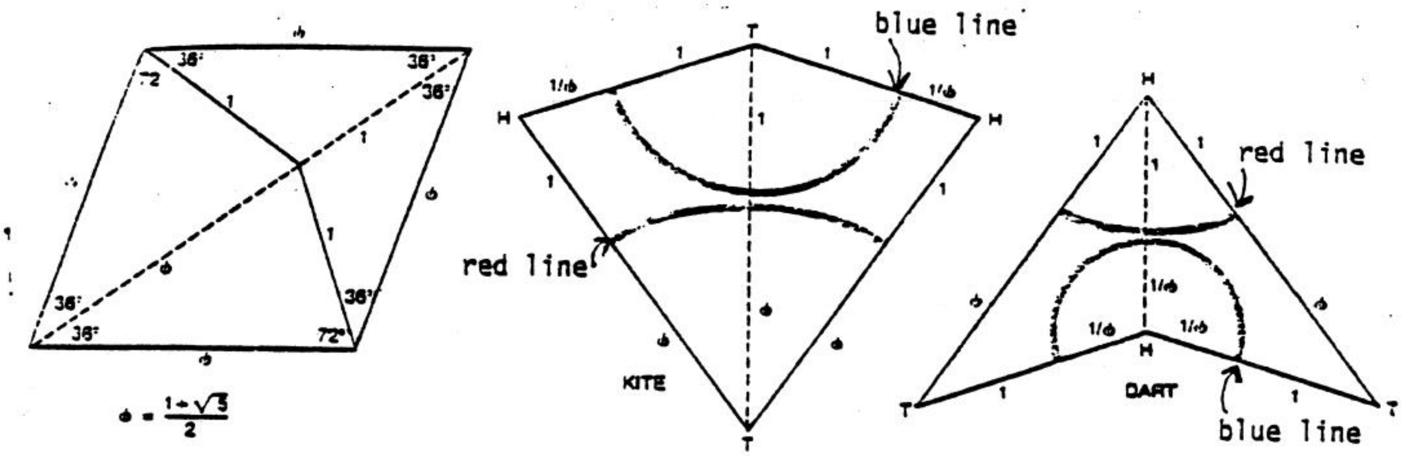
Module 13: Non-periodic Tilings

3.10 Application of the Golden Mean to non-periodic tilings of the plane

We will conclude this chapter by describing a strikingly complex and beautiful way in which the two geometric forms based on the golden mean, known as the kite and the dart and depicted in Figure 3.28 , can be used to tile the plane non-periodically.



Construction of dart and kite

A coloring of dart and kite to force nonperiodicity

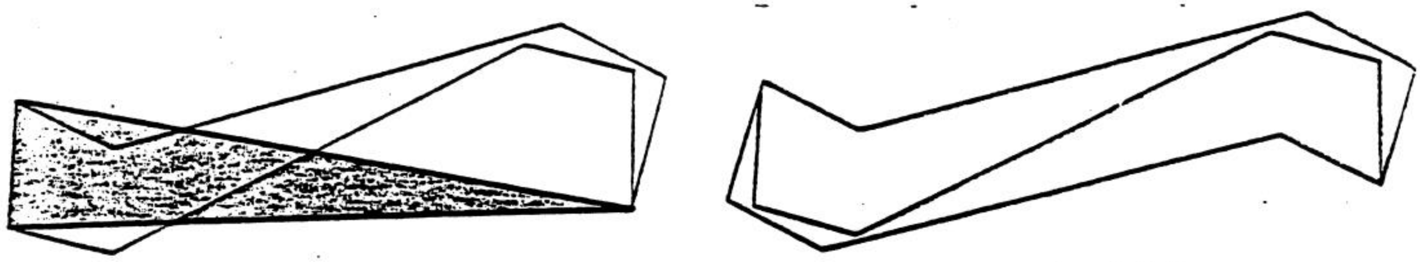
Figure 3.28

A periodic tiling is one in which the entire configuration can be translated (without rotation) to a new position which reproduces the original tiling. We say then that the tiling is invariant under translation. Escher has immortalized these tilings, as shown for example in Figure 3.29. It was until recently thought that any set of forms that tile the plane non-periodically can also tile the plane periodically. For example, the polygonal forms called enneagons shown in Figure 3.30 tile the plane both periodically and non-periodically. (Why?)

