Module 10: Spirals

1. Introduction

The *logarithmic spiral* can be thought of as a curve of life in that it is self-symmetric and occurs in the shape of sea animals called Nautilus shells, the striations of sea shells and the horns of horned animals (see Fig. 1). A logarithmic spiral is formed when a horn grows faster on the outside than the inside illustrated in Fig 1d with wooden blocks cut by a perpendicular pane. If the plane cuts the block at an angle, the growth pattern is helical. The key to the power of the logarithmic spiral is the geometric sequence. To create a logarithmic spiral we must first do surgery on a right triangle.

2. Similarity

Two figures are *similar* if each is a magnification or contraction of the other as shown in Fig. 2. Fig. 3 shows how three lengths are in proportion for a pair of similar stick figures.

3. Surgery on a right triangle

Get a piece of construction paper and draw a diagonal. Next draw a line from a vertex meeting the diagonal at right angles. Label the edges meeting at the intersection of the diagonal and the line from the vertex, by the letters a, b, c as shown in Fig. 4. Cut out the triangles along the edges with a scissors. This results in three similar triangles. Juxtapose the triangles so that their right angles are together. You can visually see that the triangles are similar and that :

$$\frac{a}{b} = \frac{c}{d}. \qquad (1)$$

4. The Log Spiral

Place the unit 1 on the 0 deg. ray of a piece of polar coordinate graph paper and any number k on the 90 deg. ray. Create a triangle by connecting 1 to k. From the vertex 2 draw a line at right angles until it intersects the 180 deg. ray at $r = k^2$. Connect this vertex with the vertex at 90 deg as shown in Fig. 5. You now have a pair of right triangles like the ones in Fig. 4 and you can see from Eq. 1 that ,

$$\frac{1}{k} = \frac{k}{k^2} \qquad (2)$$

From the vertex at $r = k^2$ draw a series of lines forming right angles at 90 deg. intervals as shown in Fig. 6a. By going in the direction of decreasing radius we get a curve both spiraling inwards and outwards as shown in Fig. 6b. The vertices of this spiral shape forms a geometric sequence as shown in Sequence 3 for k = 2:

1/4 12 1 2 4 8 16 32 (3)

These results are recorded in Table 1. The left hand column of this table records the number n of right angles, $\frac{\pi}{2}$ radians, of the vertices while the right column is the distance r from the center of the spiral at the origin to the vertex..

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$n = \frac{\theta}{\pi/2}$	r
-2	1/4
-1	½
0	1
1	2
2	4
3	8
4	16
5	32
n	2^n

This represents the Table of an exponential function base 2 shown in the graph of Fig. 7. Reading the Table backwards leads to the logarithm base 2 or $\log_2$, e.g., $\log_2 4 = 2$, $\log_2 1 = 0$, etc. The graph of $y = \log_2 x$ is also shown in Fig. 7.

The results can be summarized by ,

$$r = 2^{\frac{2\theta}{\pi}}$$

which can be rewritten,

$$r = e^{k\theta} \qquad (4)$$

is the equation of a logarithmic spiral. The logarithmic spiral embodies self-similarity. If you consider the arc subtended by any angle θ, the arc will be a similar; you can use a magnifying glass to change its scale so that the two arcs lie atop of each other. In other words, you can enlarge or contact one arc to fit exactly with the another. The logarithmic spiral is the only smooth curve with this property. We will see other curves called fractals that have this property, but those curves are nowhere smooth.

5. The Law of Repetition of Ratios

The logarithmic spiral is also intrinsic to a construction used to replicate proportions of a rectangle called the Law of Repetition of Ratios. It was used during the classical era in ancient Greek and Roman architecture and design. Its importance was recognized by Jay Hambridge who coined the term dynamic symmetry to describe this process.

