_1, b_1, y_1 incident to A and r_2, b_2, y_2 incident to B. The intersections of r_1 with b_2 and b_1 with r_2 defines a line labeled p. The intersections of r_1 with y_2 and y_1 with r_2 defines a second line labeled o. The intersections of b_1 with y_2 and y_1 with b_2 defines a third line labeled g. The three lines p, o, g meet at a third point C which mysteriously appears..

Remark : Notice that the intersections of the six line pairs define six points which lie on the hexagon $r_1 b_2 y_1 r_2 b_1\, y_2 r_1$ in which lines p,o,g connect opposite points of the hexagon.

Construction 3: Demonstrate the dual of Pappos' Theorem by choosing two points A and B and three lines through these two points leading to a third point C.

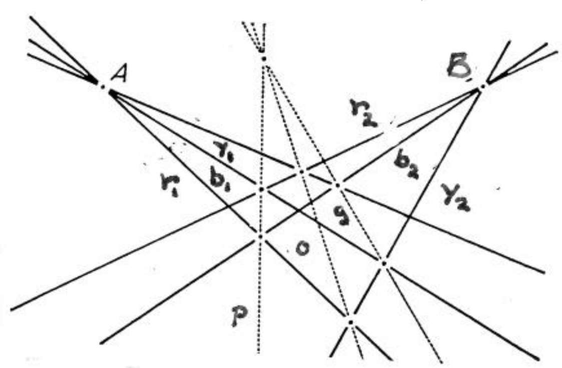

FIG. 3.1

Construction 5: Construction of a conic section

Pappos' or Pascal's Theorem can be used to draw a conic section (ellipse, circle, hyperbola, and parabola).

In Fig. 4.1, Pappos' Theorem was applied to an ellipse. In fact Pappos' Theorem works for two sets of three points lying on any conic section. Using this Theorem and starting with five arbitrary points in the plane, as shown in Fig. 5.1, we wish to construct the unique conic that goes through the points as follows:

1. Place five points arbitrarily in the plane and label them R_1, B_1, Y_1 and R_2, B_2 in any order. The problem is posed to find Y_2.
2. Connect R_1 with B_2 and B_1 with R_2. The intersection of these two lines are labeled P for purple.
3. Draw an arbitrary line through B_1. Somewhere on this line will be the missing point Y_2.
4. Connect Y_1 with B_2 , and where it intersects the line through B_2 from step 3 label this point G..
5. According to the Pappos construction, P and G lie on the third mysterious third line of Pappos' theorem. Draw this line.
6. The line from R_2 to Y_1 intersects the line between P and G at O.
7. The line from R_2 to O intersects the line through B_1 in step 3 at Y_2 , and this is the missing point. It lies on the unique conic that goes through the five starting pints.
8. Now repeat this construction for a pencil of lines through B_1 and you will construct all of the lines on the conic.

Remark: Pappos' Theorem works for all conics. In fact, even the pair of lines in Construction 2 can be thought of as the extreme case of an hyperbola, where the hyperbola degenerates to its pair of asymptotes. Since all conics are projective transformations of each other, any theorem of projective geometry that works for one conic works for the others. When Pappos' Theorem is applied to a general conic it is called Pascal's Theorem

Construction 5: Beginning with five arbitrary points, use this procedure to construct three additional points on the unique conic that goes through these points.

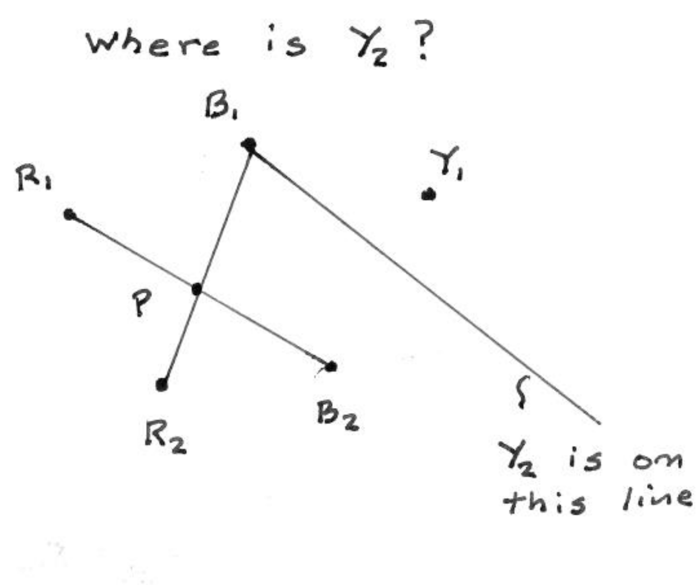

FIG. 5.1

Construction 6: Desargues Theorem

Begin with a pair of triangles with vertices labeled ABC and A'B'C' as in Fig. 6.1, with the constraint that lines AA', BB', and CC' meet a common point O. The three pairs of lines AB and A'B'; BC and B'C'; and CA and C'A' intersect at three points and these three points lie on a line (they are co-linear).

Remark: The reason why this theorem works can be seen by looking at a three dimensional version of Desargues Theorem in Fig. 6.2.

Construction 6: Choose a pair of triangles whose corresponding vertices project to a common point O and generate the line defined by Desargues Theorem.

