IX. Mirrors, Symmetry and Isometries

"Is looking glass milk good milk to drink?"
Through the Looking Glass
Lewis Carroll

1. Mirrors

Mirrors present us with a world of strange and interesting illusions. Some animals never learn that mirror images are illusions and think, rather, that they are seeing another individual. However dogs and cats are more intelligent and lose interest with the mirror as soon as they realize that they are seeing a mere image of themselves. On the other hand, chimpanzees and young children find great fascination with the fact that whereas the images they see in the mirror are themselves there are certain subtle differences. They can spend hours exploring these differences. We would like you to go back in time and try to look again at mirrors with the curiosity of a young child. We will suggest several things to do; however, you may add anything that you wish to this list. In response to each of these mirror experiments write a paragraph describing what you see.

1. Look in a mirror and wink your right eye. What does your image do?
2. Write out the words of the following poem by looking at them in a mirror

Twas brillig, and the slithy toves
Did gyre and gimble in the wabe:
All mimsy were the borogoves,
And the mome raths outgrabe.

3. Place the following objects in various orientations before a mirror:

a ball, a cube, an egg, a clock, a helical spring, a conical object, a knot, your right hand, at least three additional objects of your choosing. What do you observe?

4. Your face appears to be symmetric. Let's see just how symmetric it is. Place a mirror along the line of symmetry that divides the left side of a photograph of a human face from the right and see whether the exposed portion of the face and its mirror image combine to give a realistic or distorted picture of the entire face.

5. Look at a painting in a mirror. Does anything look strange about the painting or is it the mirror image of an equally valid painting?

6. Take a pair of mirrors as shown in Figure 1 and look at your face in the mirrors. Wink your left eye. What does your image do? How does your left hand look? Rotate the mirrors 90^{o}. What do you see?

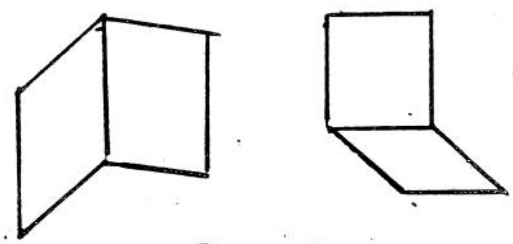
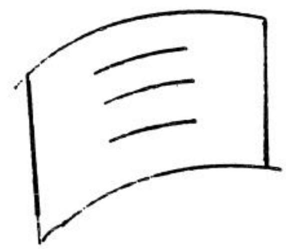


Figure 1

7. Take a curved metal sheet and look at your reflected image in the sheet. What do you see? Change the orientation of the sheet. What do you see? Try looking at objects in mirrors with different curvatures and record your observations.



8. The following sum is wrong.

```
     3414
      340
    T4813
 --------
 433T4813
```

Look at the sum in a mirror and show that it is now correct.

9. Look at these names in a mirror.

A ~~B~~ ~~C~~ ~~D~~ ~~E~~ F G H I J ~~K~~ L M N O P
Q R S T U V W X Y Z

Why is TIMOTHY not reversed?

10. Which of the letters of the alphabet will look the same when seen in a mirror? Which will not look the same no matter how you orient them?

11. Why does an ordinary mirror appear to reverse right and left but not up and down?

There is one kind of symmetry that occurs most frequently in the animal world. All animals have a plane of approximate symmetry that divides the left side of their body from the right side. This kind of symmetry is known as bilateral symmetry. It is the most frequent form of symmetry found in man-made objects and architecture as man attempts to impose his own symmetry on his artifacts. As a matter of fact, we have had to coin the words "left" and "right" to distinguish more easily between these two similar looking halves of our body. If there were some obvious asymmetry in the two halves of our body we could refer to that side without the use of words like "left" or "right." For example we could refer to the side of our body with the ear (if we had only one) or the horn, etc. Actually, our bodies are not perfectly symmetric as you may have discovered if you carried out the suggested exercise 6 of the last section in which a face is reconstructed from one half of its mirror image.

The apparent bilateral symmetry comes about due to the steady action upon our bodies of the force of gravity which distinguishes between up and down but not between left and right. The front and back protions of animals have probably evolved asymmetrically due to requirements of locomotion and finding food.

When we look in a mirror and wink our right eye, we see our image wink his left eye so we say that mirrors reverse left and right. Actually, the mirror does not reverse left and right since it is really the eye on the right side of the mirror that winks when we wink our right eye. In fact it is front and back which the mirror reverses. We imagine that we can walk behind the mirror wherein the person in the mirrors appears to have his left-right orientation reversed. Likewise an asymmetric object such as a left glove becomes a right glove in the mirror in the sense that if the left glove were carried around to the other side of the mirror it would not match up with its image, whereas a right glove would match. Is there any way that a left glove can be turned into a right glove so as to match up with its mirror image? Strangely enough the answer is yes; however, the explanation is worthy of a science fiction story rather than a mathematics book.

In order to explain how left gloves can be turned into right gloves, let us consider a similar situation that arises when a Flatlander is viewed in a Flatland mirror (which is a line in the plane). Since Flatlanders are asymmetric in their world (like left gloves in 3-D worlds), they too cannot be made to match up with their mirror images by moving them behind the mirror in the plane of their two-dimensional world. However, if we lift the Flatlander up into three dimensions and turn him around and put him back in Flatland he can now easily be seen to match up with his mirror image. The same can be done with a left glove. However, it must be lifted out of our three-dimensional world, turned around in the world of four dimensions and then put back in 3-D in which case, being now a right glove, it can be made to match the left glove's mirror image. Likewise a human being who is transported into 4-D space and then returns to 3-D space may return with all his left-hand features reversed. For example, his heart might now be on the right side of his body and any distinguishing features such as birthmarks or the part in his hair would also be reversed.

Physicists are particularly fond of symmetry. Recently it was discovered that for every elementary particle there exists a mirror image particle. For example, corresponding to an electron there exists a positron with the same size but opposite charge. Corresponding to protons there are antiprotons and for neutrons there are antineutrons. It has even been conjectured that there are mirror images of all forms of matter, known as antimatter. It is also thought that when these two mirror image forms of matter combine they disintegrate with a large explosion which appears to answer the question which Alice poses at the beginning of this chapter as to whether looking glass milk is good to drink.