. Begin with some geometric form or pattern which we call a unit and add another form or pattern called a gnomon which is required to enlarge the unit while preserving its form. In this way if we consider the unit to be a rectangle with sides in the ratio a:b and . draw a line from one vertex that meets a diagonal of the rectangle at right angles as we did in our surgery on a right triangle above. The rectangle can be divided into two rectangles one of which is a unit at smaller scale and the other being the gnomon, i.e., U = U + G as shown in Fig. 8 We see here that the two units satisfy the same rule as the three right triangles in Fig. 4,

$$\frac{a}{b} = \frac{c}{d}. \qquad (5)$$

Fig. 9 shows two classical buildings subdivided by the law of repetition of ratios.

Looking at Fig. 8a you will see that the same construction is ready to be carried out at a smaller scale so that U = U + U + G. This process can be continued indefinitely so that the unit is tiled by a sequence of whirling gnomons and one final unit, U = G + G + G + …+ U (see Fig. 6b). The result of Eq. 5 means that the scale of the gnomons decrease in a geometric progression.

If the gnomon is a square, i.e., G = S, what is the proportions of the unit (see Fig. 10a)? Let the proportions of U be x:1. Then x/1 = 1/ x-1. solving this we find that $x^2 - x - 1 = 0$ or $x = \frac{1+\sqrt{5}}{2} = \phi$, the golden mean. Show that when G = DS (double square), the proportions x : 1 of the unit is $x = 1 + \sqrt{2} = \theta$ known of as the silver mean (see Fig. 10b), and when G = U, $x = \sqrt{2}$, in other words when a rectangle of proportions $x:1 = \sqrt{2}$ (root 2 rectangle) is divided in half it yields a pair of root 2 rectangles (see Fig. 10c).

Construction 1: Use the whirling squares to construct an approximate logarithmic spiral when k = ϕ by the method shown in Appendix 1.

6. The four turtle problem

Construction 2: Follow the instructions in Appendix 2 to draw the four logarithmic spirals traversed by Tom Pizza's trained turtles.

7. Baravelle Spiral

In Fig. 11a we see that a circle is drawn tangent to an outer square (inscribed circle) and touching the vertices of an inner square (circumscribed circle). This square-within-a-square, called *an ad-quadratum square*, was much used in ancient geometry and architecture. The area of the inner square is obviously half the area of the outer square since the smaller square contains eight congruent triangles, whereas the larger square contains 16. In a sequence of circles and squares inscribed within each other, each square is 12 the area of the preceding (see Fig. 11b). Fig. 11c shows a sequence of ad-quadratum squares which are shaded to form a logarithmic spiral know as a Baravelle spiral. It is easy to construct, and with color makes an interesting design.

Construction 3: Construct a Baravelle spiral and color it to bring out its power. Details plans are given in Appendix 3.