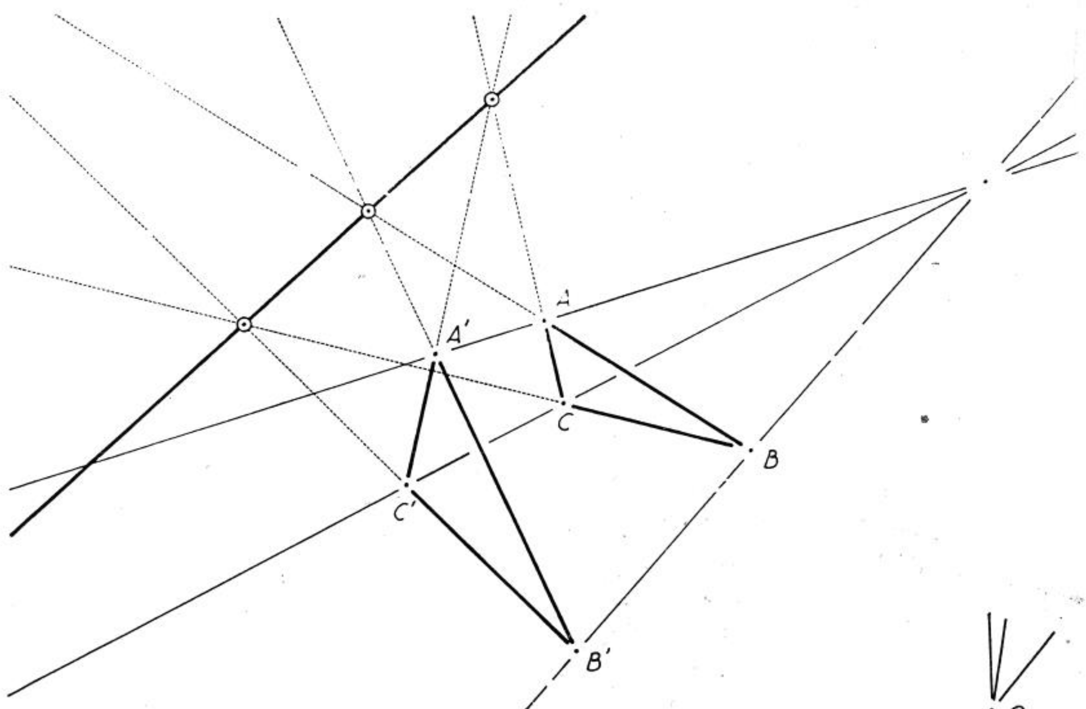

FIG. 6.1

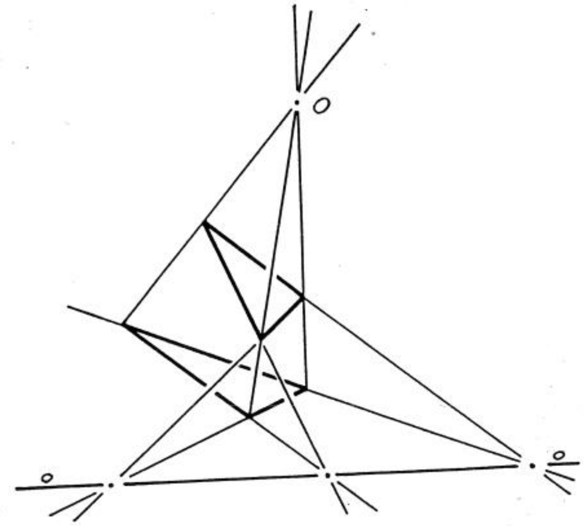

FIG. 6.2

Construction 7: An Hexagonal Net

Beginning with four non-colinear but otherwise arbitrary lines (see Fig. 7.1) we can create an hexagonal net of lines as follows:

1. Lines can be drawn through already defined points, and whenever new points are derived, additional lines can be drawn. In this way Fig. 7.2 can be derived from Fig. 7.1.
2. Continuing this process you are able to generate a net of regular hexagons as in Fig. 7.3

Remark: One line has been singled out as a kind of horizon line . The hexagons of the net are perspective drawings of regular hexagons in which opposite edges are parallel and therefore meet at the horizon line.

Construction 7: Create an hexagonal net.