3. The Geometric Theory of Mirrors and its Relation to Isometries

We have discussed how symmetry and asymmetry of 1-D, 2-D, and 3-D figures is related to mirrors. Let us now investigate the mathematics behind the transformations of points due to reflections in a mirror. We will limit ourselves to points in the plane for which a mirror consists of a line in the plane. We will describe the transformations that result from points reflected first in a single mirror, then in two mirrors either meeting at an angle or parallel to each other, and finally in three mirrors. It will turn out that we need not consider more than three mirrors to get a complete understanding of the possible transformations of points due to reflection in mirrors.

In the remainder of this section we will study mirror transformations by geometric methods and discover the laws that govern the transformations mentioned in the previous paragraph. In the next section we develop the mathematics necessary for a computer to carry out these transformations.

A. Reflections in a single mirror:

Consider a mirror M and a point P located a distance d from M as shown in Figure 3. The transformed point P' is located distance d on the other side of the mirror.

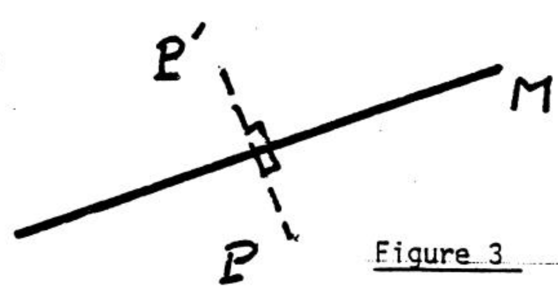


Figure 3

Rule 1: A mirror M is always the perpendicular bisector of the line between a point and its reflected image.

If we use the letter R to stand for a reflection in mirror M, then RR stands for a reflection followed by a reflection. Since a reflection followed by a reflection leaves all points P in the plane unchanged we can write:

$$RR = I$$

or

$$R = R^{-1}$$

where the reader is referred to section I for definitions of the identity transformation I and the inverse R^{-1}.

B. <u>Reflections in two mirrors</u>:

(a) Intersecting mirrors

Consider mirrors M_1 and M_2 intersecting with angle θ at O, and consider an arbitrary point P in the plane.

Reflect P first in M_1 to P',

then in M_2 to P'' as shown in Figure 4a.

Can you prove from Figure 4a that angle POP''=2θ? (Do this!)

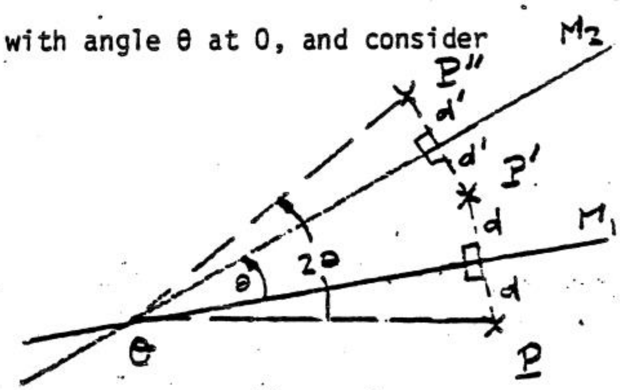


Figure 4a

Let us now reflect P first in M_2 then M_1 as illustrated in Figure 4b.

Can you prove from Figure 4b that angle POP''=-2θ? (Do this!)

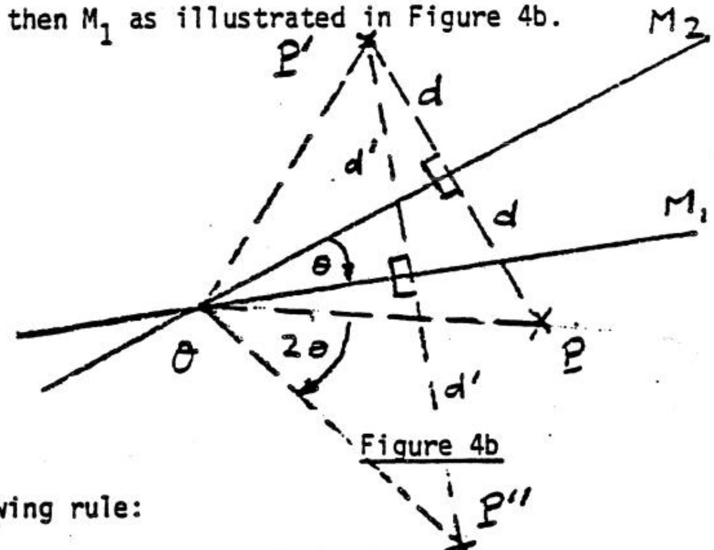


Figure 4b

We can therefore state the following rule:

Rule 2: If any point in the plane is <u>reflected</u> successively in two intersecting mirrors, the transformed point is <u>rotated</u> about the point of intersection by twice the angle between the mirrors and with the same sense of the angle between the first and second mirror of the reflections.

If we use P_1 and P_2 to stand for reflections in mirrors M_1 and M_2 respectively, and S for a rotation about the intersection 0 of the two mirrors of twice the angle between M_1 and M_2, we can make the concise algebraic statement:

$$S = R_2R_1$$

which states that S is the same as R_1 followed by R_2.

Likewise:

$$S^{-1} = R_1R_2$$

since S^{-1} is a rotation with the opposite sense of S (why?) and results from R_2 followed by R_1.

Remark 1:

Without proof, we state that any two mirrors intersecting with angle θ at point 0 will have the same effect upon an arbitrary point of the plane regardless of the orientation of the mirrors. Thus, the pair of mirrors M_1' and M_2' of Figure 5 will have the same effect on point P as M_1 and M_2, namely, they will transform P to P''.

Problem: Prove this!