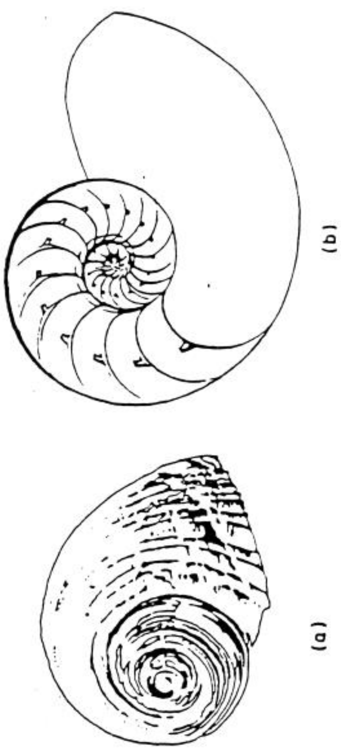
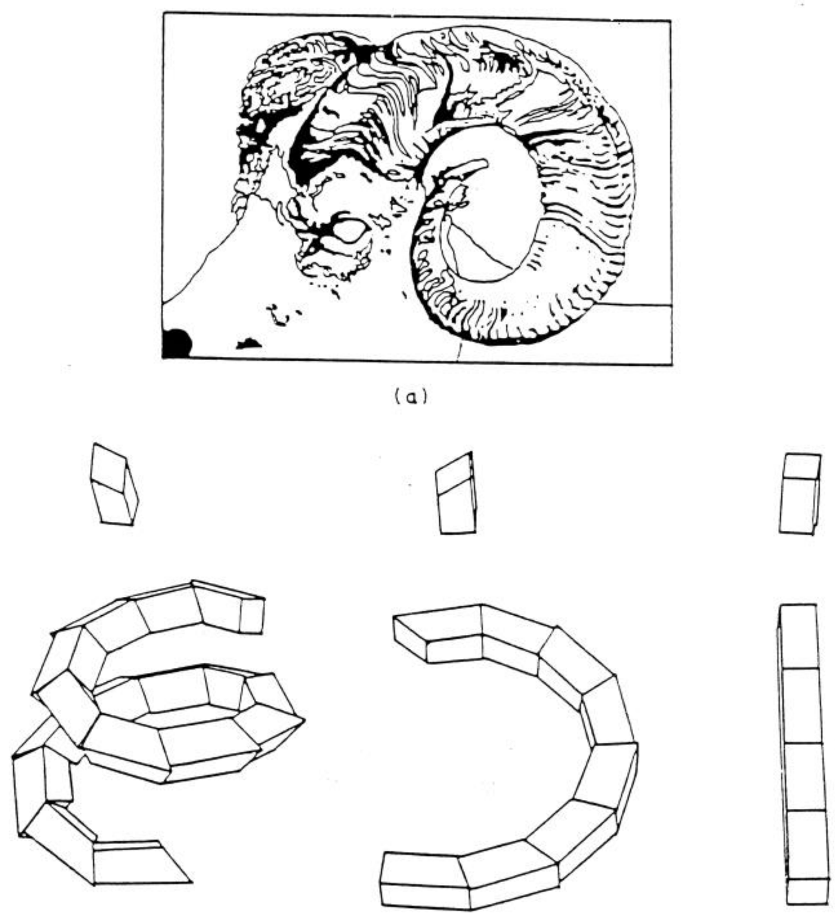


Figure 2.13 **Natural forms illustrating logarithmic spiral growth.** *(a)* **Shell forms;** *(b)* **nautilus.**

F I G. I



A logarithmic spiral is formed when a horn grows faster on the outside than the inside illustrated with rectangular wooden blocks cut by a perpendicular plane. If the plane cuts the block at an angle, the growth pattern is helical.

SIMILAR FIGURES

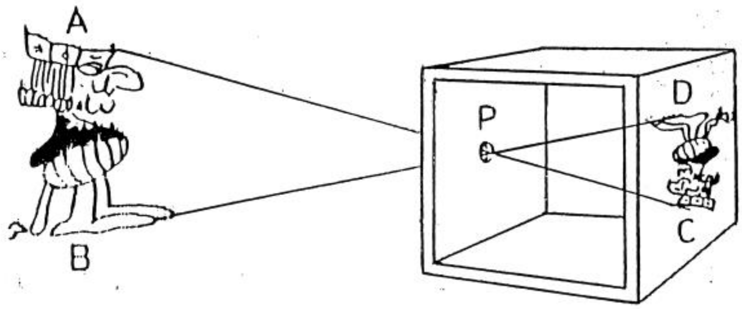
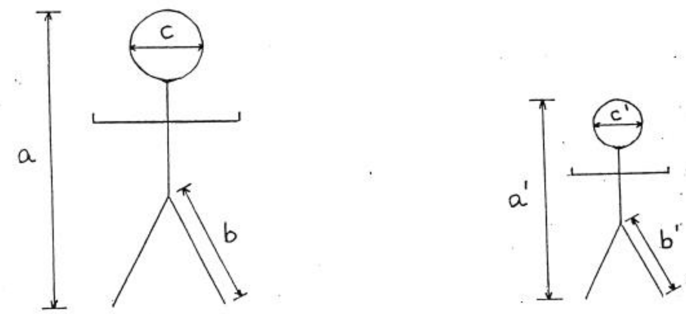


