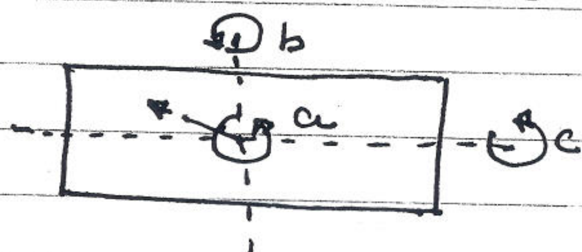
Symmetry worksheet-1

Symmetry of a Rectangle:

Get a book with rectangular shape and Subject it to four transformations:

e = leave it alone (also called the identity transformations).
a = give it a half-turn (see figure)
b = flip it lengthwise (see figure)
c = flip it sidewise (see figure)



For example transformation a followed by transformation c gives the same result as transformation b. This can be represented by the equation:

$$b = ca$$

Carry out all 16 possible combinations of transformations and record your results in a "multiplication" table like this:

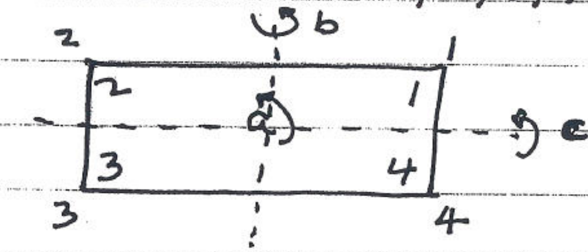
op1 \ op2	e	a	b	c
e				
a				b
b				
c				

Note: op1 me[ans] operation 1
op2 means operation transforma[tion]

Question: What observations do you make about ...

Symmetry Worksheet – 2

1. Cut out a rectangle from a piece of ~~G~~ graph paper and label the vertices 1, 2, 3, 4 on both sides and place it on an identical rectangle with vertices labeled 1, 2, 3, 4 as shown.



The identity transformation is represented by the permutation:

$$e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

Give the rectangle a half-turn and note how the vertices permute:

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

Likewise:

$$b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

If you transform the rectangle by first giving it a half-turn then flipping it in the b axis you get:

$$ba = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = c$$

2. Now create a table of permutations and compare it with your previous table.

op1 \ op2	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$
$e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$				
$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$				
$b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$				
$c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$				

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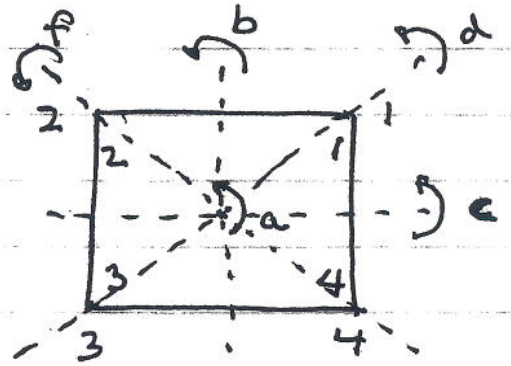
Repeat this excercise for a square where there are now 8 transformations

$e, a, a^2, a^3, b, c, d, e$

where a is a quarter-turn,
a^2 is a half-turn
a^3 is a 3/4- turn
b is a vertical flip
c is a horizontal flip
$d+f$ are flips along the diagonals
e is the identity

as shown



Create another "multipliation" table

op1 \ op2	e	a	a^2	a^3	b	c	d	f
e								
a								
a^2								
a^3								
b								
c								
d								
f								

Permutations can be represented in a very convenient form called cycle notation. To show how this works consider the symmetry transformations of a square shown on Worksheet - 3

Ex. 1:

$$b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = (12)(34) = t_2^2$$

We start with 1 which transforms to 2 and 2 transforms back to 1 so we get the 2-cycle (12). The same with 3 and 4. Since we have two 2-cycles ~~with~~ this transformation is called t_2^2.

Ex 2:

$$a^3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} = (1432) = t_4^1$$

i.e. one 4-cycle.

$$e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = (1)(2)(3)(4) = t_1^4$$

$$d = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} = (1)(24)(3) = t_1^2 \, t_2^1$$

i.e. two 1-cycles and one 2-cycle.

Remark 1: (24) = (42) but we usually represent the cycle by starting with the smallest integ.

Remark 2: (12)(34) = (34)(12) i.e. when a pair of cycles have no elements in common the cycles commute.

Remark 3: Sometimes we omit the 1-cycles with the understanding that omitted elements are 1-cycles. For example, $(1)\ (24)(3) = (24)$.

Now consider the effect of a pair of transformations. For example consider ~~applying~~ carrying out first transformation b then transformation d, i.e. db:

$$db = \underbrace{(24)}_{T_2 = d} \times \underbrace{(12)(34)}_{T_1 = b}$$

Start on the right with b. Since $1 \xrightarrow{b} 2$ and $2 \xrightarrow{d} 4$ it follows that $1 \xrightarrow{db} 4$ so we have $(1\,4$

Next transform $4 \xrightarrow{b} 3$ and $3 \xrightarrow{d} 3$ so, $4 \xrightarrow{db} 3$ and we have so far $(1\,4\,3$

Again $3 \xrightarrow{b} 4$ and $4 \xrightarrow{d} 2$ or $3 \xrightarrow{db} 2$.
This completes the 4-cycle $(1\,4\,3\,2)$
You can check that $2 \rightarrow 1$ and we start the cycle again.

You can use this cycle notation to complete the "multiplication table" for the transformations of a square.

Symmetry Worksheet - 4

Given the subgroup H of a finite group G and an element $x \in G$, the left coset of H corresponding to x is the set:

$$xH = \{ g \mid g = xh \text{ for } g \in G \text{ and } h \in H \}$$

I claim the the cosets aH for all $a \in G$ form a partition of G.