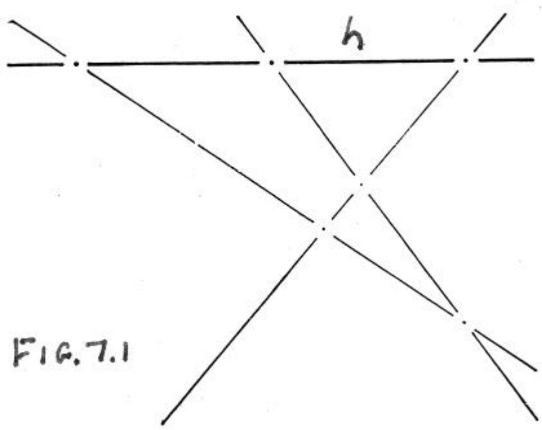

FIG. 7.1

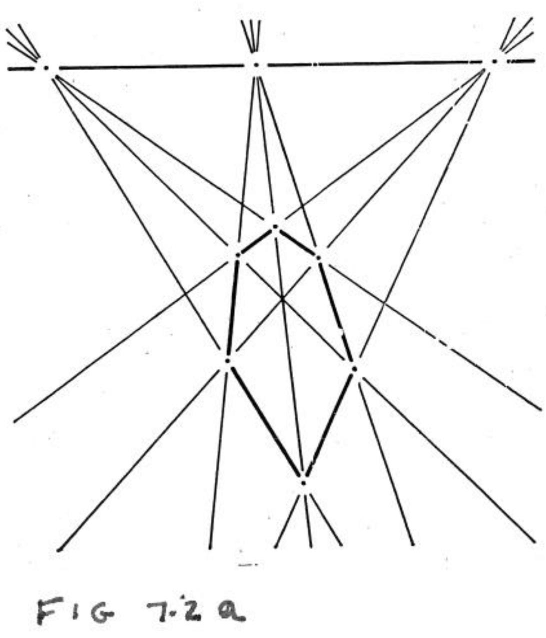

FIG 7.2 a

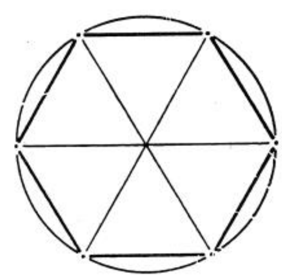

FIG. 7.2 b

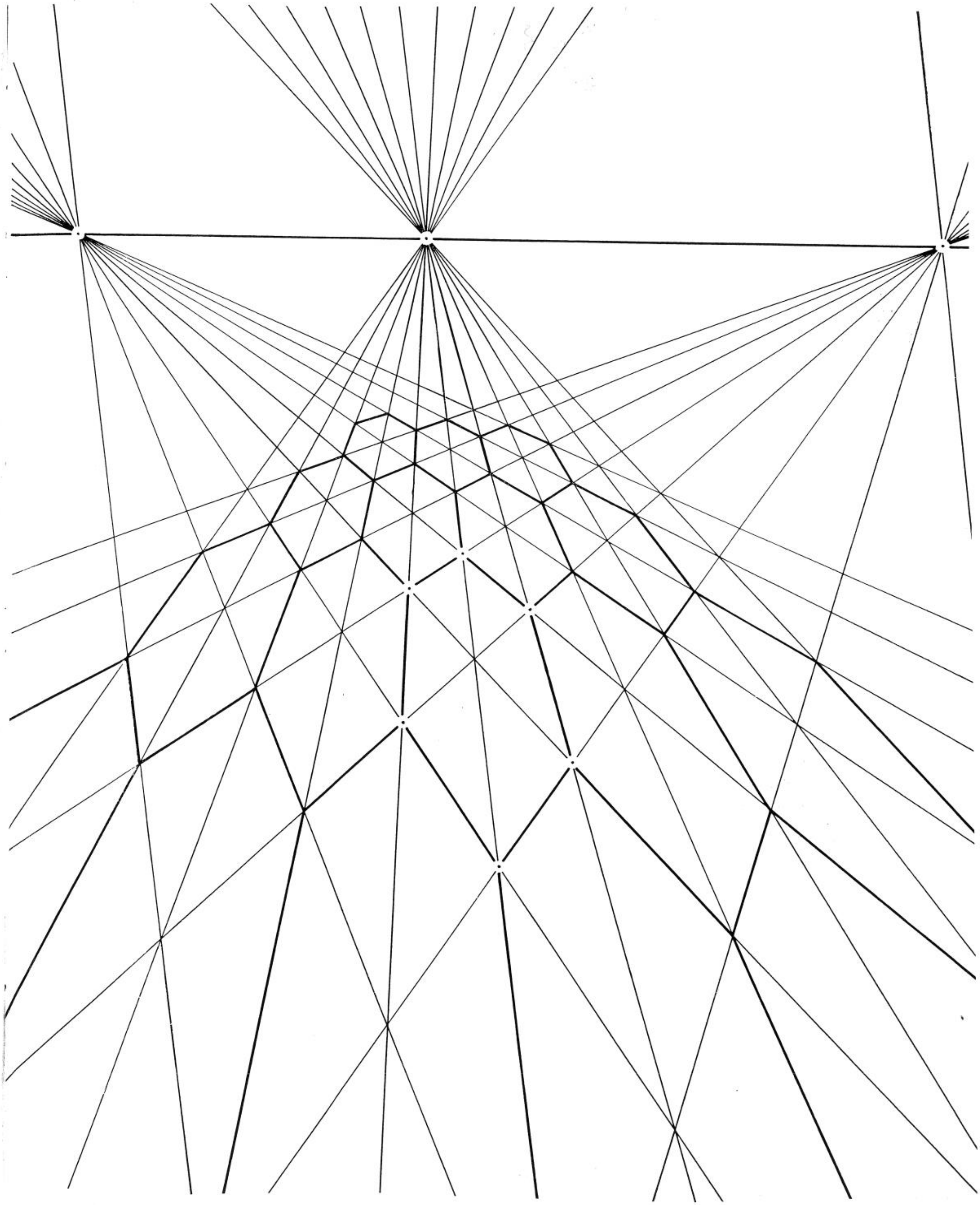

FIG. 7.3

Construction 8: A Quadrilateral net

This time we begin with the configuration of four lines and three points, A,C,D in Fig. 8.1. The three points lie on the horizontal line h This construction will define a fourth point D and a quadrilateral along with its two diagonals as follows.

1. Connect all existing points by lines. As Fig. 8.2 shows, two additional lines must be added. This defines a quadrilateral with one diagonal intersection H at D. When the other diagonal is added it intersects the horizontal line H in a fourth point, B. There are pencils of two lines incident to two of the points A, C on h
2. If h is thought to be the horizon line then the two lines in each pencil can be though of as being parallel edges of the quadrilateral in which case the quadrilateral is the projective image of a parallelogram.
3. In Figure 8.3, additional lines project from the two pencils to define a net of quadrilaterals (parallelograms) in which the diagonals of all the quadrilateral of the net intersect at the two additional points.

Remark 1: The two points A,C of h where the edges of the quadrilateral intersect intersperse the two points B,D which go through the diagonals.

Remark 2: The four points A,B,C,D on H are called *harmonic* because they satisfy the special property that their cross ratio equals -1, i.e.,

$$\lambda = \frac{AB}{BC} \div \frac{AD}{DC} = -1$$

Construction 8: From four lines and three points of your own choice construct a quadrilateral with its diagonals using the procedure described above. Compute the cross-ratio of the four points on line h and show that it equals -1. Then construct a net of at least four quadrilaterals.