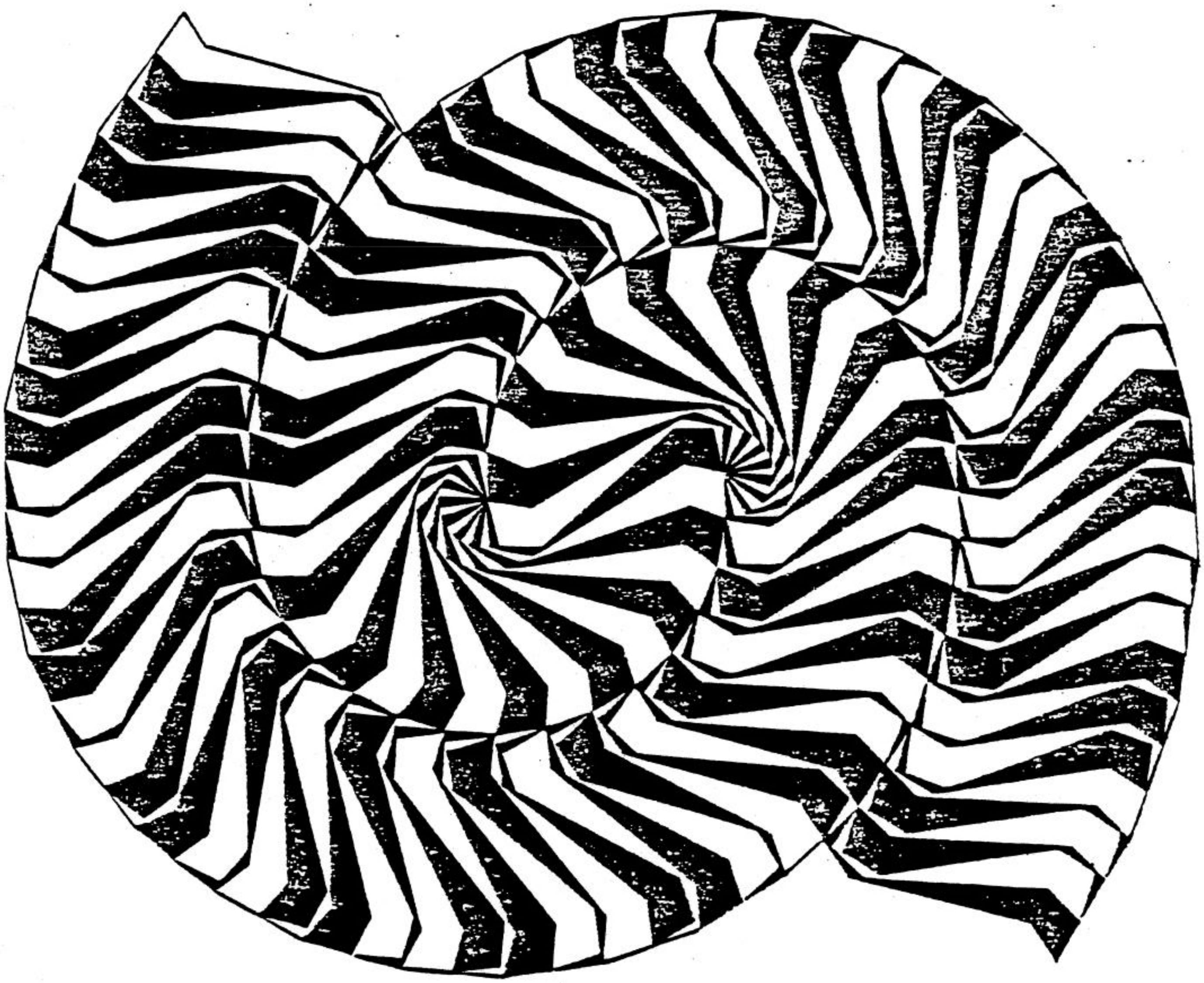
It was therefore a great interest that met Robert Berger's discovery in 1964 that there exists a non-periodic tiling of the plane for which there is no periodic tiling. However, to carry out this tiling Berger needed 20,000 dominoes.