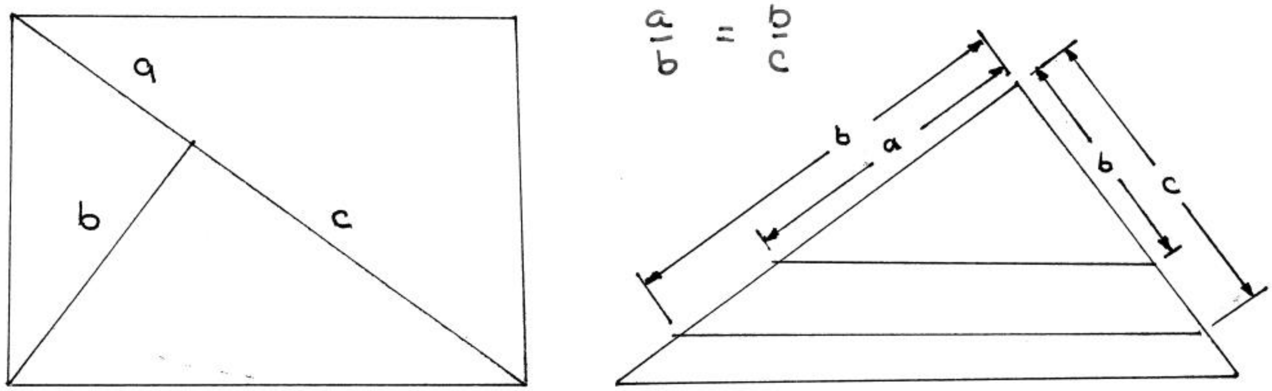
FIG. 2



$$K = \frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

K = Magnification Factor

FIG. 3



$$\frac{a}{b} = \frac{b}{c}$$

FIG. 4

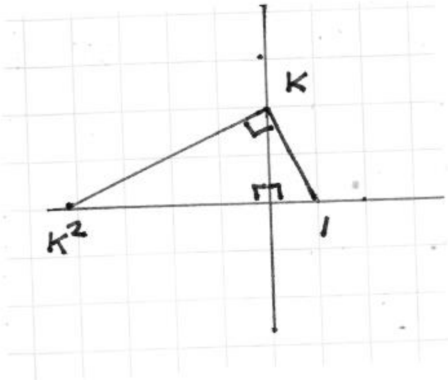


FIG. 5

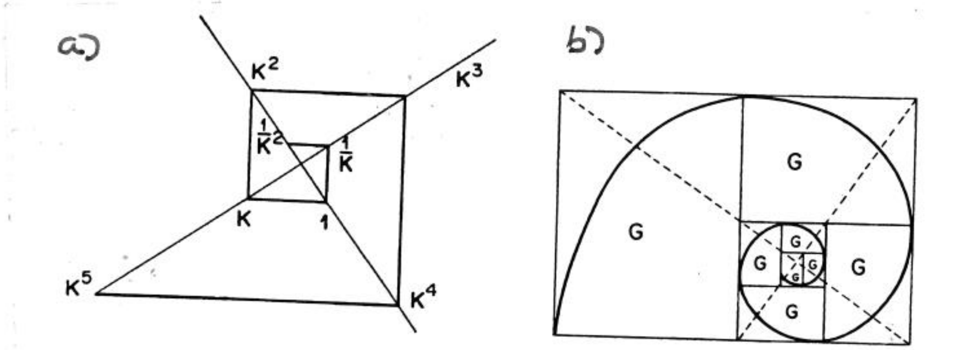


Figure 2.11 (a) Vertex points of an equiangular (logarithmic) spiral lie at a double geometric series of distances from the center; (b) a spiral is constructed from the vertex points.

