MODULE 16: PROJECT - SZILASSI POLYHEDRON

The first project involves an exploration into connections in three dimensions. It is intended, in part, as a puzzle - a puzzle which will give insight into space and connections; as a puzzle it is far from trivial. See Section 4.16 for more details. You should:

1. Use the templates on the next page provided to make a polygon. You can make it the same size as the patterns or smaller or larger, as you wish. You may also use any material you wish.

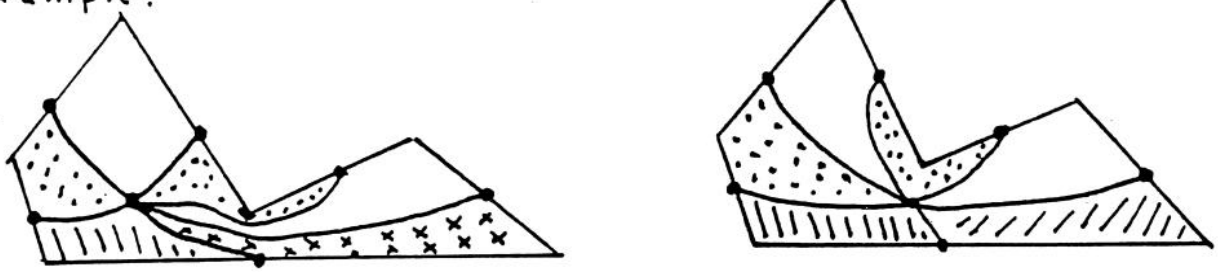
2. Assemble the parts into the shape shown on the drawing to make a "Szilassi polyhedron" . You should be able to get a solid shape with flat polygons as faces; if you use a flexible material you may be able to flex the sides and this is OK - as long as all the connections are correct and the edges don't pull apart - it's up to you how much rigidity you want.

3. Place a dot in the center of each ecdge and at a point in each face (the latter point should be chosen carefully to minimize "clutter" in the next step).

4. Connect the dot in each of the 7 faces to each of the 6 dots on the edges around that face. The lines may be straight but some will have to be curved (see sample below).

5. Color each of the regions created by (4) in such a way that no two regfions with the same color share one of the new edges (Two regions of the same color can touch at a vertex however). Use as few colors as possible.

Example:



6. Make a "presentation" of the result; label the finished object so someone who doesn't know what it is will know; include some comments about it based on your essay (see part 7). the object itself is not enough - you must present it properly.

7. Write an essay on your experiences in assembling the Szilassi polyhedron. Discuss what your THOUGHTS, FEELINGS and IMPRESSIONS were during the process of making templates, trying to figure the connections, assembling, and coloring. When you hold it in your hands and study it, what do you think of? (Negative comments are perfectly OK - what counts is YOUR impressions. Be yourself!)

As an alternative project you may construct a Csaszar polyhedron which is the dual to the Szilassi polyhedron according to the instructions and using the templates given on the next page. Color your Csaszar polyhedron with the fewest colors so that no faces that share an edge have the same color.

Grading will be based on accuracy (Correct connections), workmanship (Are the edges straight, do they meet without gaps, do the vertices come together, is the coloring done neatly, are the faces joined neatly - etc.), presentation (How well does the finished result tell a viewer what its about and does it engage a viewer's insterest?) and , lastly, your essay.