This enables us to better appreciate L.S. Penrose's discovery of two pieces based on the golden mean called a kite and a dart shown in Figure 3.28 which also tile the plane non-periodically but not periodically, if during the tiling of the plane the blue curve drawn on the kite and the dart are forced to meet only the blue curve of another kite or dart to form a continuous curve that winds through the tiling. The same holds for matching the tiles so as to insure a continuous red curve wafting through the tiling.

Most noteworthy of these tilings is the approximate pentagonal symmetry, as shown in Figure 3.31 and 3.32. They are approximate in the sense that they seem to always be striving to reproduce themselves but never quite succeeding. Wherever we look, we see a configuration that looks familiar in the sense that we have seen something just like it in one or another of the tilings. We can make this statement even more precise by stating a very remarkable theorem due to Conway. In more colloquial language the



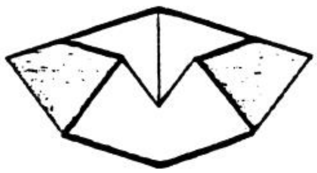
An ennagon (color at left) and a pair of ennagons (right) forming an octagon that tiles periodically



A spiral tiling by Heinz Voderberg

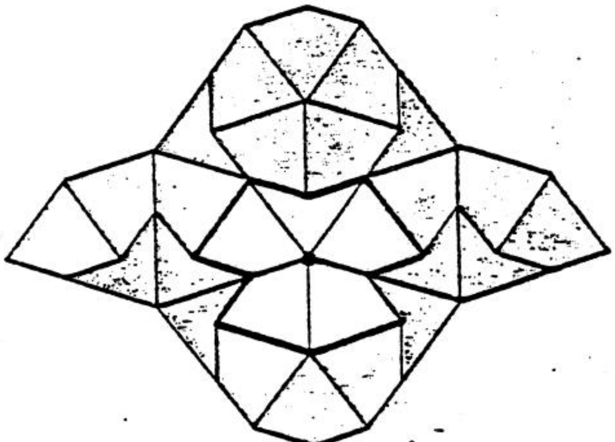
Nonperiodic tiling with congruent shapes

