:epler's musical ratios of
nental).

Obs. Error (cents)	Comp.–Kep.
−32	32
−32	0
−32	12
0	−13
−95	−9
60	−7
12	1
46	1
−1	−3
−1	−1
−47	10
21	5
58	−5
−21	16
45	26
−14	−18

Angular Traversal Harm. Mundi (sec. of arc)	
aphel.	perihel.
106	135
270	330
1574	2281
3423	3678
5810	5857
9840	23040

Module 12

6
Tangrams and Amish Quilts

The artist is like Sunday's child; only he sees spirits.
But after he has told of their appearing to him everybody sees them.

Goethe

6.1 Introduction

While driving with my family on a vacation in Lancaster County, the home of the Pennsylvania Dutch, I began to make plans for my course on the Mathematics of Design. I wanted to find a way of linking ideas from the the history of design to the world around me.

I had just been reading *Secrets of Ancient Geometry* by Tons Brunes [Bru] in which he analyzes an enigmatic eight-pointed star that will be the subject of Chapter 8 (see Figure 8.1). He describes his theory that this star, along with the subdivision of a square by a geometrical construction that he calls the "sacred cut", formed the basis of temple construction in ancient times. To construct the *sacred cut* of one side of a square with compass and straight edge, place the compass point at a corner of the square and draw an arc through the center of the square until it cuts the side as shown in Figure 6.1. This arc cuts the side of the square to a length $\frac{1}{\sqrt{2}}$ as large. Four such cuts determine the vertices of a *regular octagon* as shown in Figure 6.2.

Kim Williams, an architect living near Florence, also described to me how she had found the system related to Brunes's sacred-cut geometry embedded in the proportions of the pavements of the baptistry of the church of San Giovanni which itself is shaped like a regular octagon [Will1].

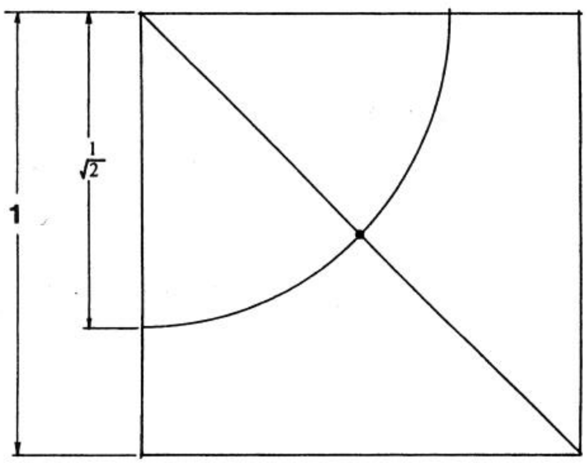


Figure 6.1 The sacred cut.

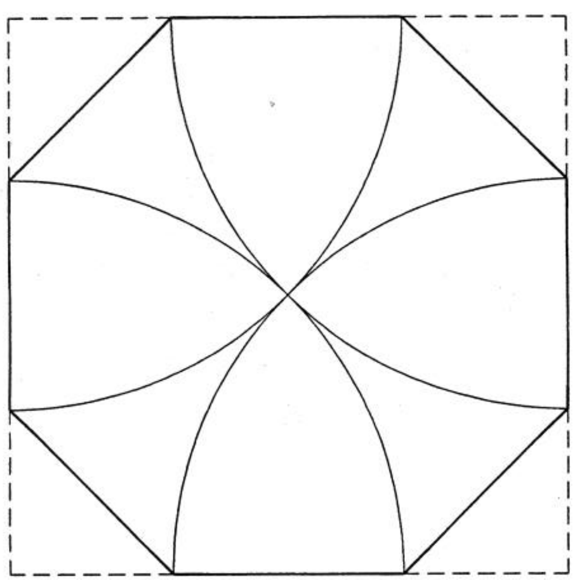
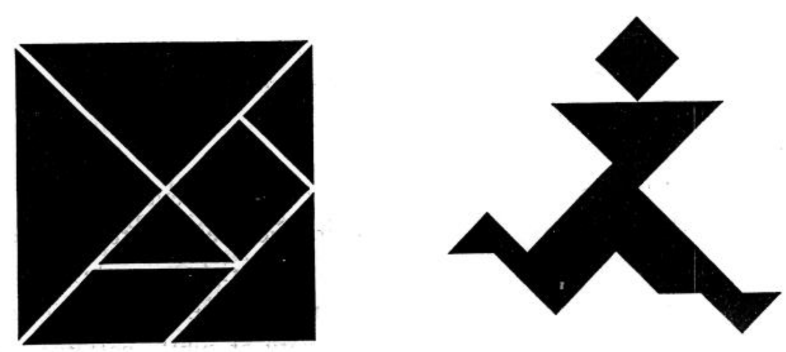


Figure 6.2 Construction of a regular octagon from four sacred cuts.

The pavements themselves had many star octagonal designs engraved in them. The star octagon, an ecclesiastical emblem, signifies resurrection. In medieval number symbolism, eight signified cosmic equilibrium and immortality.

6.2 Tangrams

Recently, I had been showing my son the fascinating tangram puzzle in which thousands of pictograms, such as the one shown in Figure 6.3a, are created from the dissection of a square into the seven pieces shown in Figure 6.3b. A tangram set can be created from a single square piece of paper by simply folding and cutting. The pieces consist of a 45-degree right triangle at three different scales along with the square and diamond formed by juxtaposing two 45-degree right triangles as shown in Figure 6.4. The side of the larger triangle is equal in length to the hypotenuse of the next smaller. Each pictogram must be formed from each of the seven pieces with no repeats and no overlaps. Enlarge the pieces, cut them out, and try your hand at constructing the pictogram shown in Figure 6.3b. Exactly 13 convex polygons (polygons with no indentations) can be constructed from the tangram set including one rectangle (other than a square) and one triangle (other than an isosceles right triangle). However, it is enough of a challenge to reconstruct the square.