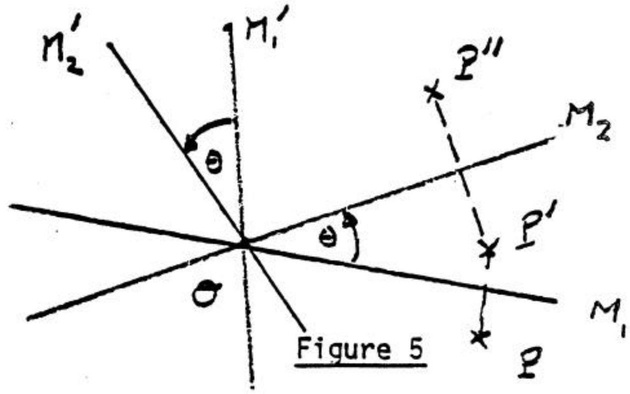
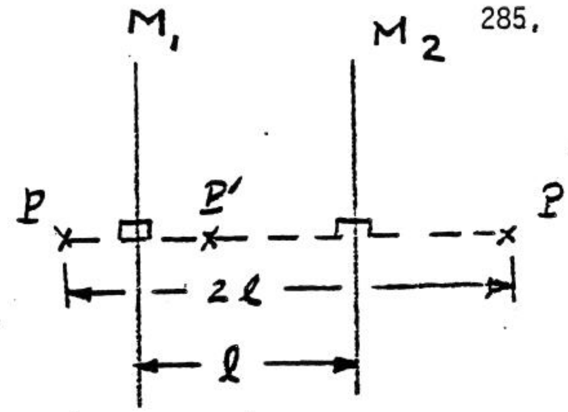


Figure 5

(b) Two parallel mirrors

Again, consider an arbitrary point, P, in the plane and two parallel mirrors M_1 and M_2 a distance ℓ apart. Reflect P first in M_1 and then in M_2 as shown in Figure 6.



Can you prove that P'' is translated from P a distance 2ℓ in a direction from M_1 to M_2, perpendicular to the mirrors? (Prove this!) Likewise, if the reflections take place first in M_2 and then in M_1 the point P is translated by 2ℓ in the other direction. Thus we can state:

Figure 6

Rule 3: Two parallel mirrors translate a point to its image point by an amount twice the distance between the mirrors in a direction from the first mirror to the second mirror of the reflections.

If we use the letter R_1 and R_2 for reflections in M_1 and M_2 and T for the translation of an arbitrary point through a distance equal to twice the distance between the parallel mirrors, then we summarize rule 3 algebraically:

$$T = R_2R_1$$

Also,

$$T^{-1} = R_1R_2 \qquad \text{(why?)}$$

Remark 2:

If two parallel mirrors a distance ℓ apart were placed anywhere without rotating them, their effect on P would be the same, namely, P would be translated a distance 2ℓ in the direction from M_1 to M_2. Thus the two mirrors M_1' and M_2' of Figure 7 have the same effect on P as M_1 and M_2. (Check this!)

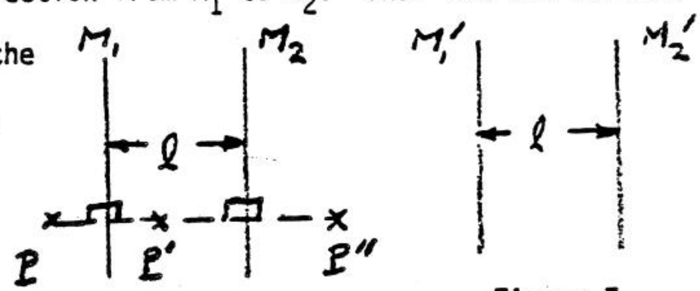


Figure 7

Remark 3:

If P is reflected in parallel mirrors M_1 then M_2, P is translated a distance twice the distance between the mirrors. If this reflected image is then reflected first in M_1 then in M_2 it too is translated by the same amount. This sets up an infinite chain of reflections, reflections of reflections, etc., familiar to anyone who has fisited a barber shop. See Figure 8.

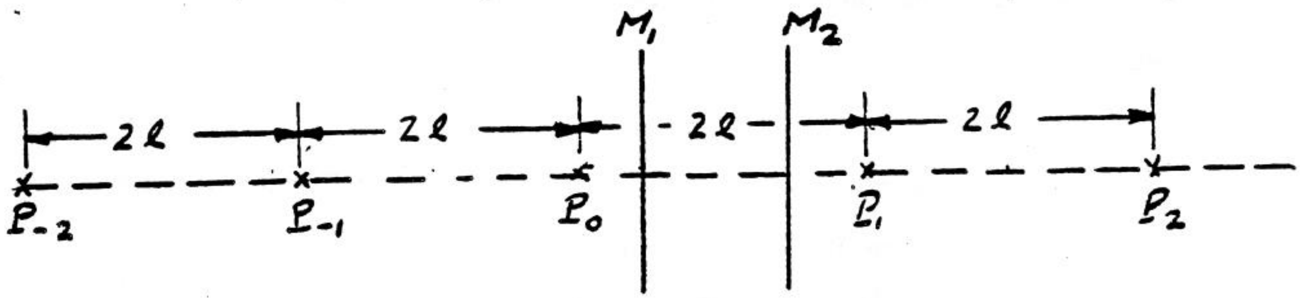


Figure 8

C. Reflections in three mirrors:

First, let's consider the different ways three mirrors can be oriented relative to each other. There are five distinct ways, and they are shown in Figure 9.