Figure 3.30



DEUCE

QUEEN



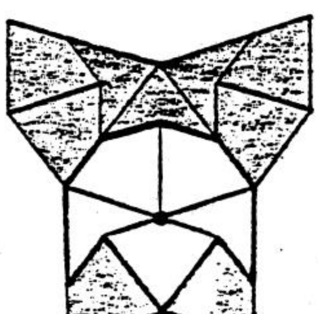


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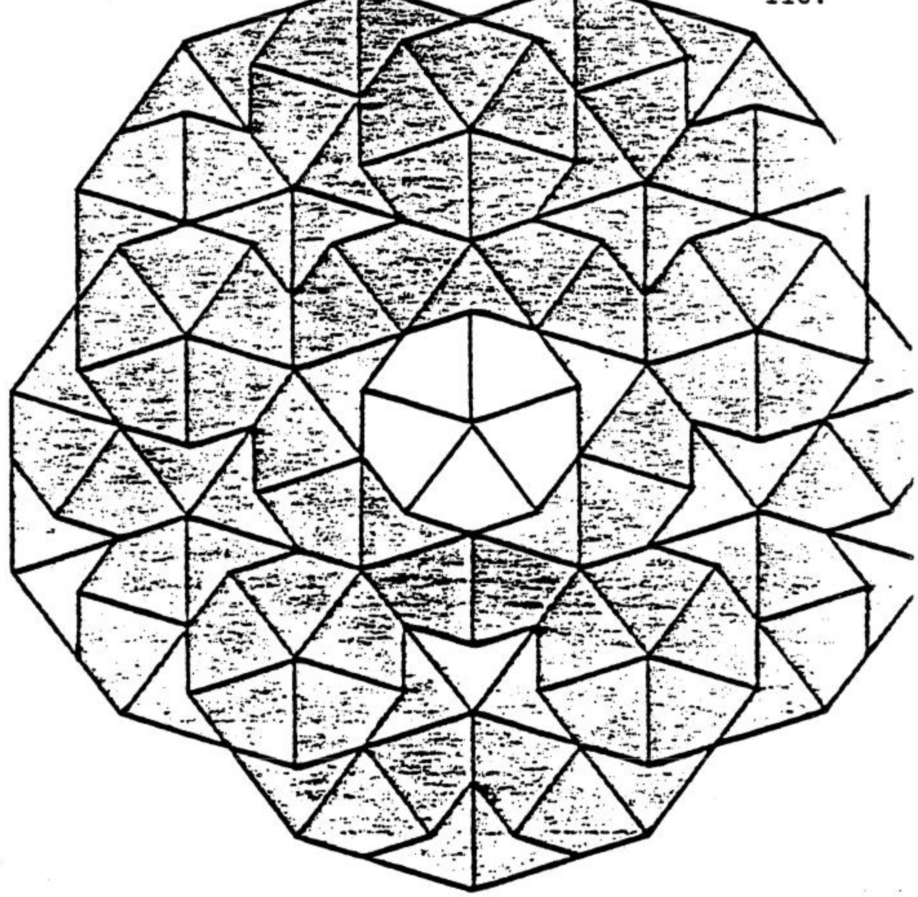