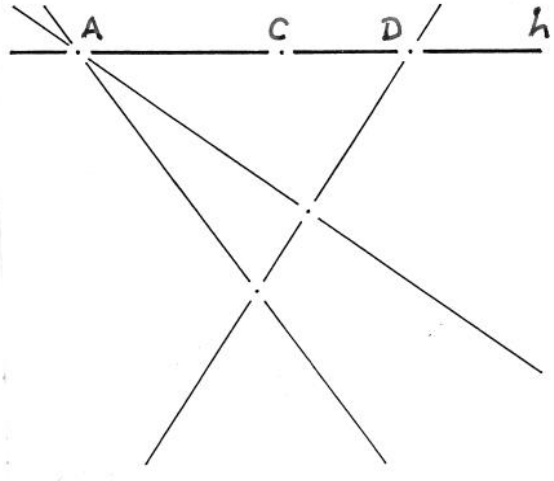

FIG. 8.1

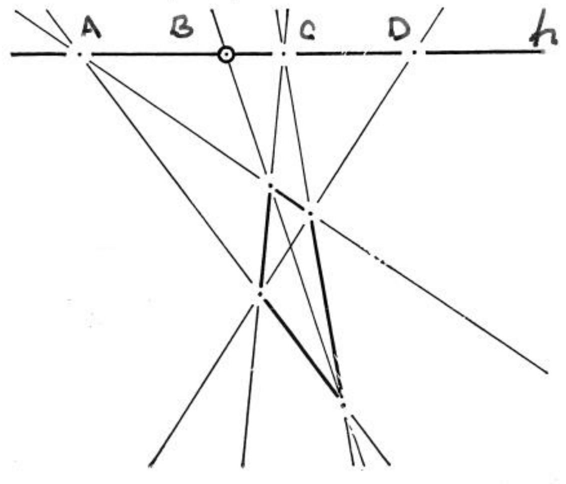

FIG. 8.2

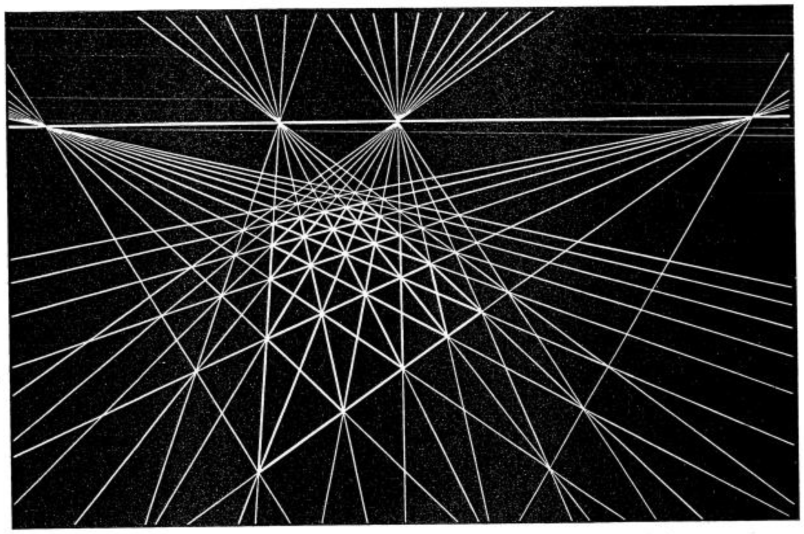

FIG. 8.3

Construction 9: Points of the Quadrilateral net going to infinity

First we must describe the notion of infinity in a projective context. Consider Fig. 9.1, a sequence of lines radiating from a point and intersecting a horizontal line. Notice that as the intersection point on the horizontal line moves from left to right the radiating line rotates counterclockwise until when the point reaches positive infinity the radiating line is horizontal. Now let the radiating line rotate slight bit more in a counterclockwise direction. Notice that the point of intersection on the horizontal line has now shifted all the way to the left and begins to come back to its starting point as the line rotates even more. In summary, the point on the horizontal line goes to positive infinity and shifts to negative infinity from which is returns.

In Euclidean geometry with the exception of parallel lines, all lines intersect at a unique point. Parallel lines, by definition, never meet, and the fifth axiom of Euclidean geometry states that given a line and a point not on the line, there exists exactly one line through the point parallel to the line. In projective geometry, this exception is eliminated and, by an axiom of projective geometry, any two lines share a common point of intersection. If the lines are parallel, that point exists but is the infinite point on the line. A new pair of lines with different orientation meet at a different point at infinity. All lines of a given orientation meet at the same point at infinity and the collection of infinite points is said to make up the *line at infinity*. Sometime the line at infinity is referred to as the *circle at infinity* since, as we saw above, the left and right infinite points on the line are identified with each other and considered to be the same point. These somewhat abstract ideas are made concrete when the infinite line is mapped to the finite horizon line on the artist's canvas. The points where parallel lines intersect are then clearly visible.

Now reconsider the configuration of the quadrilateral in Fig. 8.2. What happens to this configuration as point D moves to the right approaching infinity? The line through D approaches a line parallel to h so that one of the diagonals for each quadrilateral of the net must be horizontal as shown in Fig. 9.2. We also find that point B approaches the midpoint of the line segment AC. For the first time we have a precise measurement entering into a projective context.

Next consider the configuration of the quadrilateral in Fig. 9.3 and consider what happens to vertex 4 of the quadrilateral as it moves away from the horizontal line along the diagonal 42. Notice that as point 4 disappears into infinity, the lines 14, 24, and 34 become parallel.

In Fig. 9.4, edge 14 has been rotated so that it is no longer parallel to diagonal 42 and vertex 4 moves away from h along edge 14, goes through infinity and returnsalong the same line but on the other side of h. The direction of diagonal 42 is defined by vertex 2 and the diagonal point C on h. Therefore point 4 is uniquely defined by the intersection of the lines of action of lines 41 and 42. The opposite sides, 21 and 43, of the quadrilateral then go through the fourth point on h and we witness a quadrilateral enclosing infinity.

In this way, we are able to incorporate figures encompassing infinity in the same context as purely finite figures.

Now, in addition to letting vertex 4 move away from h along diagonal 42, let vertex 3 move away from h along diagonal 13 (see Fig. 9.5) while edge 42 rotates away from the diagonal 42. A remarkable thing happens. When vertex 4 disappears into infinity it reappears on the other side of h.

Construction 9: Carry out these constructions to see what happens as points 4 and 3 move along their diagonals to infinity and what happens when one of the edges swing away from the diagonal as in Fig. 9.5. Make a construction similar to Fig. 9.5 and locate point 4 of the quadrilateral. If you can computerize this it would be interesting to see an animation showing what happens to the quadrilateral as the points recede to infinity.