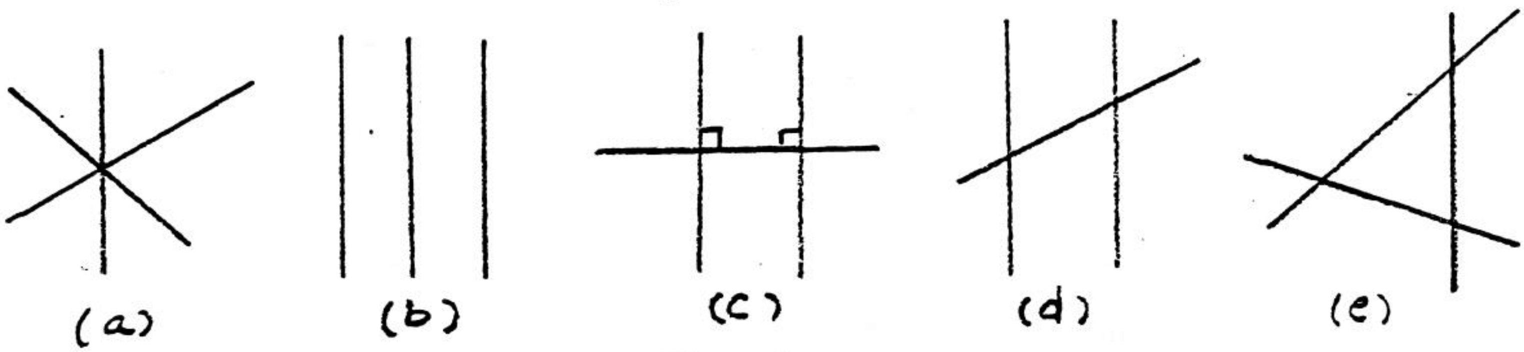


Figure 9

The mirrors can all intersect at a point, they can all be parallel, two can be parallel and the third perpendicular to them, two can be parallel and the third cuts them obliquely, and finally they can all intersect each other but not all at the same point.

(a) At first it might seek like a lot of work to analyze each of these five cases. However, the first two cases have already been analyzed by Rules 1, 2 and 3. The fact that we now have three mirrors presents no problems.

Problem:

(1) Find the transformed position of point P if it is reflected in mirrors M_1, M_2, and M_3 of Figure 10 in that order.

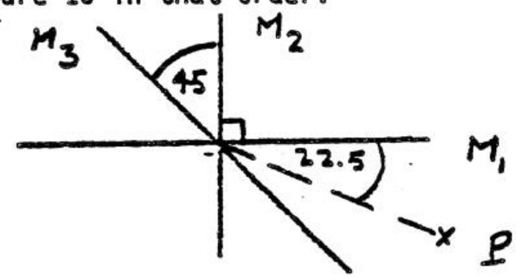


Figure 10

(2) Find the transformed position of point P if it is reflected in mirrors M_1, M_2, and M_3 of Figure 11 in that order.

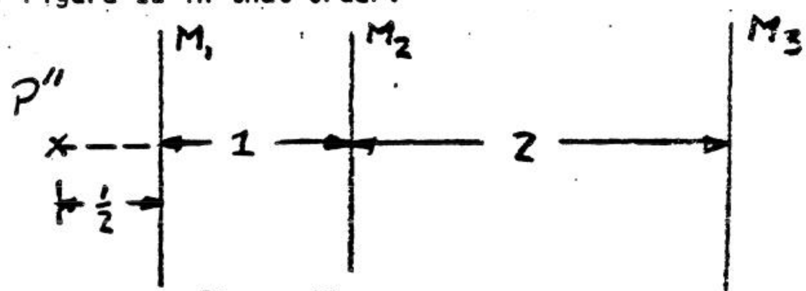


Figure 11

(b) Let us consider the third case. A point P is reflected first in the parallel mirrors M_1 and M_2 a distance ℓ apart, and then in the mirror M_3 perpendicular to M_1 and M_2 as shown in Figure 12.

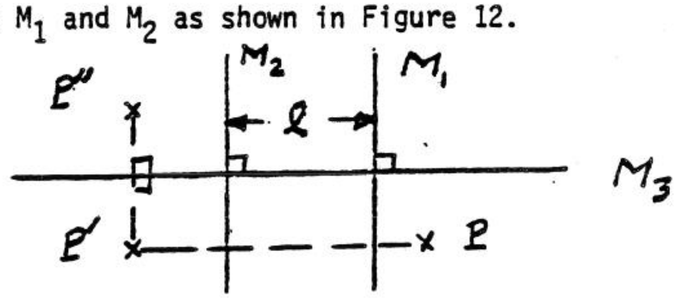


Figure 12

Thus P is translated a distance 2ℓ to P' and then reflected in M_3 to P''. This combination of a translation and a reflection in a mirror placed along the direction of the translation is called a glide reflection. Mirror M_3 is called the axis of the glide reflection. If a point P is glide reflected to P_1 and P_1 is glide reflected again through M_1, M_2, and M_3 to P_2 and P_2 is again glide reflected, ad infinitum, a pattern of glide reflections as shown in Figure 13 emerges.

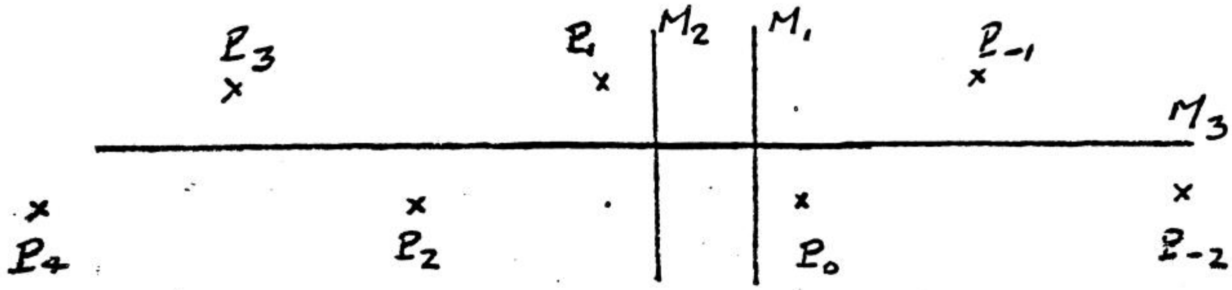


Figure 13

These are like your footsteps if you are walking in the direction of M_3.