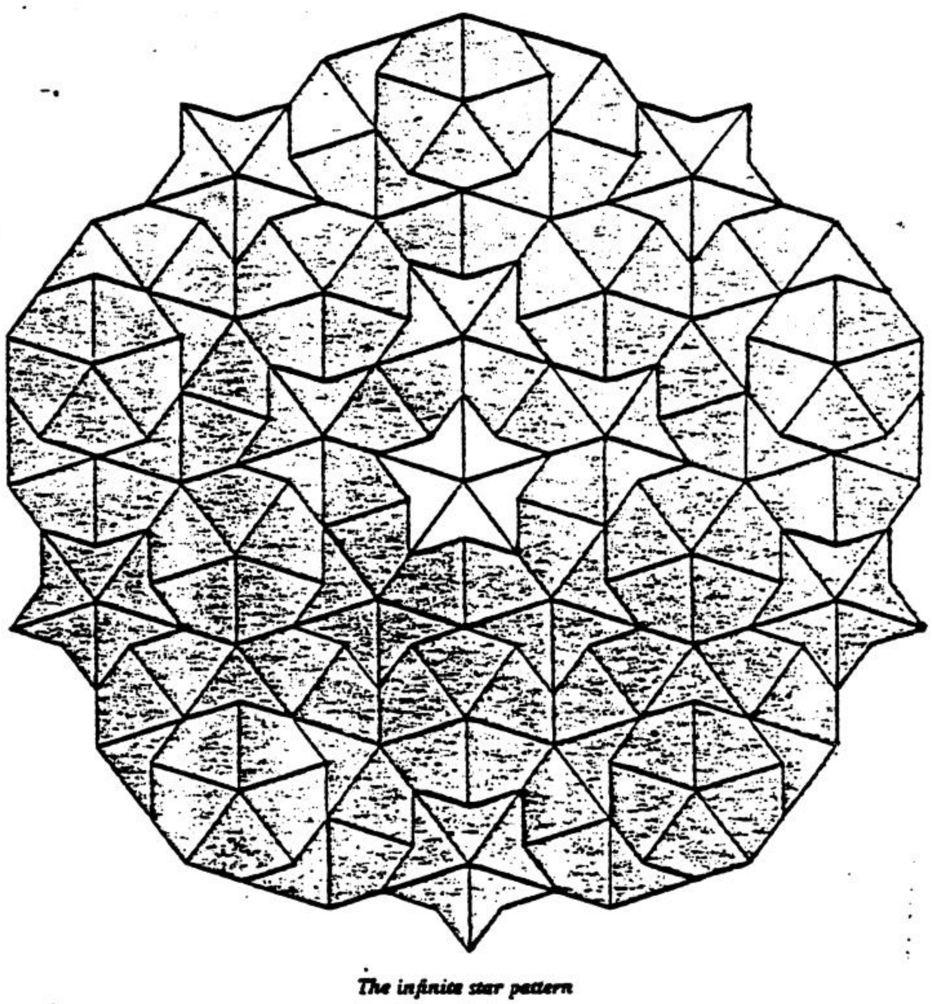
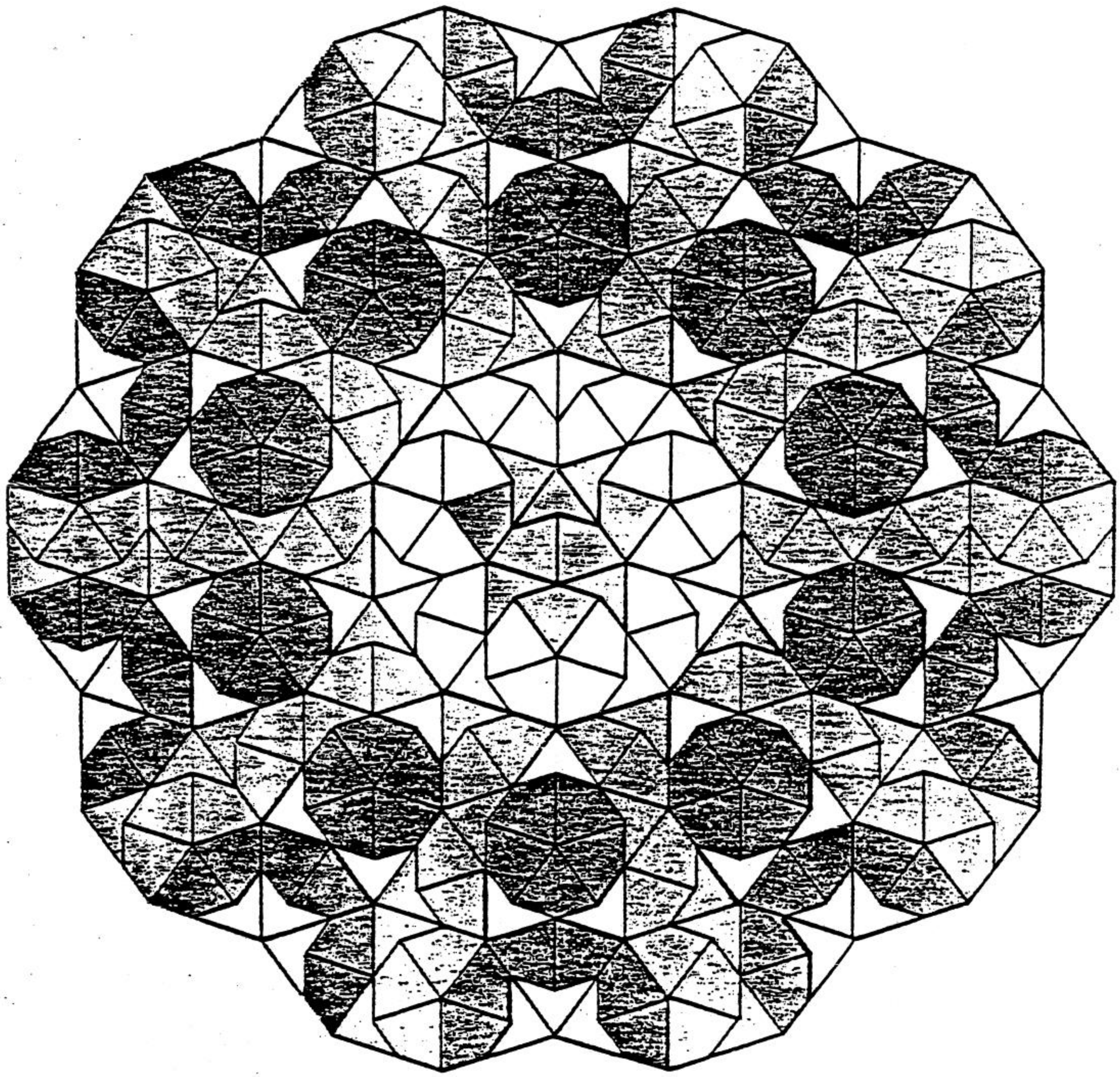
Figure 3.31 Some starting positions for Penrose tilings