FIG. 6

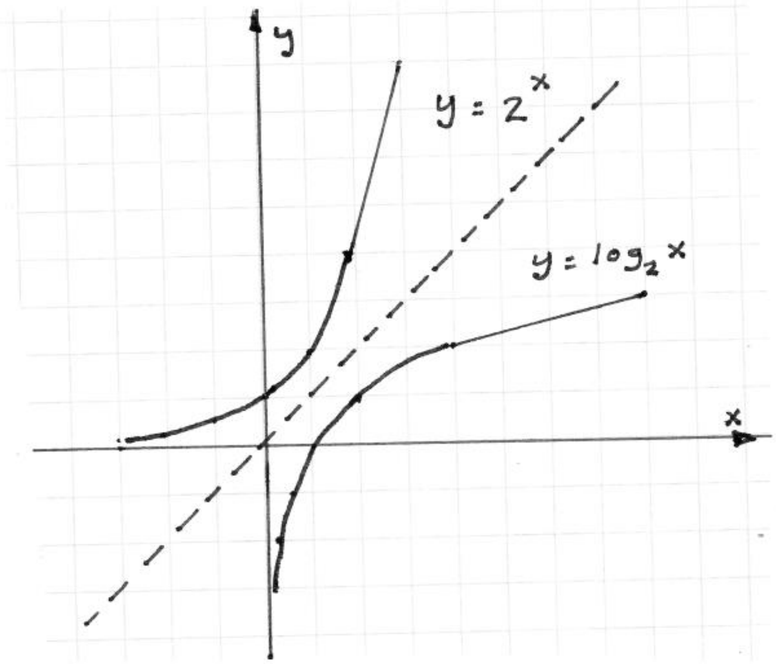
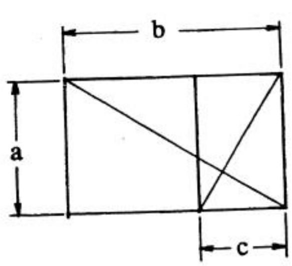


FIG. 7



$$\frac{a}{b} = \frac{b}{c}$$

FIG. 8

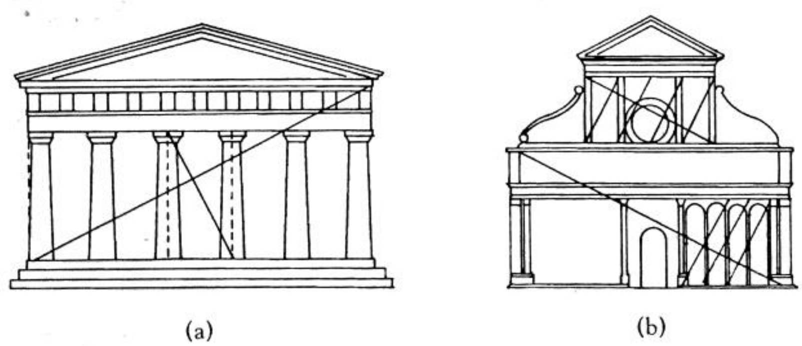


Figure 7.9 Use of the law of repetition of ratios to proportion of (a) a Greek temple, and (b) S. Maria Novella in Florence.