(c) Now we are in a position to consider the fourth case. We will show how this case reduces to case 3. The simultaneous reflections in M_1, M_2, then M_3 can be thought of as first a reflection of P in M_1 to P' and then a rotation about point O of Figure 14 of an amount twice the angle θ from M_2 to M_3 to P'', i.e., $R_3R_2R_1 = (R_3R_2)R_1 = S_{2\theta}R_1$

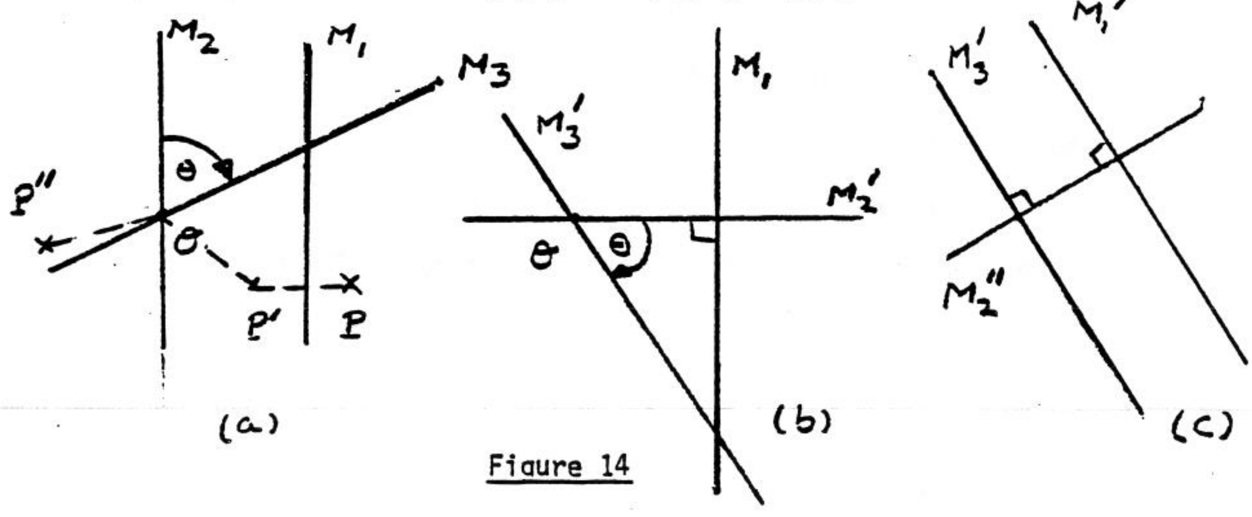


Figure 14

But since the rotation about determined by M_2 and M_3 does not depend on the orientation of M'_2 and M'_3 (see Remark 1), the mirrors in Figure 15b have the same effect on P as the mirrors in Figure 15a.

Now the simultaneous reflections through M_1, M'_2, and M'_3 of Figure 15b can be thought of as a rotation of 180^0 about the intersection of M_1 and M'_2 followed by a reflection in M'_3, i.e., $R_3R_2R_1 = R_3(R_2R_1) = R_3S_{180^0}$. But by Remark 1 M_1 and M_2 can also be rotated until M''_2 is perpendicular to M'_3 as shown in Figure 15c. This reduces case 4 to case 3, where M''_2 is the axis of the glide reflection.

Problem:

Show that case 5 also reduces to case 3. For the mirrors oriented as in Figure 16 find the axis of the glide reflection.

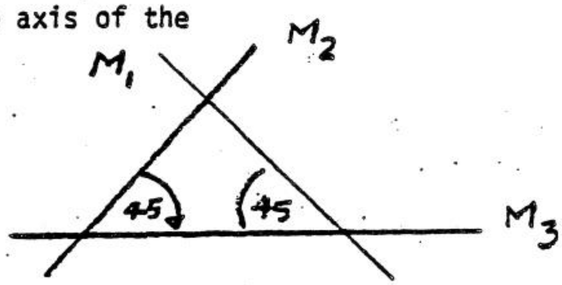


Figure 16

D. A major theorem about isometries:

We have described how reflections in one, two, or three mirrors give rise to all the isometries of the plane, namely, reflections, rotations, translations, and glide reflections. Now we summarize all of the results of this section by an interesting theorem.

Theorem:

Any isometry of the plane can be carried out by a series of no more than three reflections.

In other words any two figures which are congruent under an isometry can be made to coincide by no more than three reflections.

It will be sufficient to match up the two congruent triangles shown in Figure 17. The procedure is as follows:

1. Choose two corresponding vertices, for example A and A' and find the perpendicular bisector M_1 of AA'

2. Reflect ΔABC in a mirror at M_1 to ΔA'B''C'' as shown in Figure 17 .

3. Reflect ΔA'B''C'' in the perpendicular bisector M_2 of two other corresponding vertices, say, B' and B'', to ΔA'C'''B' .

4. Reflect ΔA'B' C''' to ΔA'B'C' in a mirror placed on A'B' as shown in Figure 17 .

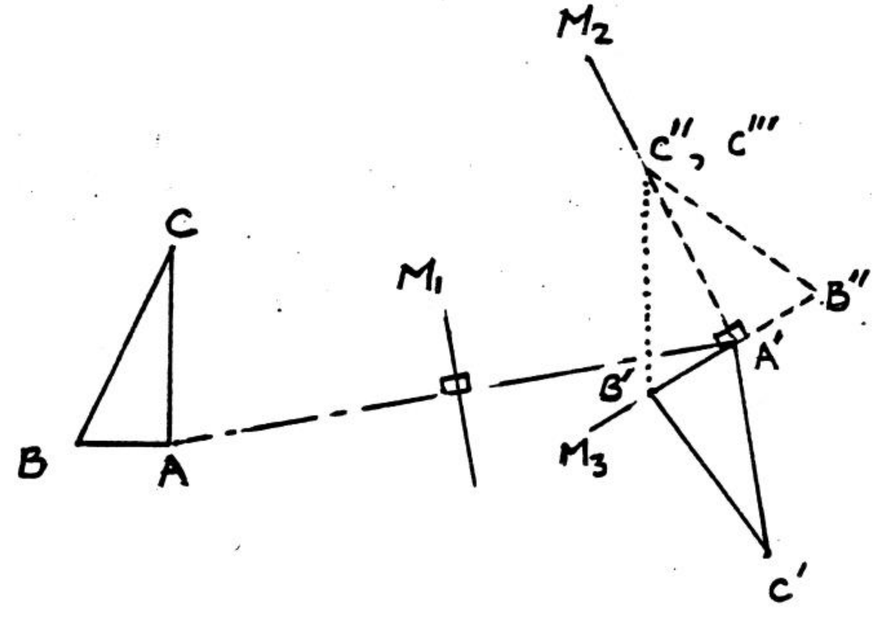


Figure 17

4. The Analytic Theory of Isometries

In this section we will demonstrate how the isometries discussed in the previous section can be carried out analytically. The methods that we come up with are programmable for a computer graphics facility. We will assume that the student knows how to multiply matrices, and we will not attempt to justify every mathematical statement although we will try to make the subject seem plausible. We will also restrict ourselves to isometries of the plane.