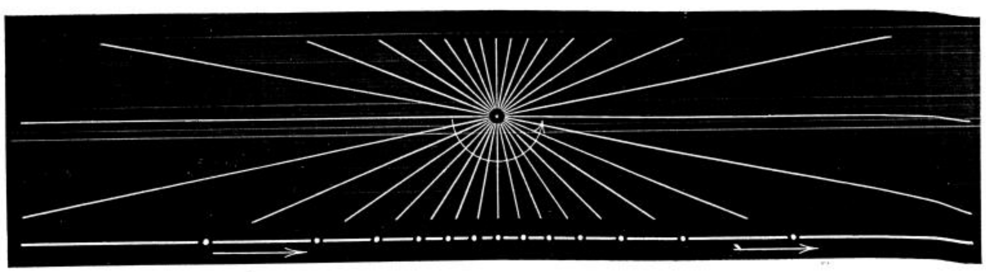

FIG. 9.1

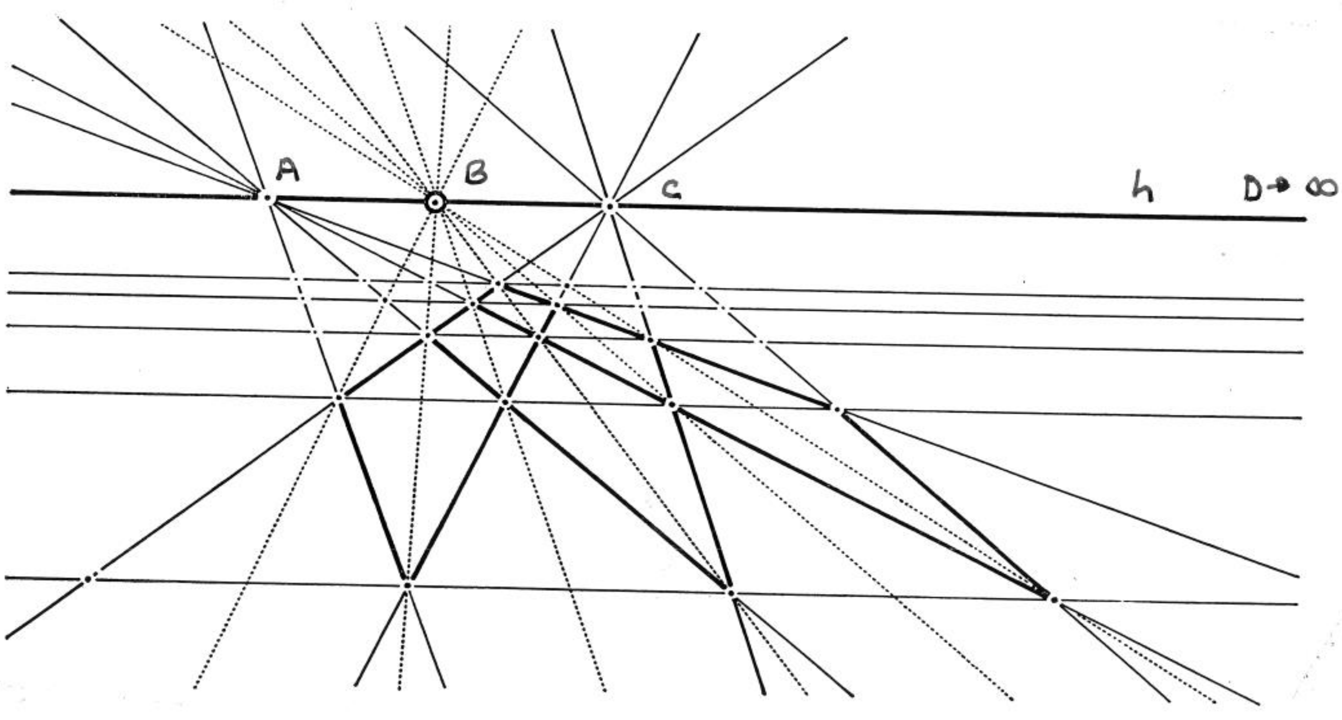

FIG. 9.2

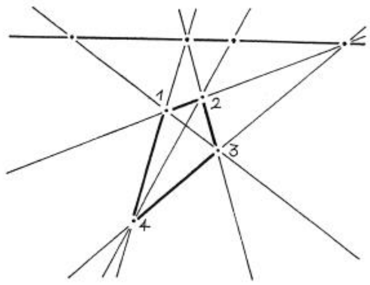

FIG. 9.3

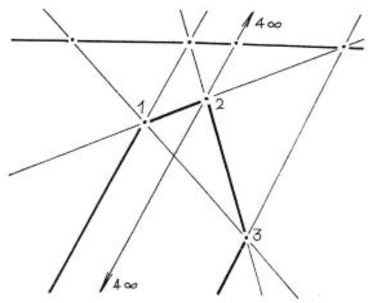

FIG. 9.4

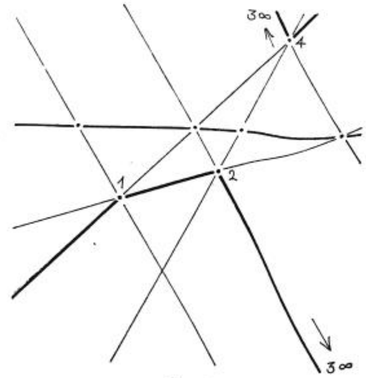

FIG. 9.5

Construction 10: Branchion's Theorem

Branchion's Theorem is the dual to Pascal's Theorem.

In Pascal's Theorem, we saw in Construction 4, that the points on an ellipse or circle are numbered 1,2,3,4,5,6 and the six points are connected in any order to define a hexagon. Opposite edges of the hexagon meet at three points on a line (the Pascal line) with the line dependent on the order.

In Branchion's Theorem the points on a conic are numbered 1 through 6 in any order, and six lines are drawn tangent to the six points and connected to form a hexagon. Opposite points on the hexagon meet at a common point (the Branchion point) with the point dependent on the order..

Construction 10.1: Fig. 4.3 shows all 12 hexagon configurations that are possible when six points are evenly spaced around a circle and connected in different order. In which of the configurations does the Pascal line go to infinity.

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In Fig. 10.2 three circles are shown with six points evenly distributed around the circumference of a circle. Branchion's theorem is illustrated for three different orderings of these points. Tangent to each of the points is a line numbered the same as the points. When line 1 intersects line 2 the point of intersection will be a vertex of the hexagon and that vertex will be labeled 1, line 2 intersects line 3 at vertex 2, etc. By Branchion's Theorem, opposite vertices of the hexagon, i.e., 1and 4; 2 and 5; 3 and 6 meet at a common point, the Branchion point. You will notice that the ordering in Fig. 10.2c is leads to a hexagon that goes through infinity.

Construction 10.2: Place six points evenly around the circumference of a circle and construct a Branchion hexagon for two different orderings of the points and show that opposite vertices meet at the Branchion point.