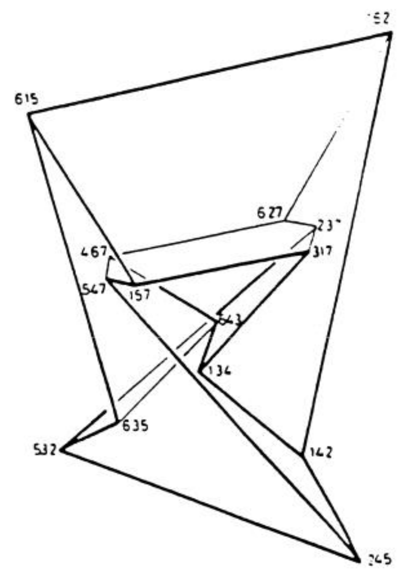
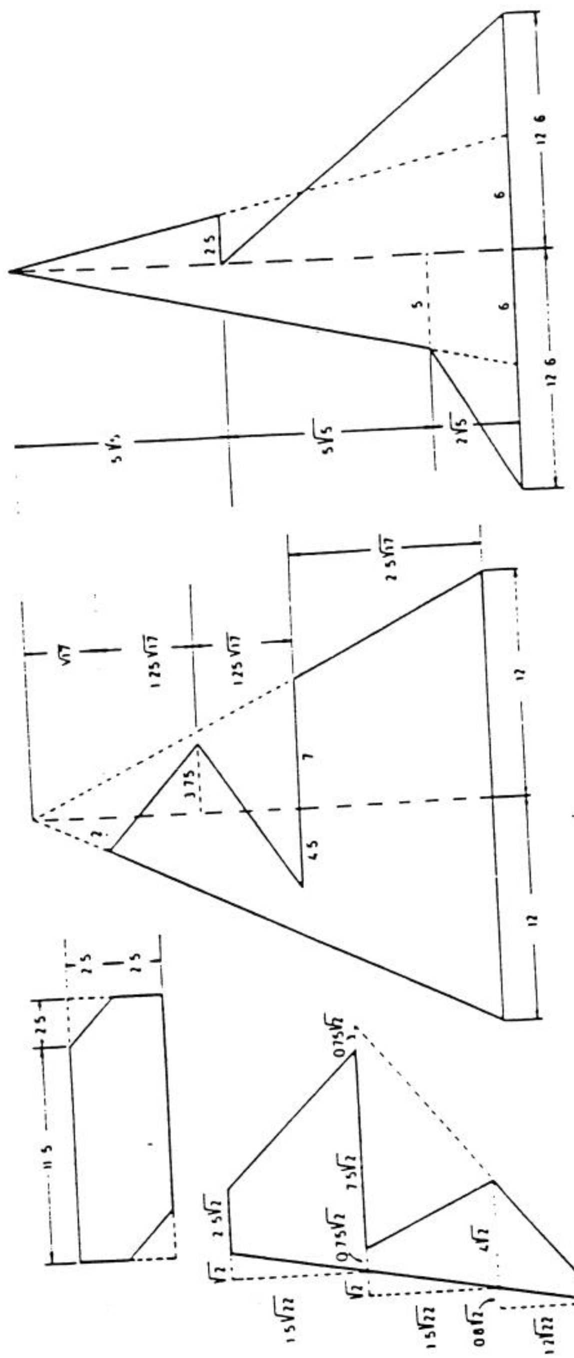


Figure 4 Toroid with seven faces topologically isomorphic with Heawood's seven colours toroidal map •