FIG. 9

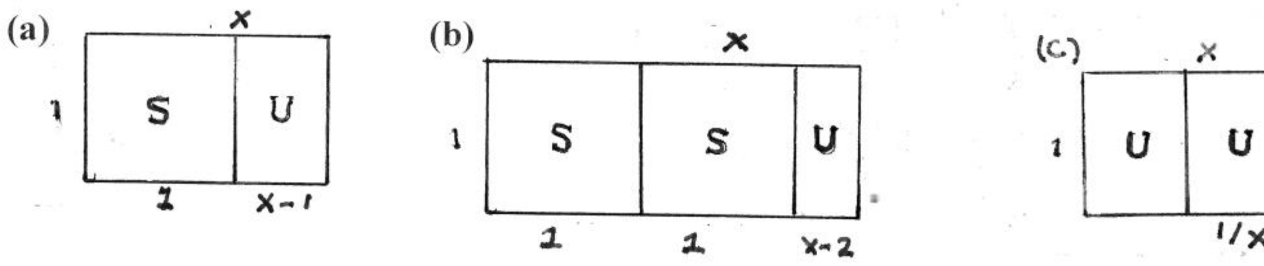
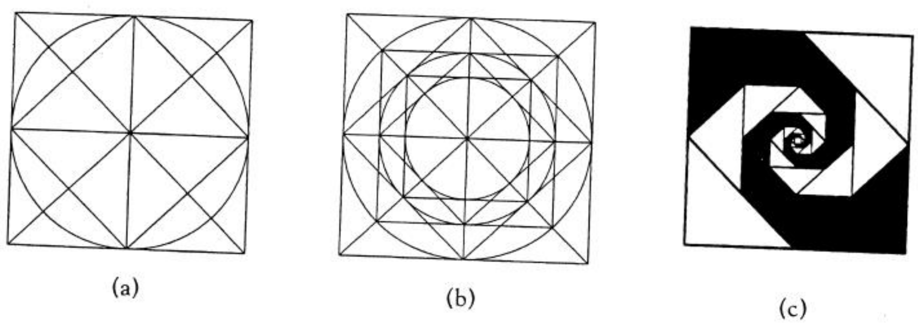


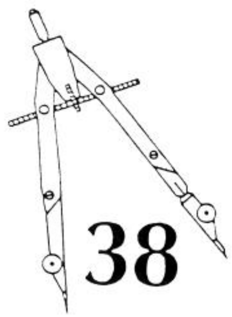
FIG. 10



(a) An *ad-quadratum* square, the area of the inside square is half the area of the outer square; (b) geometric series of ad quad squares forming a Baravelle spiral (logarithmic) (c).

FIG. 11

APPENDIX I



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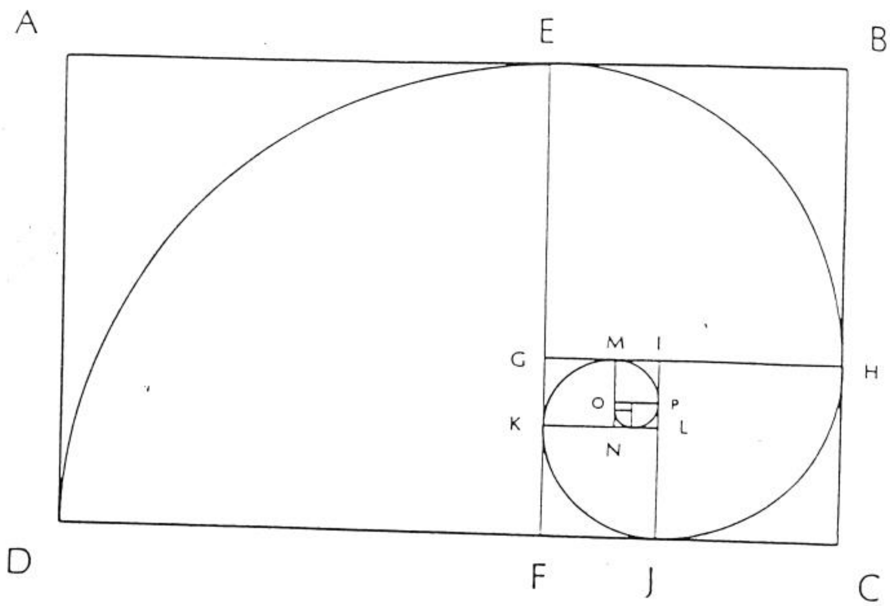


Construct a Golden Spiral Given a Golden Rectangle

Given Golden Rectangle ABCD

1. Open the compass to measure AD, place the metal tip on A and cut an arc on $\overline{AB}$.
2. Label the point of intersection E.
3. Without changing the setting, place the metal tip on D and cut an arc on $\overline{DC}$.
4. Label the point of intersection F.
5. Draw $\overline{EF}$.

Note: AEFD is a square.