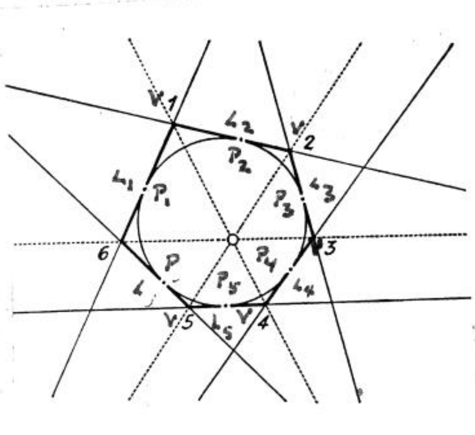

FIG. 10.1a

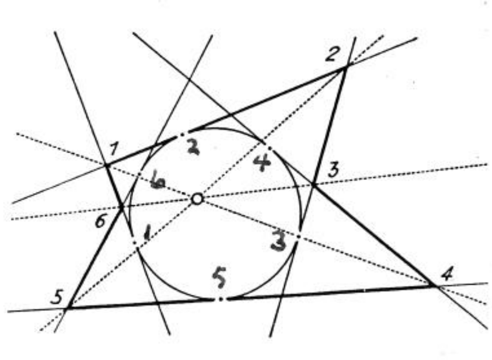

FIG. 10.1b

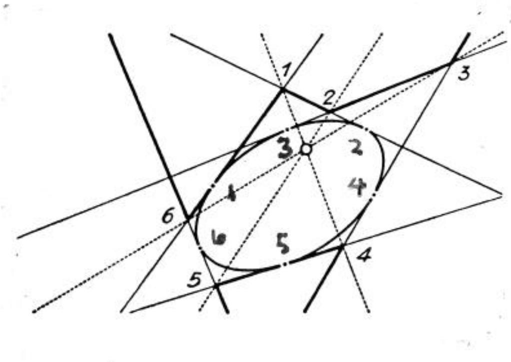

FIG. 10.1c

Construction 11: Pole and Polar of an ellipse.

Perhaps the simplest and most elegant construction in projective geometry and its application to design is the notion of the pole and polar of a conic. We will consider the conic to be an ellipse or a circle.

The pole is a point either outside, inside, or on an ellipse. Associated to the pole will be a line called the polar.

In Fig. 11.1, the point is outside the ellipse or circle. The pole is defined to be the line that is incident to the two tangents to the ellipse from the pole.

Any line from the pole A intersects the ellipse at S and T and the polar at A' as shown in Fig. 11.2.. It turns out that the four points A,S,A',T are harmonic, their cross-ratio equals -1.

In Fig. 11.3, the pole is as point on the ellipse or circle. The polar is then the line tangent to the ellipse at the pole.

In Fig. 11.4, the pole is within the ellipse. To find the polar, draw any two lines through the pole. Where each line intersects the ellipse, draw a pair of tangent lines. Draw the line that joints these two pairs of tangent lines. This is the polar line. In fact the pair of tangent lines for all lines through the pole intersect on the polar as can be seen in Fig. 11.5..

Construction: For the ellipse in Fig. 11.6, draw the polar lines for the case where the pole is outside, inside, and on the ellipse.

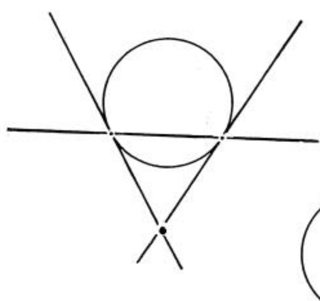

FIG. 11.1

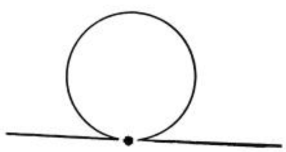

FIG. 11.3

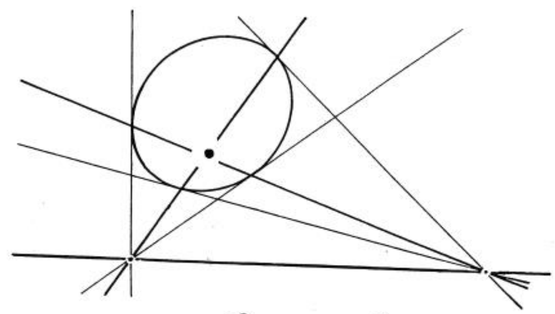

FIG. 11.4

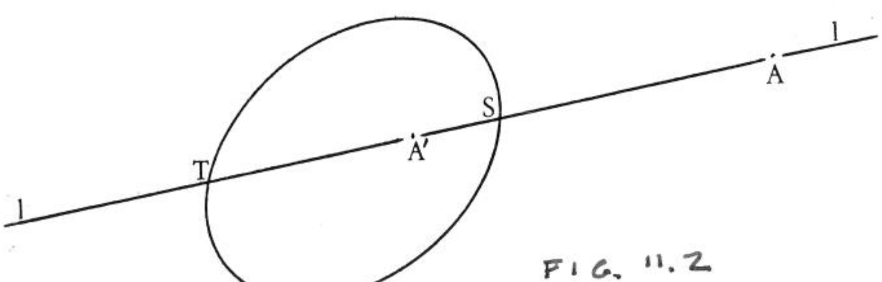

FIG. 11.2

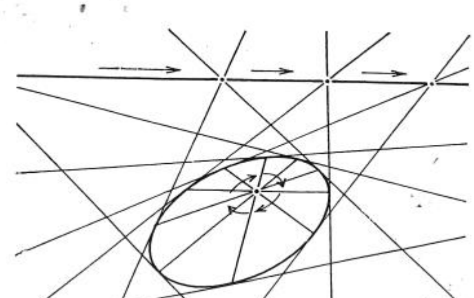

FIG. 11.5

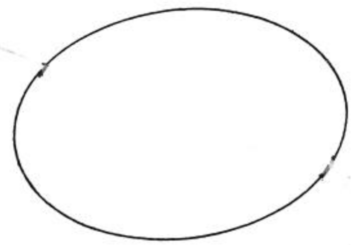

FIG. 11.6

Construction 12: Inversion in a circle

If we consider the relationship between pole and polar in the context of a circle, this leads to a wonderful transformation known as inversion in a circle from which many interesting designs can be derived.