(a) (b)

Figure 6.3 (a) The tangram set; (b) a pictogram constructed with the tangram set.

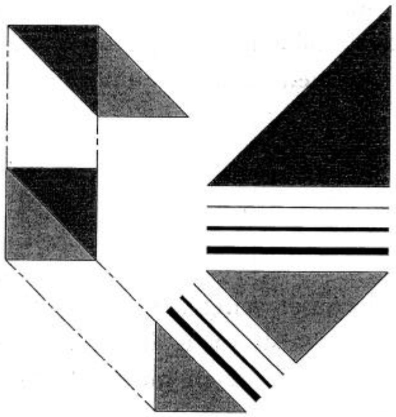


Figure 6.4 The 45 degree right triangle is the geometric basis of the tangram set.

6.3 Amish Quilts

On our vacation to Pennsylvania Dutch country we were able to explore the countryside, visit working farms, and delve briefly into the rich history of the people. The Amish and Mennonites settled in Pennsylvania during the eighteenth and nineteenth centuries as refugees from religious persecution in Germany and found a haven of freedom and rich farm lands in Lancaster county. While the Mennonites are devoutly religious and live simple lives devoid of materialistic pursuits, they do enjoy a few of the comforts of modern society. The Amish, however, attempt to insulate themselves as much as possible from outside influences and live a plain existence in which they farm without electricity, drive horsedrawn carriages, and wear unostentatious clothing. Amish women live extremely proscribed lives caring for the house and children. One of the few outlets for their creativity is the practice of quilt making [Ben].

The oldest known quilts date to about 1850. However quilting designs have changed only slightly through the years. Geometric patterns consisting of squares, star octagons, diamonds and 45-degree right triangles are used in simple designs. While the geometric patterns are the manifest content of the quilts the fabric is stitched with a variety of subtle patterns such as tulips, feathers, wreaths, pineapples, and stars.

I purchased a quilt with the design shown in Figure 6.5a. I was amazed to see that it consisted almost entirely of pieces from the tangram set.

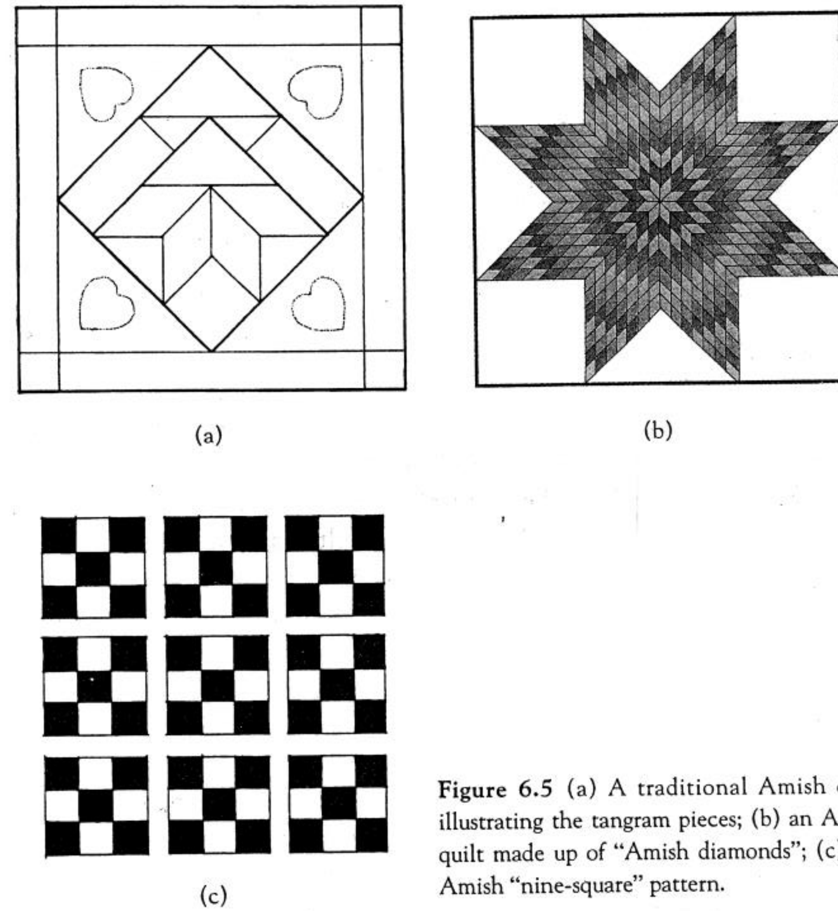


Figure 6.5 (a) A traditional Amish quilt illustrating the tangram pieces; (b) an Amish quilt made up of "Amish diamonds"; (c) the Amish "nine-square" pattern.

You can see that it has 45 degrees right triangles at three different scales, squares, and diamonds that have the same internal angles as the tangram diamond, namely 45 degrees and 135 degrees. However, the Amish quilt diamonds differ from the Tangram diamonds by having all equal edge lengths. The ratio of the diagonals of the Amish diamond is $1+\sqrt{2}:1$, an important number that will be considered in greater depth in the next chapter. This is identical to the ratio of line segments into which the sacred cut divides the edge of a square. I shall refer to these as *Amish diamonds*.