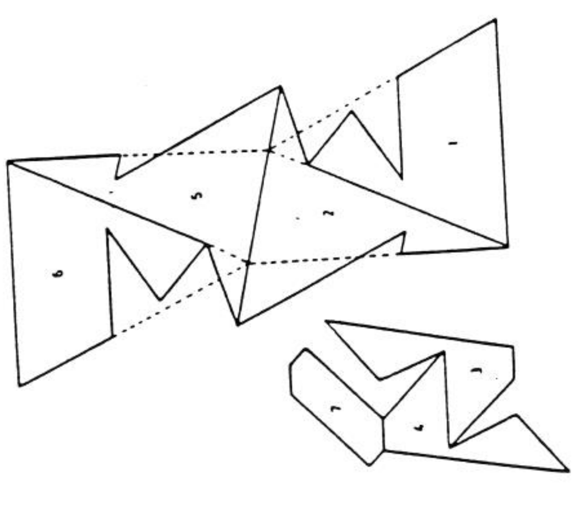


Figure 5 Data for making seven faces toroid •

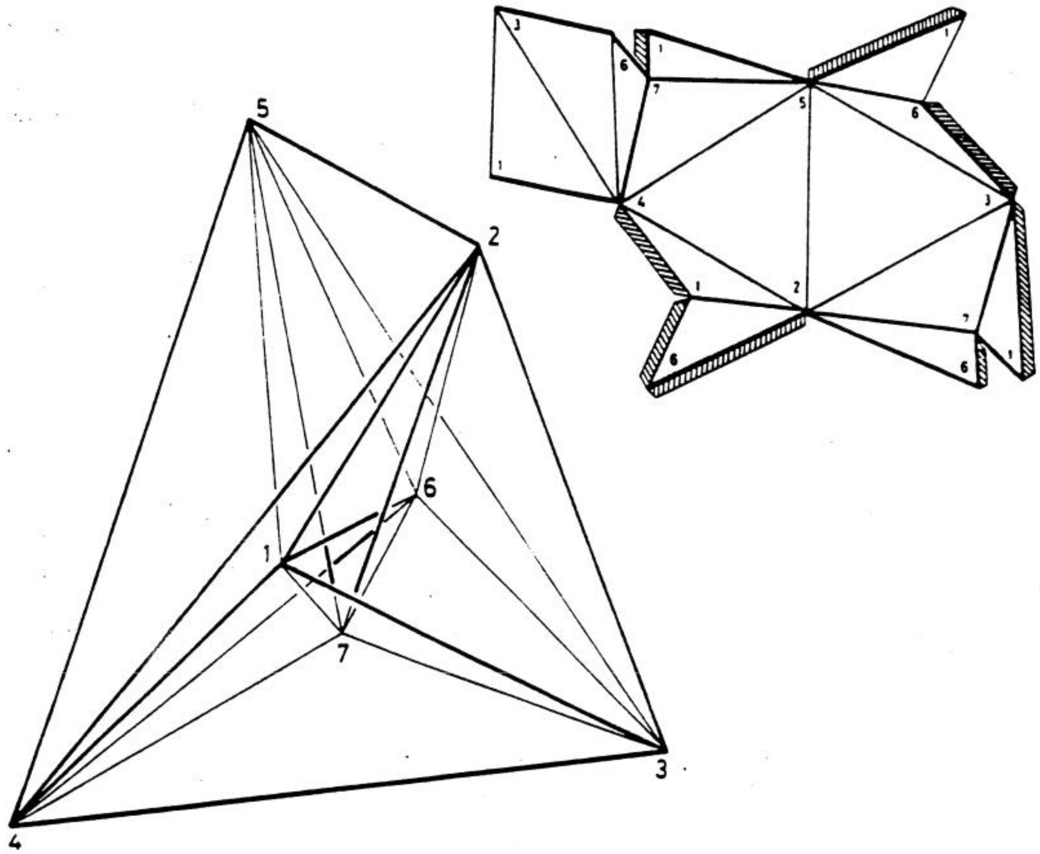


Figure 2 A toroid without diagonal, C_1 variant • Un toroïde sans diagonale, variante C_1.

Data of the polyhedron having no diagonal

edge • arête	C_1 edge length • longueur de l'arête	C_1 interfacial angle • angle dièdre
(1-6)	10	76°8'
(2-5)	24	70°32'
(3-4)	24	54°26'
(2-4) = (5-3)	24	51°3'
(2-3) = (5-4)	24	52°43'
(3-7) = (4-7)	12.89	340°8'
(2-7) = (5-7)	17.15	74°25'
(1-5) = (6-2)	18.69	339°19'
(1-2) = (6-5)	12.55	156°51'
(1-4) = (6-3)	12.55	204°28'
(1-3) = (6-4)	17.36	41°40'
(1-7) = (6-7)	5.86	243°30'

faces of the polyhedron • les faces du polyèdre						
(1-6-2)	(1-4-2)	(2-4-5)	(1-3-4)	(1-5-7)	(5-4-7)	(4-6-7)
(6-1-5)	(6-3-5)	(5-3-2)	(6-4-3)	(6-2-7)	(2-3-7)	(3-1-7)