A. Representation of isometries by matrices:

As we saw in Chapter I isometries are members of a class of transformations known as linear transformations that can be represented by matrices in the sense that if a point P and its transformed point P', as shown in Figure 18, are represented by column vectors, $P = \begin{pmatrix} x \\ y \end{pmatrix}$ and $P' = \begin{pmatrix} x' \\ y' \end{pmatrix}$, then

$$P' = AP \text{ where A is the 2x2 matrix } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

In other words:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

or, written out,

$$x' = ax + by$$
$$y' = cx + dy$$

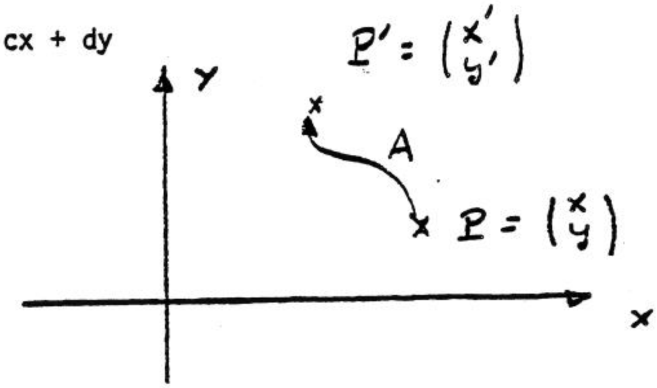


Figure 18

Example:

If $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

find out how Mr. Flatlands, shown in Figure 19, whose vertices are $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}$ transforms.

But, $A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$A \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \end{pmatrix}$$

Also, these matrix transformations have the property that lines are transformed into lines. Thus the result of the transformation is shown in Figure 19.

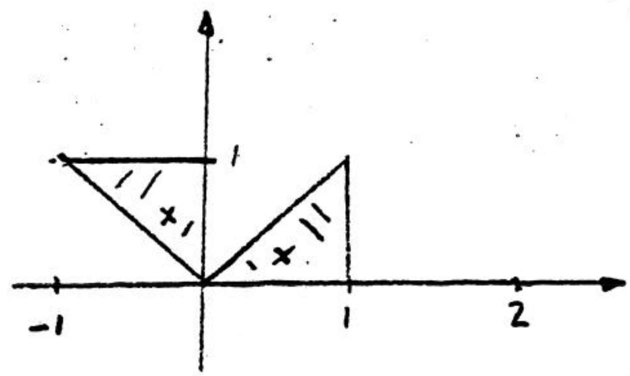


Figure 19

Thus we see that $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the matrix of a counterclockwise rotation about the origin.

Problems:

What transformations are represented by the following matrices?
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} .707 & -.707 \\ .707 & .707 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Show what these matrices do to Mr. Flatlands.

B. <u>How to construct a matrix to carry out a desired rotation or reflection</u>:

We first note the way in which a matrix transforms the special points $\binom{1}{0}$ and $\binom{0}{1}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$$

We notice that $\binom{1}{0}$ transforms to the first column of the matrix, while $\binom{0}{1}$ transforms to the second column.

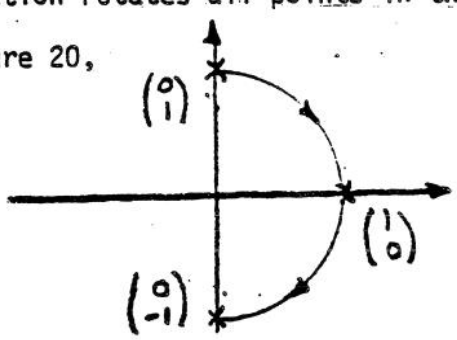
Conversely, we can show that if $\binom{1}{0} \longrightarrow \binom{a}{c}$ and,

$$\binom{0}{1} \longrightarrow \binom{b}{d}$$

that $\binom{a}{c}$ and $\binom{b}{d}$ must be the first and second columns of the transformation matrix respectively.

Example 1:

If the transformation rotates all points in the plane 90^{0} clockwise, then, as shown in Figure 20,



<u>Figure 20</u>

$$\binom{1}{0} \longrightarrow \binom{0}{-1}$$

$$\binom{0}{1} \longrightarrow \binom{1}{0}$$

Therefore, the transformation matrix is:

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Example 2:

If the transformation rotates all points counterclockwise through θ degrees. Then, as shown in Figure 21,

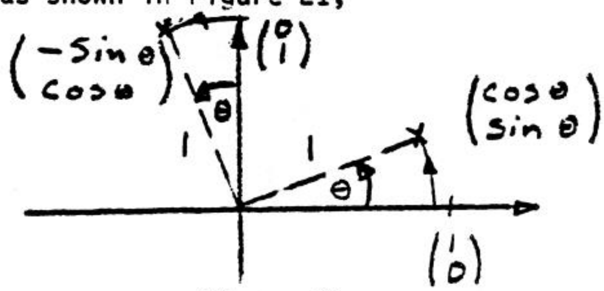


Figure 21

$$\begin{pmatrix}1\\0\end{pmatrix} \longrightarrow \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$$

$$\begin{pmatrix}0\\1\end{pmatrix} \longrightarrow \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$$

Thus,

$$A = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

What is A for $\theta = 90^{o}, 180^{o}, 45^{o}$?

Problems:

Find the matrices that carry out the following transformations: on a typical point.

1. Reflect in a mirror placed at angle 45^{o} clockwise from the x-axis.
2. Rotate counterclockwise by 90^{o} and then reflect in a mirror placed on the x-axis.
3. Reflect first in the x-axis and then in the y-axis.
4. Reflect first in the y-axis and then in the x-axis.
5. In problems 2-5 compute the matrix of each component of the compound transformation and show that the product of the component matrices equals the matrix of the component transformation.

C. Representations of translations by matrices:

We have seen how to express rotations about the origin and reflections in mirrors through the origin by matrices. Is there a way to express a translation by a matrix? If we try to construct a 2×2 matrix to do translations we will not succeed. However, if we change the way we represent points in the plane we will be able to use 3×3 matrices to represent translations.