I also purchased a larger quilt which utilizes the pattern of the star octagon shown in Figure 6.5b. It is made of the tangram diamonds in rhythmically alternating colors so that it appears to be pulsating energy into a room. Finally I purchased two potholders in the basic Amish nine-square pattern (Figure 6.5c). I was soon to discover that the nine-square was intimately related to Brunes's star figure (see Section 8.3). At last I had the connections that would give unity and substance to my course.

I shall now summarize some of the geometric connections related to these personal discoveries, as I reported them to my class.

6.4 Zonogons

A regular octagon can be tiled with two squares and four Amish diamonds in two different ways, as shown in Figure 6.6 (if the tangram diamonds are used, the octagon will not be regular).

This is an example of a more general result that says an n-zonogon can be tiled by $\frac{n(n-1)}{2}$ parallelograms in two distinct ways [Kap3], e.g. when $n = 4$ by $4 \times \frac{3}{2} = 6$ parallelograms. An n-zonogon is a parallelogram with n pairs of parallel and congruent edges, i.e., the edges of its parallelogram tiling are oriented in n vector directions as shown in Figure 6.6 for the 4-zonogon with its four vector directions. The central angle of the regular octagon is represented by $\theta = \frac{360}{8} = 45$ degrees in Figure 6.7a, while the two different types of two parallelogram derived from the 4-zonogon are shown in Figure 6.7b to have angles of $1\theta, 3\theta$ and 2θ, 2θ. We are using this notation to represent the two distinct angles of a parallelogram (the other two angles are repeated). Notice that the angles add up to 4θ, or 180 degrees, whereas the angles surrounding each vertex in Figure 6.6 sum to 8θ [Lal3]. This can easily be generalized to n-zonogons and their derived parallelograms [Kap3].

A key property of n-zonogons is that their edges line up in a series of n sets of parallel edges or *zones*. The edges of each zone are oriented in the direction of one of the n vectors that define the zonogon. You can observe this in Figure 6.6 for the 4-zonogon. If the length of one of the vectors is shrunk to zero, then one of the zones is eliminated and the n-zonogon collapses to a $(n-1)$-zonogon. Alternatively, each of the n vectors can be expanded or contracted, with the effect that the shape of the zonogon is distorted without altering the internal angles of its parallelograms.

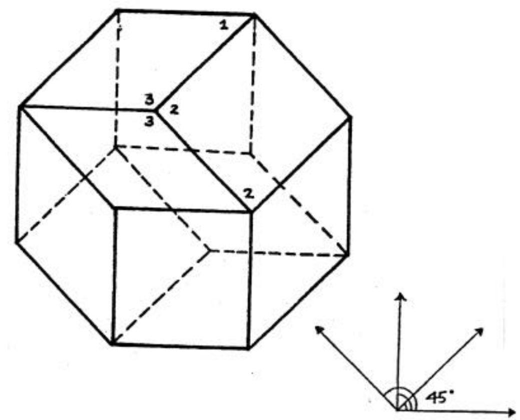


Figure 6.6 A regular octagon tiled with two squares and four Amish diamonds in two ways.

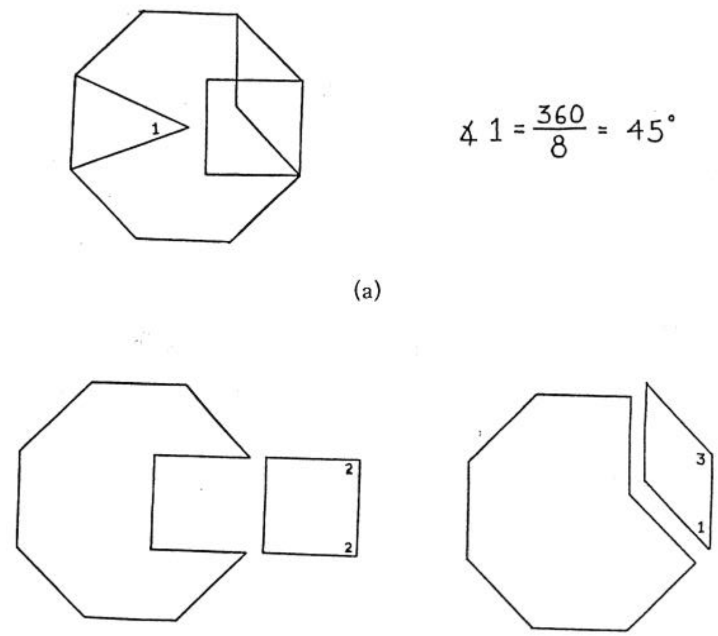


Figure 6.7 (a) The parallelograms defined by a 4-zonogon; (b) the two angles of the parallelograms add to 4.