The infinite sun pattern



The infinite star pattern



The cartwheel pattern

Figure 3.31 The Batman starting position theorem can be described as follows. Let's say that you are residing in a finite region of diameter d, of a "Penrose tiling" (or universe). Let's call this finite region your town. If you are suddenly transported to another universe (a different tiling) how far must you wander to find an exact replica of your town? Conway proved that you need not wander more than a distance of 2d from your new position, although the exact distance is unpredictable!

Once again the golden mean has resulted in a set of tiles that fit together (to fill the infinite plane) in a repetitious way (i.e., Conway's Theorem) but non-monotonously (since no two tilings are alike outside a finite region). Finally we illustrate the mysterious pentagram of Pythagoras once again in Figure 3.33 and ask you to locate the kite and dart hidden in this figure.

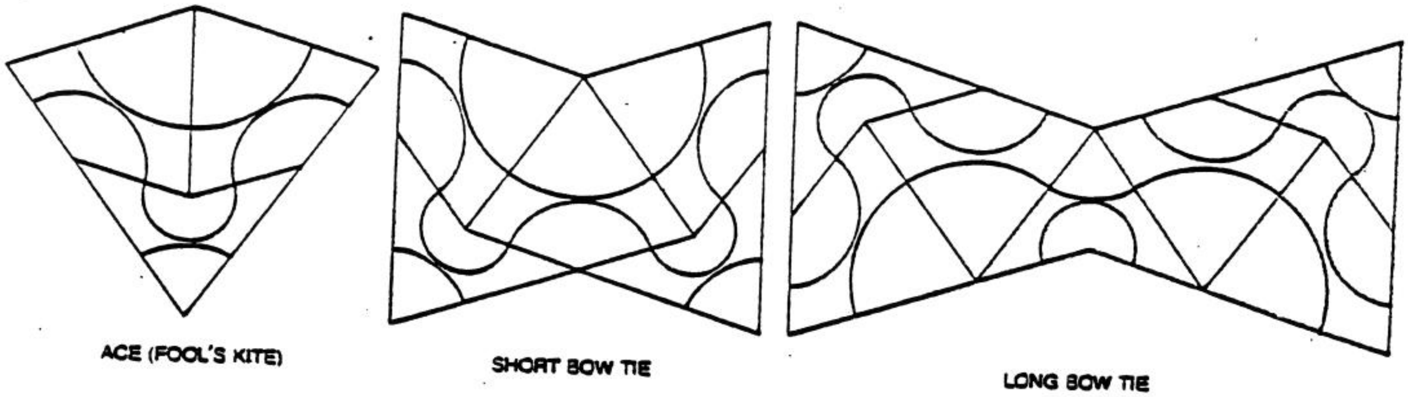
Construction 3.3:

Construct a pattern of at least fifty kites and darts. The following discussion by Martin Gardner in Scientific American [ ] may be helpful.

To appreciate the full beauty and mystery of Penrose tiling one should make at least 100 kites and 60 darts. The pieces need be colored on one side only. The areas of the two shapes are in the golden ratio. This proportion also applies to the number of pieces you need of each type. You might think you need more of the smaller darts, but it is the other way around. You need 1.618... as many kites as darts. In an infinite tiling this proportion is exact.