Let us represent the point $\binom{x}{y}$ by the triple of numbers $\begin{pmatrix} cx \\ cy \\ c \end{pmatrix}$ where c can be any number greater than zero. Thus,

$$\binom{x}{y} \equiv \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x \\ \frac{1}{2}y \\ \frac{1}{2} \end{pmatrix} = \text{etc.}$$

You will notice that a point P now has many representations. For example:

$$\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \equiv \binom{3}{2}. \quad \text{Also} \begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix} \equiv \binom{3}{2} \quad \text{while} \begin{pmatrix} 1 \\ 2 \\ \frac{1}{2} \end{pmatrix} = \binom{2}{4}$$

What points are represented by $\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 2/3 \end{pmatrix}$?

Now consider Mr. Flatlands again with vertices $\binom{0}{0}, \binom{1}{0}, \binom{1}{1}, \binom{\frac{1}{2}}{\frac{1}{4}}$.

He is represented in the new system by $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \\ 1 \end{pmatrix}$.

How does the following matrix, A, transform this triangle?

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

Indicate your result in Figure 22.

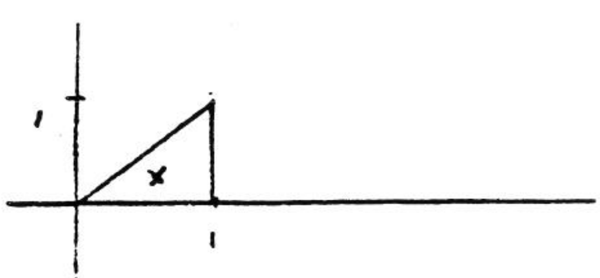


Figure 22

Now you can see how any translation in the plane can be carried out via a 3 x 3 matrix.

Problem:

Write the matrix to translate any point one unit to the right and two units down. Test your matrix on the triangle on Mr. Flatlands.

Now again consider Mr. Flatlands and transform him by matrix,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

Indicate your result in Figure 23.

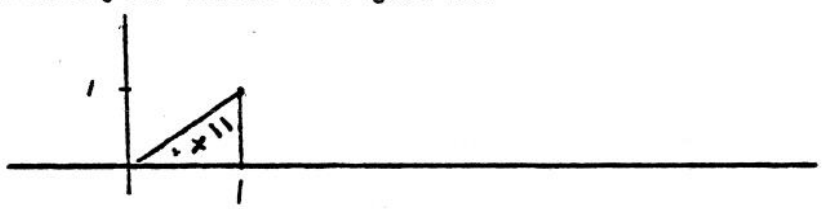


Figure 23

Based on this result, what conjecture can you make about the nature of the transformation, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{pmatrix}$.

What does the transformation $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ do to Mr. Flatlands?

What about $\begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{pmatrix}$?

Summarizing these results, we have seen that given the matrix

$$\begin{pmatrix} a & b & e \\ c & d & f \\ o & o & g \end{pmatrix},$$

first performs a rotation or reflection in the plane specified by the submatrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then translates to the right or the left or up or down by amounts given by e and f, and finally expands or contracts by an amount determined by $\frac{1}{g}$.

D. Rotations around a point other than the origin:

Let's say we wish to rotate any point P by 90^0 about a center C of rotation located at (2,1) to the new location P' as shown in Figure 24.

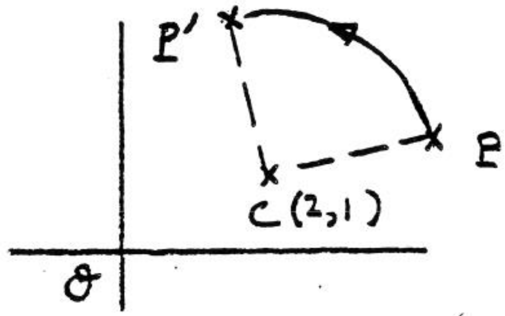


Figure 24

You may remember that the matrix $S_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ rotates points about the origin θ. Although we can't use this matrix to rotate P_c about C we can translate C and P, without changing their orientation, so that C is at the origin, perform the rotation about and then translate the resulting points and P_0' back to C and P_c' again without changing their orientation. These three steps are shown in Figure 25. In this figure T represents the translation back to the origin; S, the rotation; and T^{-1}, the translation back to C.

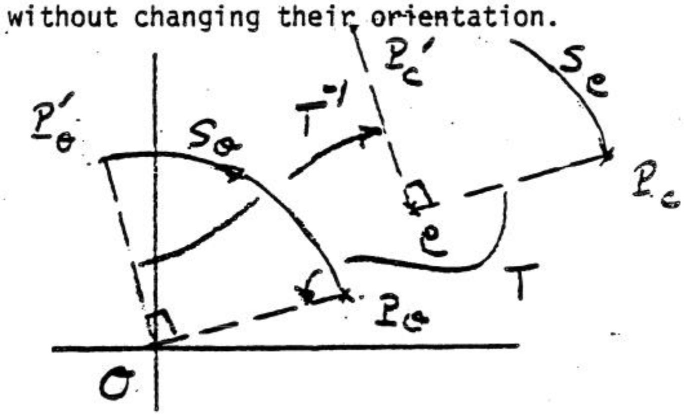


Figure 25

Thus the rotation about C is represented by:

$$S_c = T^{-1} S_\theta T$$

$$= \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem:

What is S_c for a rotation of 180^0 about C(1,-1) of the point P(2,-2). Show the result graphically.

E. A final mystery transformation:

So far we have found a purpose for every element of our 3 x 3 matrix except the first two elements of the last row. Let us now see what effect these elements have on points in the plane.

Consider two parallel line segments (railroad tracks) whose end points are $L_1 : \binom{1}{0}, \binom{1}{1}$ and $L_2 : \binom{2}{0}, \binom{2}{1}$ shown in Figure 26.

Plot the transformed lines under

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{pmatrix}.$$

Are the transformed lines still parallel? Can you find an interpretation for this mystery transformation? Where do the points (1,M) and (2,M) transform to as $M \to \infty$. Can you find an interpretation for this result?