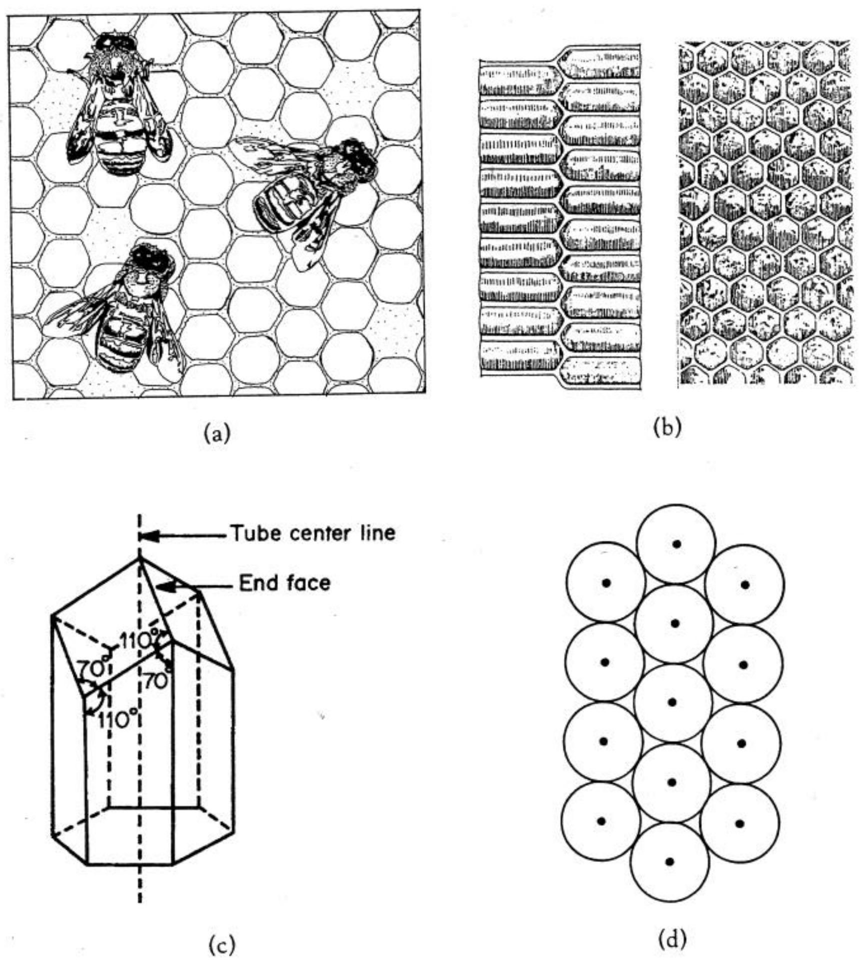


Figure 6.8 The rhombic dodecahedron as the structure of the beehive.

6.5 Zonohedra

The two sets of parallelograms that tile the 4-zonogon in Figure 6.6 can be seen to be a schematic drawing of a twelve-faced, space-filling polyhedron known as a *rhombic dodecahedron* (RD) [Kap3]. This polyhedron is representative of a class of polyhedra with opposite faces parallel and congruent known as *zonohedra*. If the two sets of three-connected edges are

removed from this figure it is easy to see that it represents an *hexagonal prism*. In fact if all the faces of the RD are taken to be rhombuses with diagonals in the ratio $\sqrt{2}:1$ the polyhedron that results is precisely the form that caps the hexagonal prisms that make up the structure of the beehive (see Figure 6.8). It also represents the configuration of the garnet crystal. Here are the links to the natural world that I was looking for. Similar to zonogons, n-zonohedra are defined by n *zones* of parallel edges. Likewise, if the zonohedron is made of linear rods, one zone of parallel rods can be eliminated and the sticks reconnected, the result being a zonohedron of one order less [Kap3]. This concept was cleverly used by Steve Baer [Boo1], [Bae] to create zonogon shaped houses which could be easily renovated by changing their size and shape in a manner forbidden by geodesic domes. Change a single edge of a geodesic dome and all edges must change size accordingly. However, transformations of zonohedra can be limited to one zone at a time.

6.6 N-Dimensional Cubes

Just as the n-zonogon can be subdivided into parallelograms, an n-zonohedron can be subdivided into two interlocking sets of,

$$C(n,3) = \frac{n(n-1)(n-2)}{6} \qquad (6.1)$$

parallelopipeds where $C(n,3)$ is the number of ways one can choose three objects from a group of n where order is not important. If this is done, then n edges are incident at each vertex giving a projection of an n-dimensional cube in 2 or 3 dimensions. But what do we mean by an n-dimensional cube?

Let's consider a 4-dimensional cube, or tesseract as it is called, the boundary of which, in one of its two-dimensional projections, is a 4-zonogon. We see it pictured in Figure 6.9 as the fifth in a series of 0,1,2,3, and 4-dimensional cubes. The 0-dimensional cube (see Figure 6.9a) is a point with no degrees of freedom. The surface of a 1-dimensional cube (line segment) is gotten by translating the 0-dimensional cube (point) parallel to itself (see Figure 6.9b). One has freedom to move left or right along the line. The surface of a 2-dimensional cube (see Figure 6.9c) is gotten by

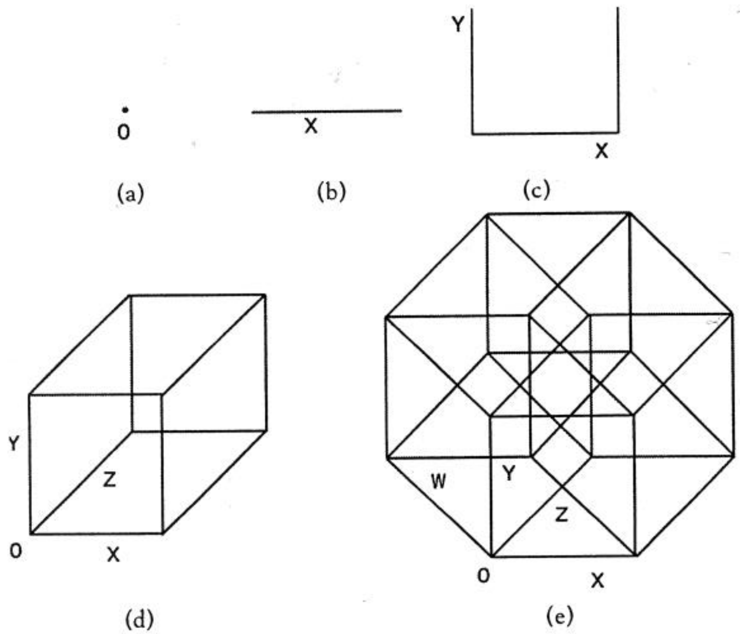


Figure 6.9 Diagrams of 0,1,2,3, and 4 dimensional cubes.