1. Consider a circle with radius r and center at O with point A outside of the circle. If A is considered the pole then line UV is the polar as shown in Fig. 12.1.
2. Where the line through O and A cuts the polar line is where A' is located. We can say that A' maps to A under inversion in the circle and vice versa.
3. Since triangle OUA' is similar to OUA it follows that,

$$\frac{OA'}{r} = \frac{r}{OA} \qquad (1)$$

If the radius is considered to be one unit, then OA and OA' are inverse distances from the center of the circle, i.e.,

$$OA' = \frac{1}{OA} \qquad (2)$$

4. It is easy to find a polar when the pole A lies within the circle. On the line OA locate A' by using either Eq. 1 or 2. The polar line will be perpendicular to OA' at point A' as shown in Fig. 12.2.
5. A circle of points (poles) are taken within the circle in Fig. 12.3. Each point has a polar constructed as in step 4. Notice that the envelope of lines maps out the arc of a circle exterior to the circle. Note: The envelope of a curve is the set of tangent lines to each point on the curve.
6. Next consider any curve within a unit circle. How can we use Eq. 2 to map this curve to the exterior of the circle? We will demonstrate this mapping for three points A,B, and C on the inner curve (see Fig. 12.4). We will be able to find the envelope of the outer curve by determining the polars a,b,c corresponding to points A,B,C as we did in step 5.
7. To find the polars a,b,c first locate A',B',C' using Eq. 1. The polar lines will be at right angles to lines OA', OB', OC'. If enough points are taken on the inner curve, the outer curve will be accurately defined by its envelope. To find the polar envelope we only have to move around this curve with a set-square, keeping one arm of the set-square through O, and the right angle on the curve. In this way we can draw as many tangents to our polar curve as we wish or find the time for.
8. Fig. 12.5 shows a collection of points around a circular arc within the circle of inversion. By the above construction each point is associated with a polar line with the envelope of these lines being an outer circle.
9. In Fig. 12.6, three such collections of circular points within the circle of inversion lead to three sets of external points and three envelopes of lines defining three outer circles. This net of circular lines and points makes a lovely picture.

10. Several lovely inversions of curves are shown in Fig. 12.7.
11. In this inversion, every point on the circumference of the circle of inversion remains invariant; points in the interior of the circle are transformed to external points and vice versa; convex figures are transformed to concave and vise versa; lines tangent to the circle of inversion are transformed to circles touching the circle of inversion and containing its center and vice versa; and angles between lines are preserved under this transformation.