~~Check~~ Problem: ~~Chec~~

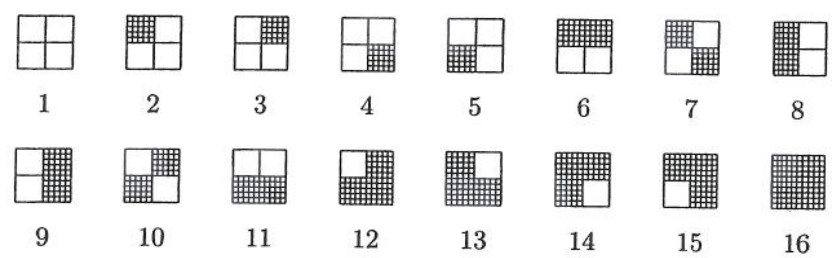
Check this claim for the group, S, of transformations of a square when $H = \{e, a^2\}$. Use the multiplication table for S to compute:

$$aH,\ a^2H,\ bH,\ cH,\ dH,\ fH,\ eH$$

e.g. $bH = \{be, ba^2\} = \{b, c\}$

Symmetry Worksheet - 5

Consider a square divided into 4 subsquares with each subsquare colored either ~~red~~ black or white. There are 16 possibilities shown below:



Questions:

1. Two a these patterns are considered to be identical if you can get one from the other by one of the 8 ~~tran~~ symmetry transformations of a square, i.e. a rotation or a flip.

   How many distinct patterns are there (i.e non-identical)? Illustrate one pattern from each identical class.

2. For each of the 8 ~~tran~~ symmetry transformations of a square, which of the patterns are invariant (do not change when subjected to the transformation)? For each transformation $\{e, a, \cancel{b, c, d}\ a^2, a^3, b, c, d, f\}$ list the pattern number which are invariant.

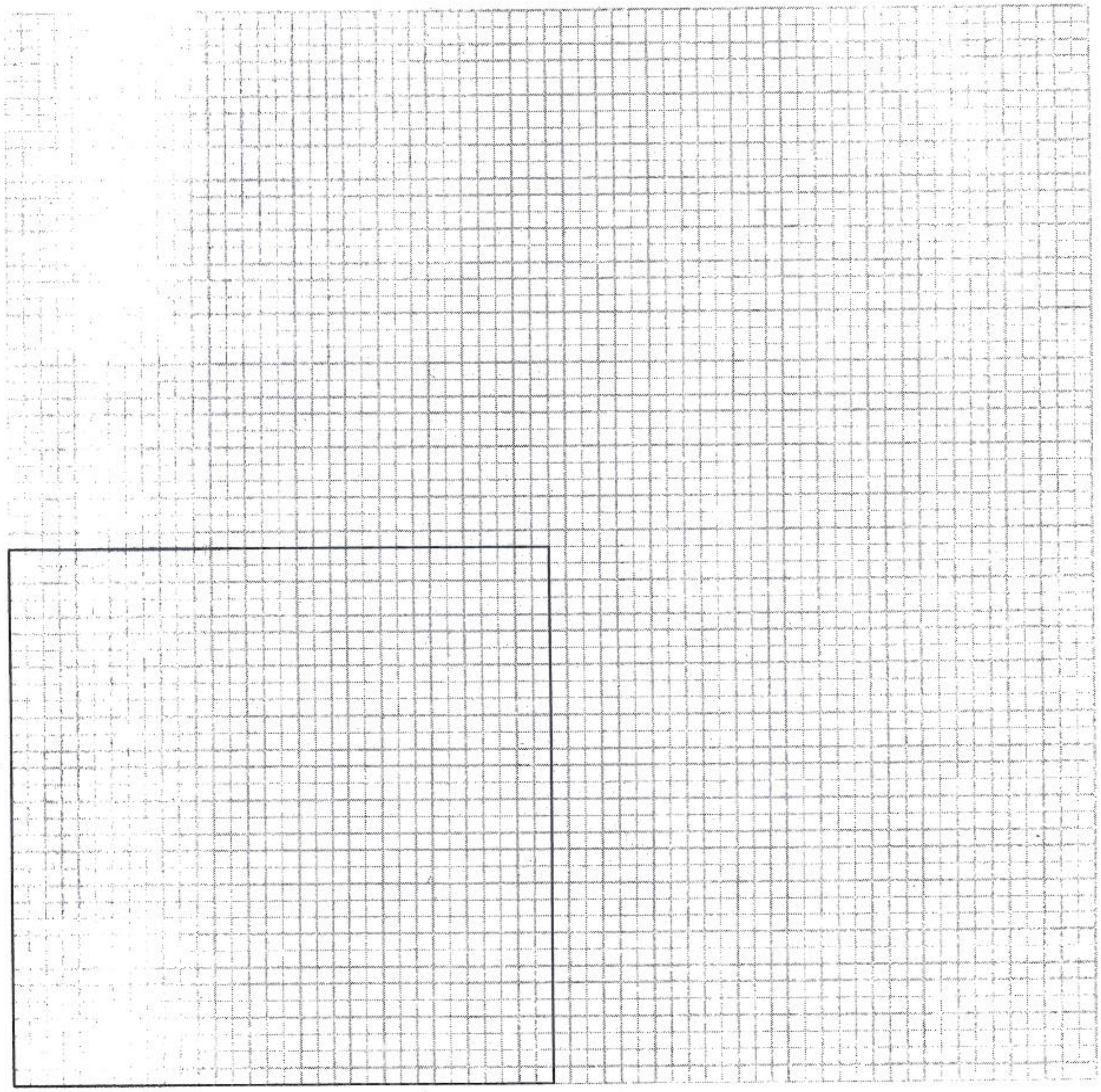


Figure 8.19. Template for the IFS drawing.