translating the line segment parallel to itself to obtain a square. Movement is possible on the surface of the square: left–right or up–down. A 3-dimensional cube (see Figure 6.9d) is obtained by translating a square parallel to itself, resulting in a surface with freedom of movement: left–right, up–down, in–out. Finally, the 4-dimensional cube (see Figure 6.9e) is obtained by translating the 3-dimensional cube parallel to itself. You can see that now 4 degrees of freedom are possible: left–right (x), up–down (y), in–out (z), and movement in the elusive fourth direction (w). Of course Figure 6.9e is only the projective image of a 4-dimensional cube the same way that Figure 6.9c is only a projection of a 3-dimensional cube. In an actual 4-dimensional cube there would be no intersecting lines, planes, or volumes just as a 3-dimensional cube has no crossing edges despite the crossing edges that appear in its 2-dimensional projection.

As predicted by Equation (6.1), the tesseract has two sets of 4 intersecting cells projected into the 4-zonogon. Notice the star octagon in Figure 6.9e, reminiscent of my Amish quilt.

6.7 Triangular Grids in Design: An Islamic Quilt Pattern

A 3-zonogon is shown in Figure 6.10. The two sets of 3 parallelograms that tile the hexagon can be seen to be an ordinary cube in perspective. The hexagon is also subdivided into a triangular grid. This triangular grid is useful as a design tool.

In Figure 6.11b we see a triangular grid developed from a family of closely packed circles shown in Figure 6.11a and created as shown in Appendix 6.A. Repeating patterns can be created by deleting lines from Figure 6.11. Two examples are shown in Figure 6.12, and additional designs recreated from a square grid of circles are shown in Appendix 6.A.

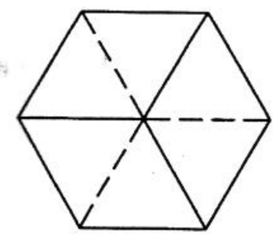
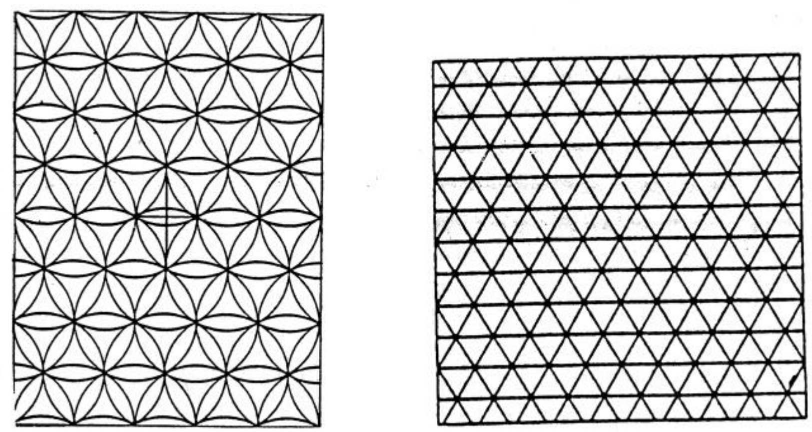
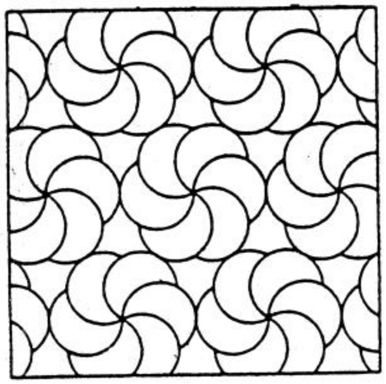


Figure 6.10 A 3-zonogon viewed as either a 3-dimensional cube or as a triangular grid.



(a) (b)

Figure 6.11 (a) A triangular grid of closely-packed circles; (b) a triangular grid.



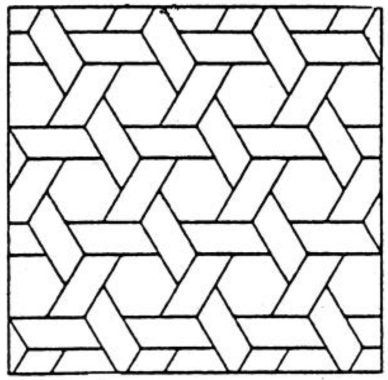


Figure 6.12 Two patterns conforming to (a) the grid of close packed circles; (b) the triangular grid.

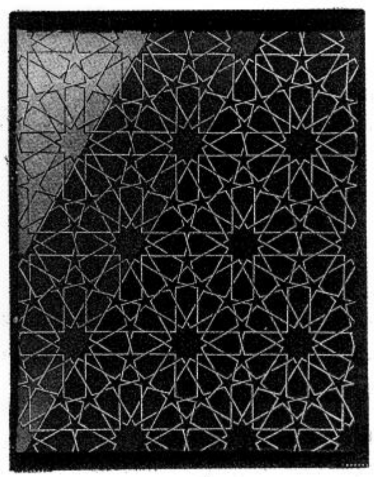


Figure 6.13 "Cairo Quilt" by Margit Echols© 1994, Cotton, 90" × 110", machine pieced, hand quilted.