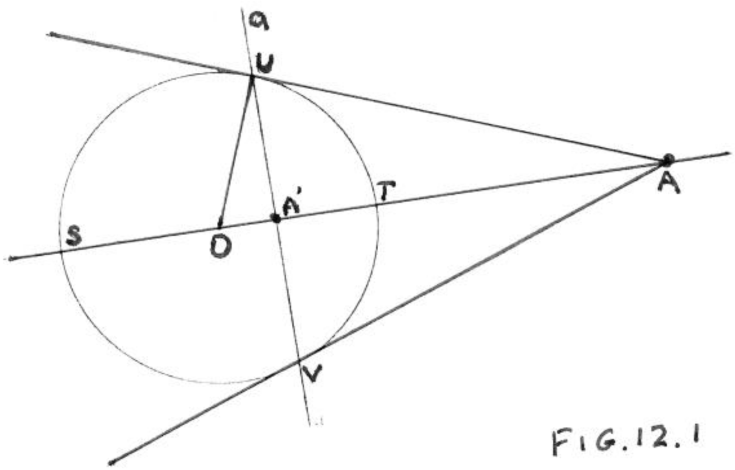

FIG. 12.1

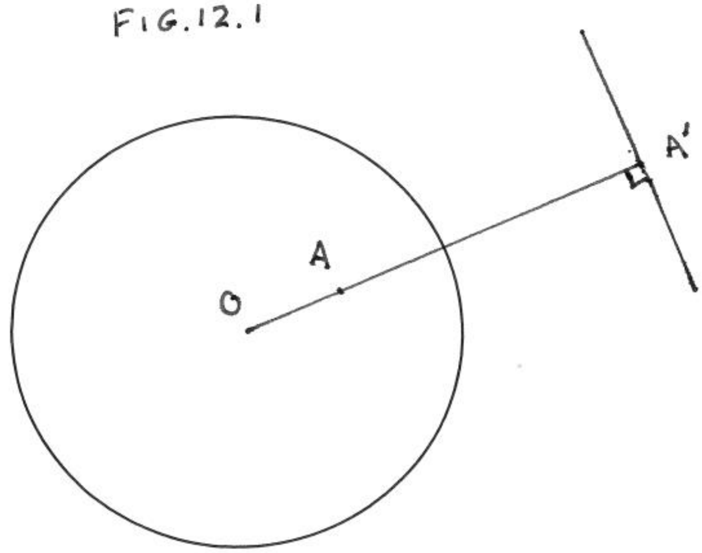

FIG. 12.2

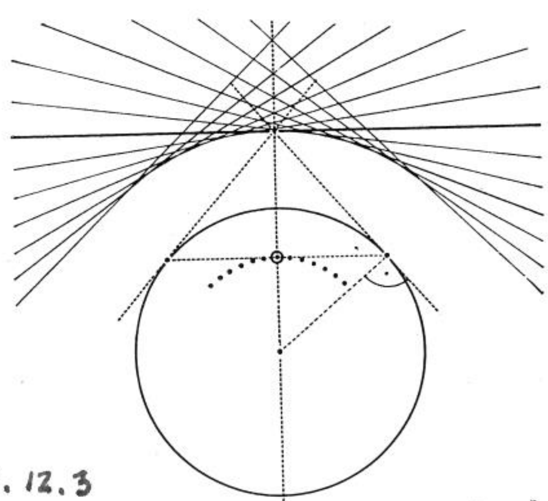

FIG. 12.3

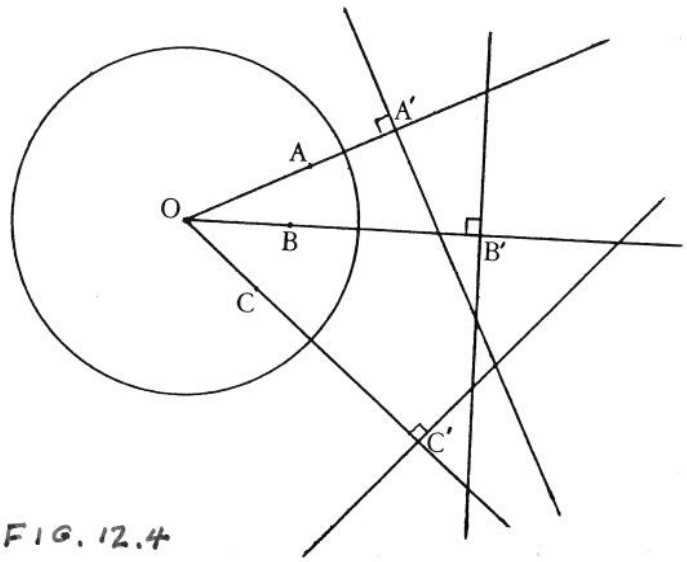

FIG. 12.4

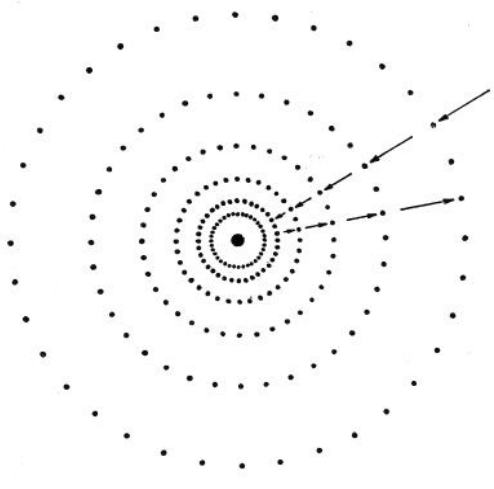

FIG. 12.5

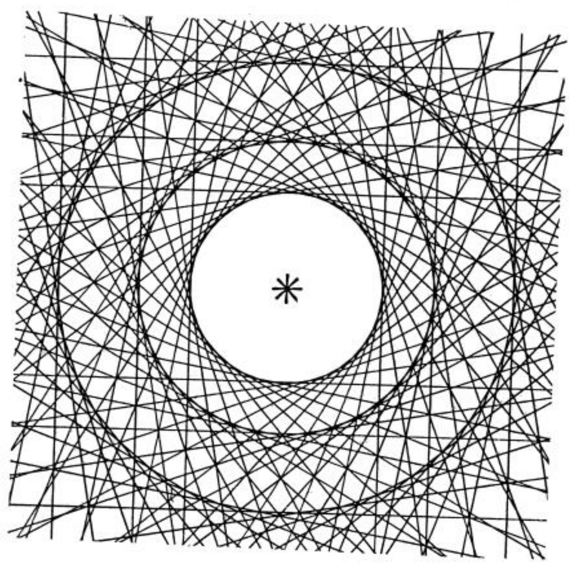

FIG. 12.6

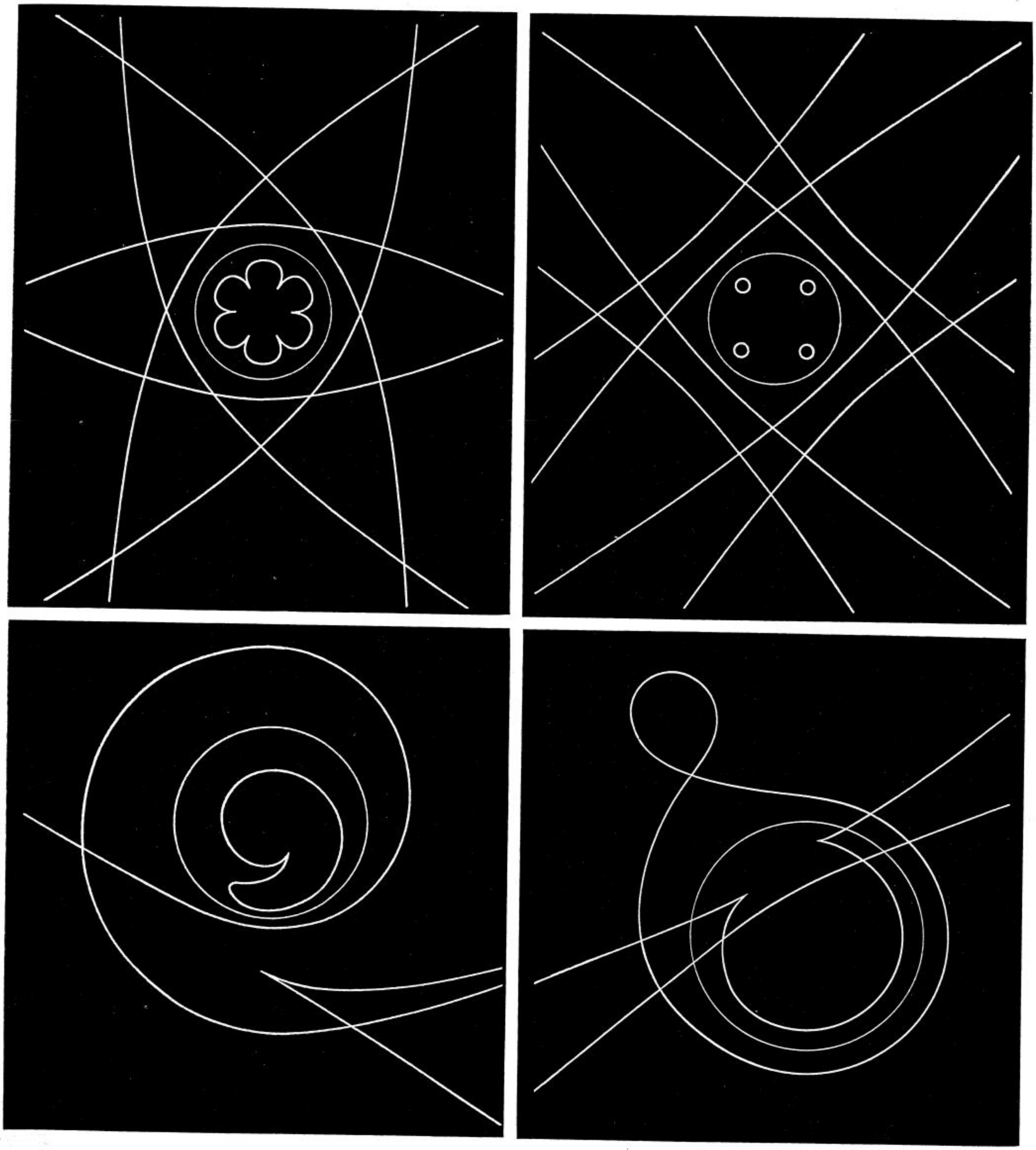

FIG. 12.7