THE HEAWOOD SURFACE 5

A TOROID WITH SEVEN CONGRUENT AND PAIRWISE ADJACENT REGIONS

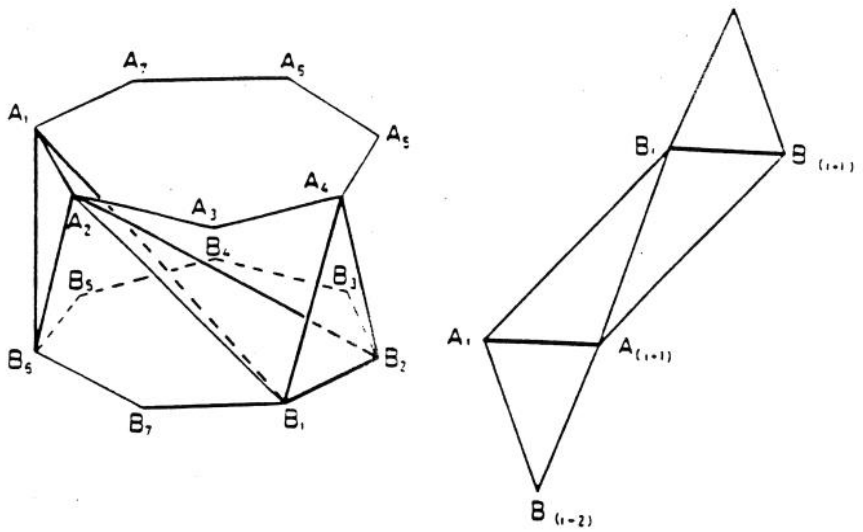


Figure 7 Realization of Heawood's map with toroid constructed of seven regions any of which is congruent and neighbouring with each other. Every region consists of four triangles • Une application du réseau de Heawood sur un toroïde dont chacune des sept régions est adjacente à toutes les autres et dont toutes les régions sont congruentes. Chaque région est constituée de quatre triangles.

5. Various problems arose in the last century in connection with the colouring of maps, e.g. the four-colour problem. In 1890 it was proved by Heawood that seven colours are sufficient for colouring any map drawn on a torus. At the same time he also showed that seven colours are necessary too, by drawing on the torus a map consisting of seven regions, any two of which were adjacent, so that in order to get the required colouring each region had to be given a different colour. By the polyhedron described here Heawood's seven-colour map can be constructed from seven simple planar hexagons, that is Heawood's toroidal map topologically isomorphic to this toroid.

Heawood's seven-colour map can also be realized with a toroid each face of which is a regular polygon. (In this case, of course, the regions no longer consist of a single face.) The network of this construction (due to Stewart [3], p. 199), together with the separate *interior* and *exterior* halves of the toroid, may be seen in **Figure 6**. The numbers in the polygons denote the colours. Examination of **Figure 6** or of a model prepared on this basis confirms that every colour is in fact adjacent to all of the other colours.

A similarly interesting construction is the toroid in which the regions are not only adjacent, but also congruent. Each such region consists of four triangles which are congruent in pairs. For its construction, let us consider the regular heptagon $A_1A_2 \dots A_7$.

We rotate it by the angle $(5/2) \cdot (2\pi/7)$ about its centre, and then shift it in the direction perpendicular to its plane. In this way we obtain the regular heptagon $B_1B_2 \dots B_7$. For each $i = 1, 2, \dots, 7$, the figure consisting of the triangles $A_iA_{i+1}B_{i-2}$, $A_iB_iA_{i+1}$, $B_iB_{i+1}A_{i+1}$ and $B_iA_{i+3}B_{i+1}$ is coloured one colour and is considered as one region (**Figure 7**). (If some index does not lie between 1 and 7, 7 is either added to it or subtracted from it so as to yield a number between 1 and 7, i.e. indices are taken modulo 7.) If the i-th region is rotated by the angle $2\pi/7$ around the axis joining the centres of the two regular heptagons, we obtain the $(i + 1)$-th region. These regions are therefore indeed congruent, and together form a toroid. Examining the indices of the edges bordering the regions one can see that each of them is indeed adjacent to all of the others. For example, the neighbour of the i-th region along the edge $\overline{A_iB_i}$ *is the* $(i + 1)$-th region, and its neighbour along the edge $\overline{A_iB_{i-2}}$ is the $(i - 3)$-th region.

For the construction of the polyhedron, we may arbitrarily fix the distance of the planes of the two regular heptagons or, for example, the sides of the isosceles triangle $A_iA_{i+1}B_{i-2}$. From these, the other data may be calculated. Table 3 provides the data of three variants of the toroid, differing in their edge lengths.

This polyhedron is a regular toroid in class T_1; its faces belong to two congruence classes and its solid angles to one congruence class, i.e. they are congruent. It has $7 \cdot 4 = 28$ faces.

Table 3
The toroid with seven congruent, pairwise adjacent regions

edge • arête	1st variant • 1re variante		2nd variant • 2e variante		3rd variant • 3e variante	
$i = 1, 2, \dots, 7$	edge length • longueur de l'arête	interfacial angle • angle dièdre	edge length • longueur de l'arête	interfacial angle • angle dièdre	edge length • longueur de l'arête	interfacial angle • angle dièdre
$\overline{A_iA_{i+1}}$	6	64°1'	6	51°45'	6	43°21'
$\overline{A_iB_{i-2}}$	6	150°13'	8	152°13'	10	153°1'
$\overline{A_iB_i}$	13.48	51°12'	14.48	65°11'	15.68	74°33'
$\overline{A_{i+1}B_i}$	10.04	332°15'	11.35	325°13'	12.84	320°43'