Margit Echols [Ech] has developed geometrical principles suited to the particular requirements of the art of quilting. One of her quilts is based on an Islamic pattern generated from the triangular grids of Figure 6.11a and b. Her quilt pattern, illustrated in Figure 6.13 contains twelve pointed stars. Three pairs of bounding edges of the star, when extended, traverse the

entire pattern and form a triangular grid. Notice that pentagonal star-like figures make a surprise appearance in the final design.

Echols has the following to say about the art of quilt making:

> "The quiltmaker is faced with tremendous restrictions inherent in both the laws of geometry and the technology of patchwork. How is it we can bear the time it takes to make a quilt? — Besides the obvious rewards of accomplishing technical challenges, of making colors sing, the tactile sensuality of textiles, and the meditative quality of repetitive handwork — there is the pleasure of problem solving of putting the puzzle together, of playing the game, a serious game of a battle against chaos which has deep intellectual appeal."

6.8 Other Zonogons

For a 5-zonogon, the central angle is $\theta = \frac{360}{10} = 36$ degree and the two species of parallelogram are 1θ, 4θ and 2θ, 3θ (adding up to 5θ). These parallelograms have interesting properties since the ratio of their edge length to one of their diagonals is related to the golden mean, a number whose value is $\tau = \frac{1+\sqrt{5}}{2}$. These parallelograms will be discussed further in Section 20.4 and will arise in Section 25.2 in the context of quasicrystals. Designs with these parallelograms, such as the one in Figure 6.14, have approximate five-fold symmetry.

The design possibilities are all the richer for tiling a 6-zonogon. Tiling the 6-zonogon by its parallelograms, 1θ, 5θ; 2θ, 4θ; and 3θ, 3θ where $\theta = \frac{360}{12} = 30$ degrees, results in perspective diagrams of the *rhombic triacontahedron* (30 parallelogram faces) and the *truncated octahedron* (with 6 square and 8 hexagon faces) shown in Figure 6.15. By successively removing zones the 6-zonohedron (rhombic triacontahedron) collapses to a 5-zonohedron (rhombic icosahedron), then to a 4-zonohedron (rhombic dodecahedron), and finally to a 3-zonohedron (parallelopiped). In the last phase of this transformation there are two possible parallelopipeds, type 1 and type 2, that are the building blocks for all the other zonohedra derived from the 6-zonohedron, much as parallelograms are building blocks for zonogons

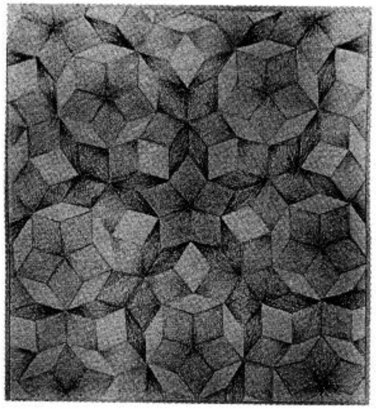
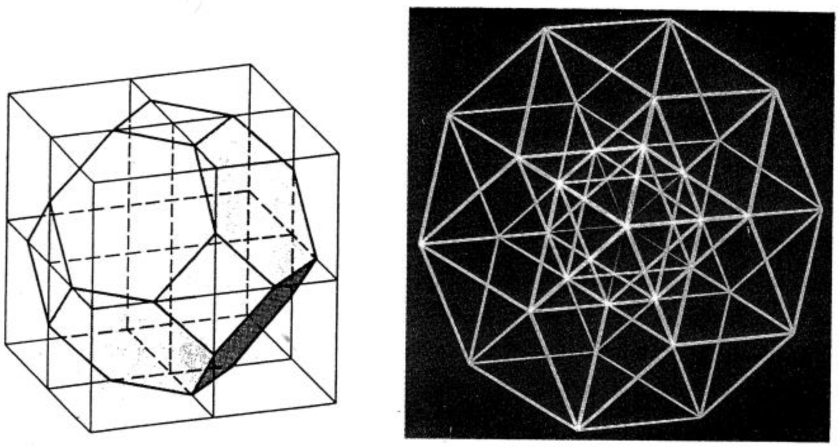


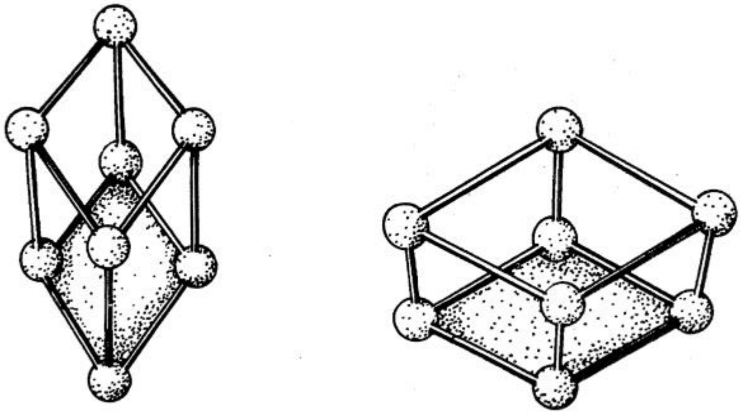
Figure 6.14 A pattern with approximate five-fold symmetry made up of the two parallelograms of the 5-zonogon from the Mathematics of Design class of Jay Kappraff.