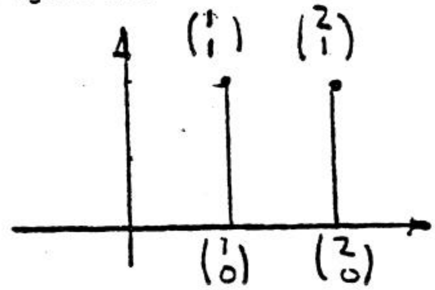


Figure 26

F. Projections:

When we observe an object such as a desk or the window or ceiling of the classroom we always see a distorted image of the object. For example, lines of the object that are parallel are seen to be either parallel, converging, or diverging depending on where the object is in relation to our eye. In fact we are all familiar with railroad tracks which appear to converge to a point in the distance. We say that we are viewing the objects or the railroad tracks "in perspective."

Perspective works very simply. Every time we view a scene from the world we are actually looking at an image of the scene projected by our eye (assume one eye is closed for simplicity) onto a screen situated in front of us, perpendicular to the ground. (See Figure 27.) Lines of the object parallel to our line of sight behind the screen seem to come together in the distance like railroad tracks, whereas lines of the object parallel to our line of sight in front of the screen appear to diverge. In appendix A we describe in greater detail how we view objects in perspective.

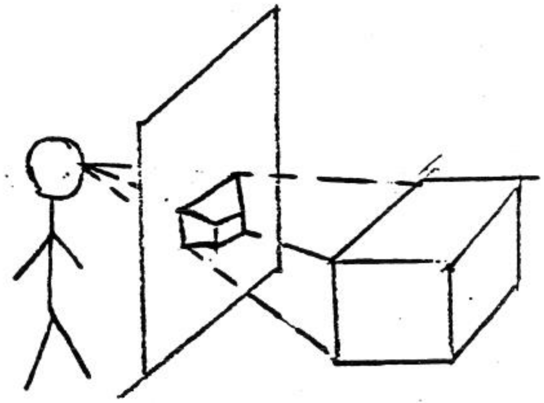


Figure 27

We can show that the mystery matrix of the previous section projects points from a plane into itself. It is difficult to get a physical understanding of this transformation. However, if we consider the three-dimensional analog of the mystery matrix, namely, $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{k} & 0 \end{pmatrix}$

we can get a better understanding of the nature of projective transformations. However we must first introduce a new way of denoting points in 3-D space, analogous to the extended way of writing points in 2-D, namely, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \equiv \begin{pmatrix} cx \\ cy \\ cz \\ c \end{pmatrix}$. Thus the point $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \equiv \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 4 \\ 0 \\ 2 \\ 2 \end{pmatrix}$ etc.

If we imagine that the eye is located at (0,0,k) on the z-axis of an x,y,z-Cartesian coordinate system, the 4 x 4 matrix transforms points of a solid object in 3-D space onto the x,y-plane, as shown in Figure 28 in such a way that each point, P of the object is projected along the line of sight from the eye until it pierces the x,y-plane at P'

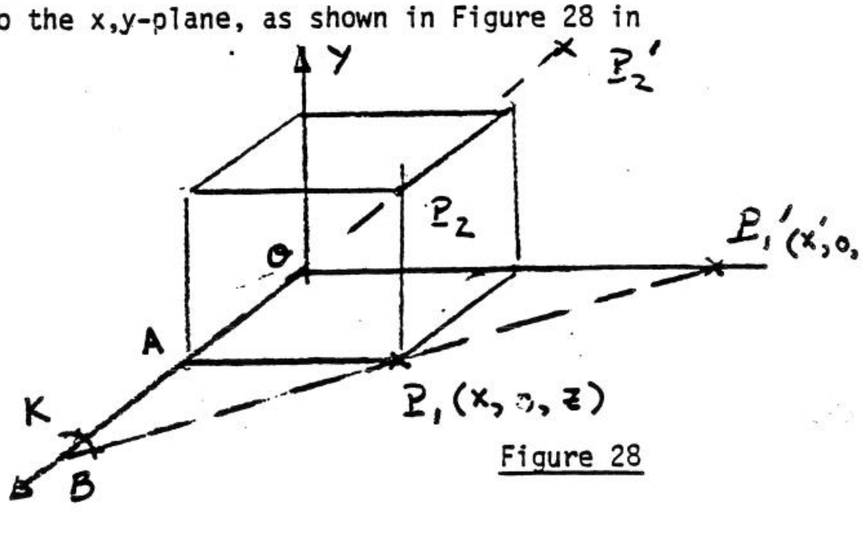


Figure 28

To prove that this transformation projects points, consider a point with coordinates (x,y,z).

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{k} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \\ 1 - \frac{z}{k} \end{bmatrix}$$

but

$$\begin{bmatrix} x \\ y \\ 0 \\ 1 - \frac{z}{k} \end{bmatrix} \equiv \begin{bmatrix} \frac{kx}{k-z} \\ \frac{ky}{k-z} \\ 0 \\ 1 \end{bmatrix} \qquad \text{(why?)}$$

thus,

$$x' = \frac{kx}{k-z}, \quad y' = \frac{ky}{k-z}, \quad z' = 0$$

For example the point $\underline{P_1}$ with coordinates $(x,0,z)$ in figure 28 is transformed to P_1' with coordinates $(x',0,0)$. But from figure 28, triangle $\Delta\ 0\ B\ P_1'$ is similar to $\Delta\ A\ B\ P_1$

thus,

$$\frac{k}{k-z} = \frac{x'}{x} \qquad \text{(why?)}$$

or

$$x' = \frac{kx}{k-z}$$

which justifies the x' tranformation equation. The y' equation can be proven in a similar manner.

Problem:

(a) Determine and graph the transformed points of the unit cube at the origin of the coordinate system lying in front of the viewing screen (x,y-plane), with vertices: (1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,0,1), (0,1,1), (1,1,1), and (0,0,0). Let k = 2

(b) Do the same for the unit cube placed 5 units behind the viewing screen with vertices: (0,0,-6), (+1,0,-6), (0,1,-6), (1,1,-6) (0,0,-5), (1,0,-5), (0,1,-5), (1,1,-5). Also let k = 2 in the projection matrix.