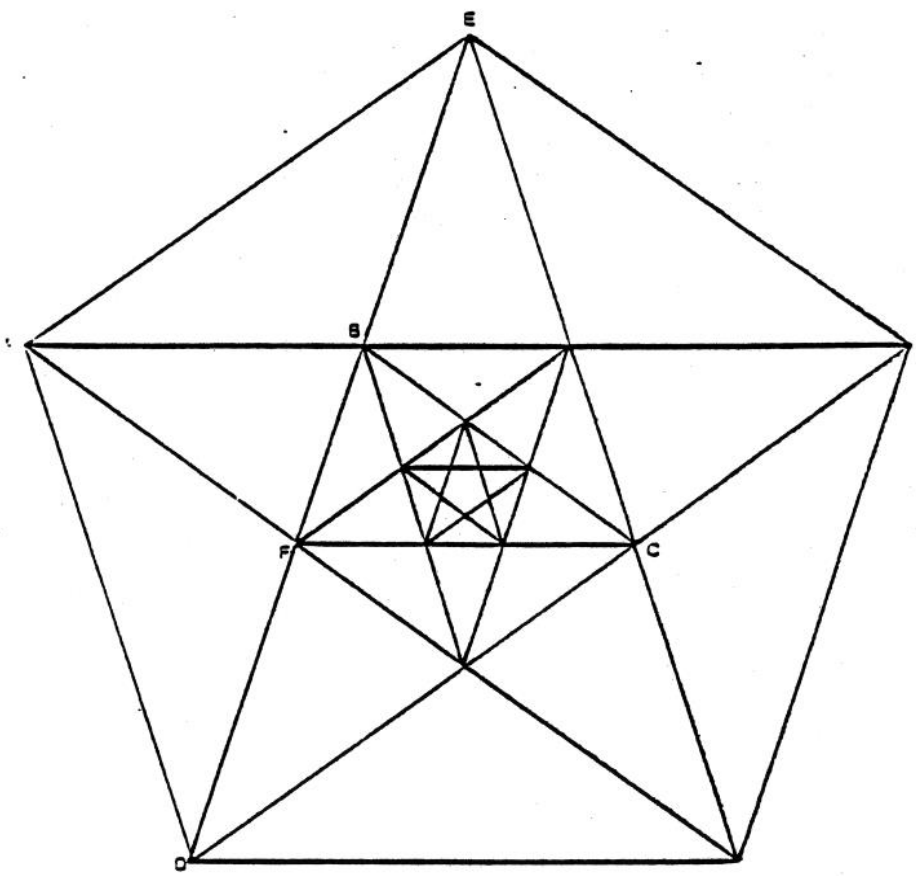
A good plan is to draw as many darts and kites as you can on one sheet, with a ratio of about five kites to three darts, using a thin line for the curves. The sheet can be photocopied many times. The curves can then be colored with, say, red and green felt-tip pens. Conway has found that it speeds constructions and keeps patterns stabler if you make many copies of the three larger shapes in the lower illustration on this page. As you expand a pattern you can continually replace darts and kites with aces and bow ties. Actually an infinity of arbitrarily large *pairs* of shapes, made up of darts and kites, will serve for tiling any infinite pattern.

A Penrose pattern is made by starting with darts and kites around one vertex and then expanding radially. Each time you add a piece to an edge you must choose between a dart and a kite. Sometimes the choice is forced, sometimes it is not. Sometimes either piece fits, but later you may encounter a contradiction (a spot where no piece can be legally added) and be forced to go back and make the other choice. It is a good plan to go around a boundary, placing all the forced pieces first. They cannot lead to a contradiction. You can then experiment with unforced pieces. It is always possible to continue forever. The more you play with the pieces, the more you will become aware of "forcing rules" that increase efficiency. For example, a dart forces two kites in its concavity, creating the ubiquitous ace.



Aces and bow ties that speed constructions

Figure 3.32



The Pythagorean pentagram

Figure 3.33