(a) (b)

Figure 6.15 (a) The truncated octahedron; (b) the rhombic triacontahedron.

(see Figure 6.16). All faces of this family of zonohedra are congruent rhombuses and have diagonals in the ratio, $\tau : 1$ and for this reason they are called *golden iso-zonohedra* [Miyazaki 1980]. Each zonohedron can be tiled by the number of parallelopipeds given by Equation (6.1). For example, the rhombic dodecahedron with $n = 4$ is tiled by 4 parallelopipeds, 2 of type 1 and 2 of type 2 as shown in Figure 6.17. The rhombic triacontahedron,



(a) (b)

Figure 6.16 A golden parallelopiped of type 1 and type 2.

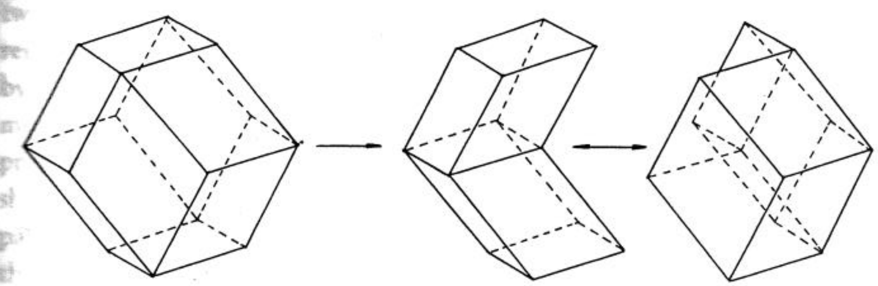


Figure 6.17 A rhombic dodecahedron tiled by by two golden parallelopipeds of type 1 and two parallelopipeds of type 2.

shown in Figure 6.18, with $n = 6$, is tiled by 20 parallelopipeds, 10 of type 1 and 10 of type 2.

The 6-zonogon can also be viewed as a distorted 2-dimensional projection of a 6-dimensional cube, and as for the 4-zonogon, it too has a star dodecagon (12 pointed star) at its center (see Figure 6.19). We also encountered this star in Figure 3.4c in connection with tone cycles of musical thirds, fourths, fifths, and wholetones. The cover of *Connections* [Kap3] shows the extraordinary result of truncating a 6-dimensional cube at one of its vertices.

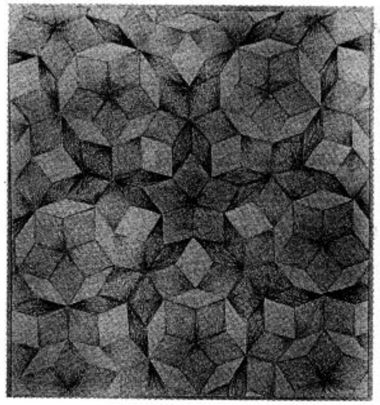
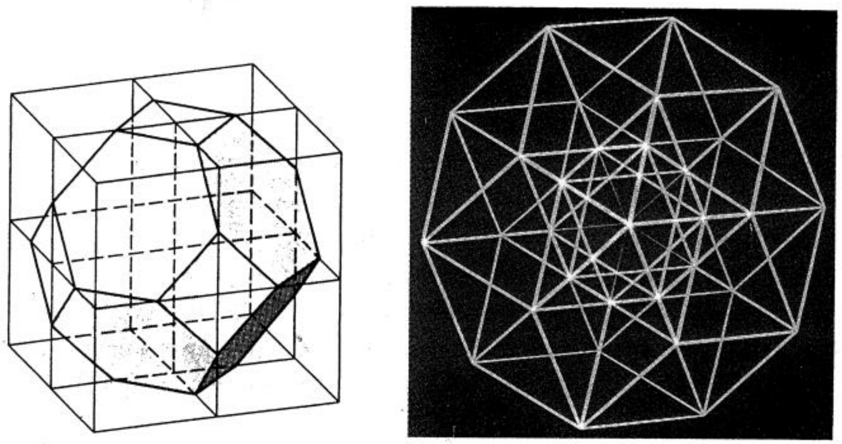


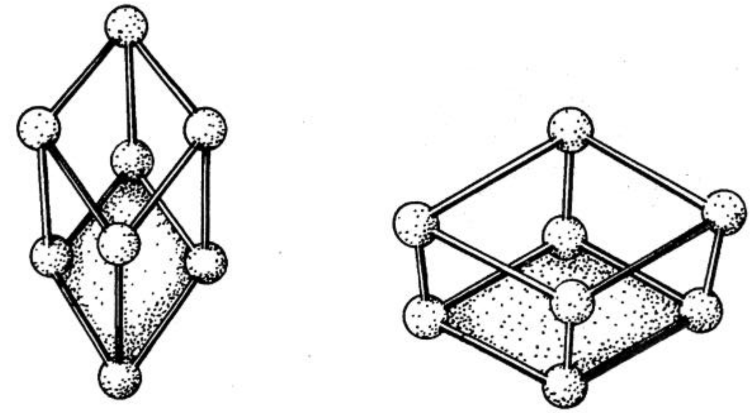
Figure 6.14 A pattern with approximate five-fold symmetry made up of the two parallelograms of the 5-zonogon from the Mathematics of Design class of Jay Kappraff.



(a) (b)

Figure 6.15 (a) The truncated octahedron; (b) the rhombic triacontahedron.