Repeat Steps 1-5 in succession to get:

6. Square BHGE from rectangle BCFE.
7. Square CJIH from rectangle CFGH.
8. Square FKLJ from rectangle FGIJ.
9. Square GMNK from rectangle GILK.
10. Square MIPO from rectangle MILN.

This creates the pattern of *whirling squares*.

11. Place the metal tip on O and draw arc PM.
12. Place the metal tip on N and draw arc MK.
13. Place the metal tip on L and draw arc KJ.
14. Place the metal tip on I and draw arc JH.
15. Place the metal tip on G and draw arc HE.
16. Place the metal tip on F and draw arc ED.

Now the spiral is a Golden Spiral.

Note: the spiral may be enlarged by increasing the size of the Golden Rectangle using the method described on pages 44 through 45 and continuing to cut the arc that joins the non-consecutive vertices of each new square.

APPENDIX 2

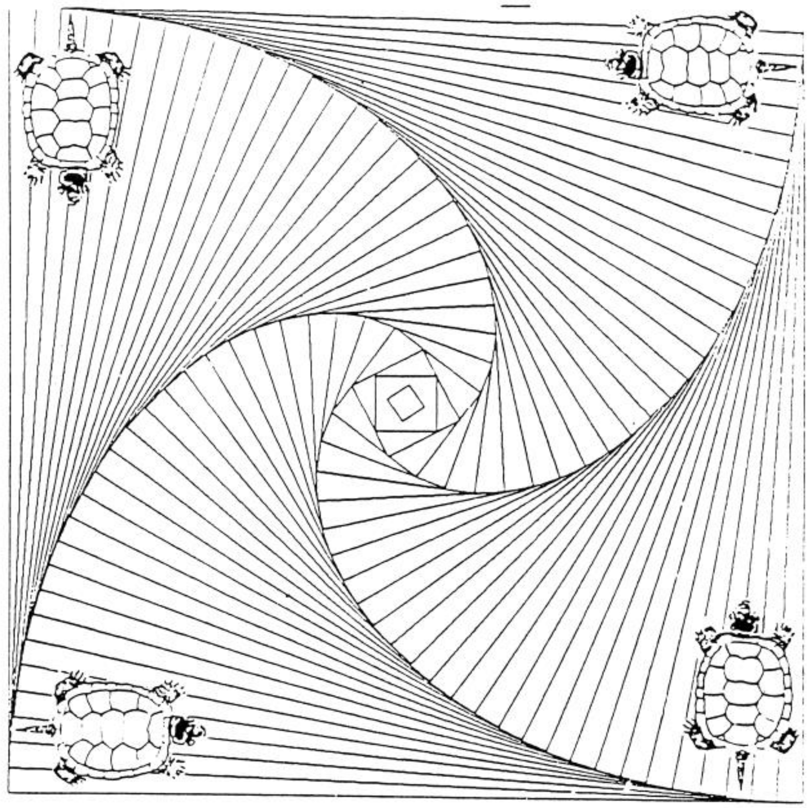


Figure 2.12 Four turtles, Abner, Bertha, Charles, and Delilah, traverse the sides of a square but are constrained to follow each other at all times. Their paths must be logarithmic spirals whose common center is the center of the square.

Problem 2.4

Tom Pizza has trained his four turtles so that Abner always crawls toward Bertha, Bertha toward Charles, Charles toward Delilah, and Delilah toward Abner. One day he put the four turtles in *ABCD* order at the four corners of a square room. He and his parents watched to see what would happen.

"Very interesting son," said Mr. Pizza. "Each turtle is crawling directly toward the turtle on its right. They all go the same speed, so at every instant they are at the corners of a square." (See Figure 2.12.)

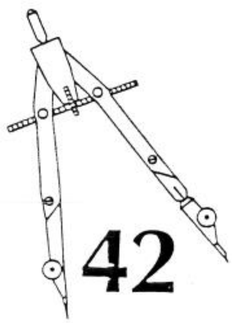
"Yes Dad" said Tom, "and the square keeps turning as it gets smaller and smaller. Look! They're meeting right at the center!"