(see Figure 6.16). All faces of this family of zonohedra are congruent rhombuses and have diagonals in the ratio, $\tau : 1$ and for this reason they are called *golden iso-zonohedra* [Miyazaki 1980]. Each zonohedron can be tiled by the number of parallelopipeds given by Equation (6.1). For example, the rhombic dodecahedron with $n = 4$ is tiled by 4 parallelopipeds, 2 of type 1 and 2 of type 2 as shown in Figure 6.17. The rhombic triacontahedron,



(a) (b)

Figure 6.16 A golden parallelopiped of type 1 and type 2.

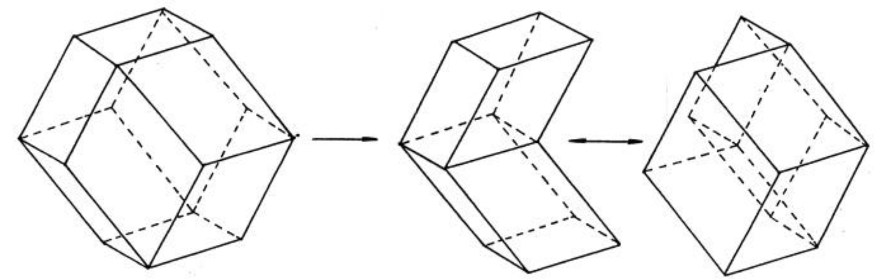


Figure 6.17 A rhombic dodecahedron tiled by by two golden parallelopipeds of type 1 and two parallelopipeds of type 2.

shown in Figure 6.18, with $n = 6$, is tiled by 20 parallelopipeds, 10 of type 1 and 10 of type 2.

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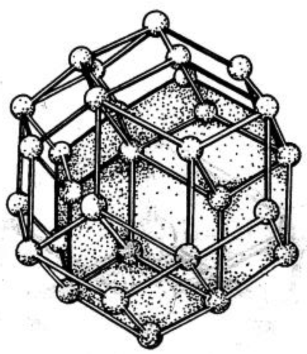

Figure 6.18 A rhombic triacontahedron tiled by golden parallelopipeds.

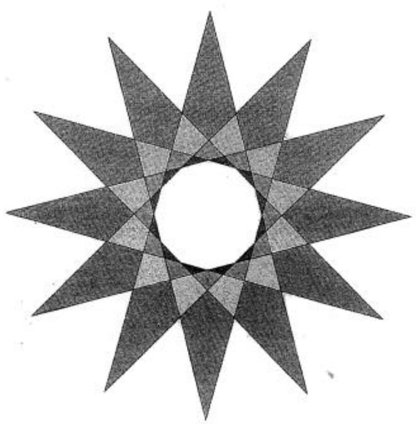

Figure 6.19 A star dodecagon.

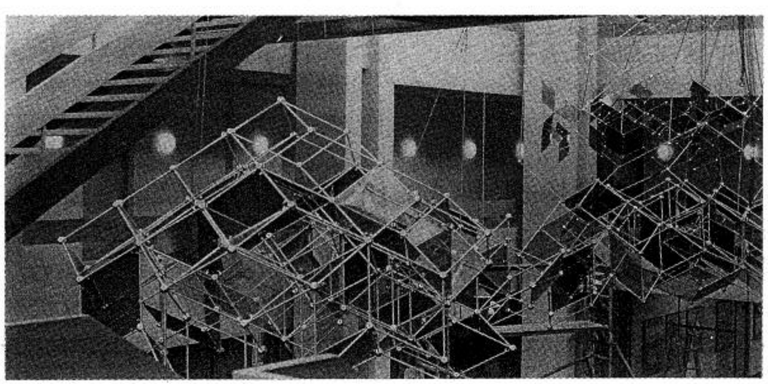

Figure 6.20 A 60-foot long rhombic triacontahedron sculpture with a quasicrystal interior at Denmark's Technical University in Copenhagen by Tony Robbin.

Alan Schoen [Schoe] has created a puzzle called Rombix in which multicolored tiles which are composites of 8-zonogons are used to create a prescribed set of designs. The artist Tony Robbin [Rob] has built a 60-foot sculpture, shown in Figure 6.20, based on quasicrystal geometry for the three story atrium at Denmark's Technical University in Copenhagen. The dome is a rhombic triacontahedron with a quasicrystal interior.

6.9 Conclusion

The concept of the zonogon is the key to understanding the system of design used by the Amish. It leads to a means of visualizing the two-dimensional projection of an important class of three-dimensional polyhedra known as zonohedra, and it places an understanding of the nature of three-dimensional geometry firmly in the context of higher-dimensional geometry. The 4- and 5-zonogons define systems with a repertoire of two parallelograms, the first related to $\sqrt{2}$ and the sacred cut, the second related to the golden mean. A system of architectural proportions developed by the Le Corbusier, known as the Modulor, is based on the golden mean [Kap3]. In the next chapter we shall explore the $\sqrt{2}$ system of proportions in greater depth. We shall also see that these two systems share a unifying structure with roots in the musical scale. The number of parallelograms proliferate for zonogons of a higher order which inhibits their usefulness to serve as systems of proportion.

My visit to the Amish country, examination of the quiltwork of Margit Echols, and the structures of Tony Robbin have reinforced my feeling that artists, and practitioners of the folk arts have infused their work with patterns that share themes of common interest to mathematicians and scientists.

Appendix 6.A

6.A1 *Steps to Creating a Triangular Grid of Circles*

1. Begin with a point at the center of a circle of arbitrary radius.
2. From an arbitrary point on the circumference of this circle draw another circle of the same radius through the center of the first circle