Assume that each turtle crawls at a constant rate of 1 centimeter per second and that the square room is 3 meters on the side. How long will it take the turtles to meet at the center? Of course, we must idealize the problem by thinking of the turtles as points.

Mr. Pizza tried to solve the problem by calculus. Suddenly Mrs. Pizza shouted: "You don't need calculus, Pepperone! It's simple. The time is 5 minutes."

What was Mrs. Pizza's insight? If you cannot provide the requisite insight to solve this problem, you can always diagram the paths of the turtles in small increments of time, drawing four sides of the square at the end of each interval. The result is the pattern shown in Figure 2.12.

APPENDIX 3



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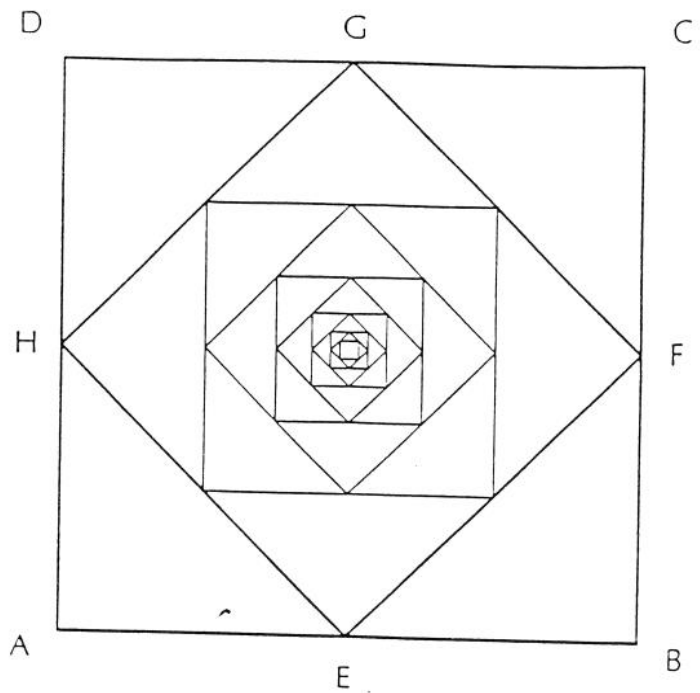
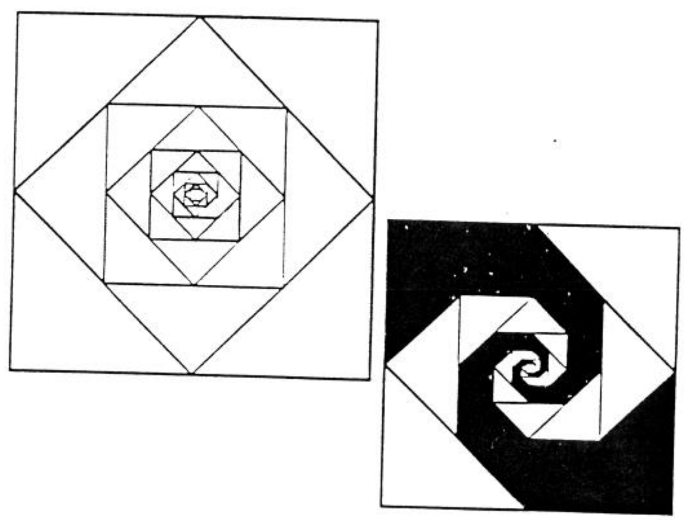
Construct a Baravelle Spiral within a Square





Given square ABCD

1. Locate the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$.

2. Label them E, F, G and H, respectively.

3. Join them with line segments to form square EFGH.

4. In the same way, join the midpoints of the sides of EFGH to form another square.

5. Repeat the process until the final square is the desired size.

6. The "spiral" shape becomes visible when the triangles are shaded as illustrated in the diagram.

Now the resulting figure is a Baravelle Spiral.