Module 4: Star Polygons

1. Introduction

Star polygons have great symbolic significance in all religions. They are also carriers of much important mathematical information. Fig. 1 shows the Sri Yantra star of the Hindu religion. In Module 1 we will see the basic role of star hexagons in projective geometry

2. A number theoretic property of star polygons

Consider the following two stars: Fig. 2a illustrates a star pentagon {5,2} while Fig. 2b is a star hexagon {6,2}. The notation {n,k} means that the star polygon has n vertices and every k-th edge is connected by an edge. Figures 2a and b differ in one important respect. Whereas, {5,2} can be drawn in a single stroke, to draw {6,2} you have to take your pencil off of the paper.

Question: By drawing a number of star polygons, can you find a simple condition between integers n and k that will guarantee the star can be drawn in a single stroke (without taking your pencil off of the paper). Drawing a variety of 8 and 12 pointed stars will be helpful in this exploration.

Construction 1: Using the circle in Fig. 3 that has 360 divisions, create an interesting star design of your own.

3. The star pentagon and star octagon

The star pentagon or pentagram as it is called {5,2} and star octagon {8,3} have particular interest (see Fig. 4a and b). The pentagram was the mystical symbol of Pythagoras' academy while the star octagons was an important symbol in Christianity and Islam. The {8,3} star was basic to the design of the Dome of the Rock, the second most important Islamic place of worship.

The ratio of the diagonal of the pentagon to its side is $\phi:1$. The diagonals of the star pentagon (see Fig. 5a) also divide themselves in the golden ratio, $\phi:1$ where $\phi=\dfrac{1+\sqrt{5}}{2}$ the golden mean.

To construct a pentagon : Begin with a line segment. Divide it into the golden section $\phi:1$ (see page 83 of <u>Connections</u>). When the side of the pentagon has length 1 its diagonal has length ϕ. It is now easy to use compass and straightedge to construct the pentagon. Try your hand at doing t his.

To construct an octagon, from each vertex of a unit square, draw an arc through the center of the square as shown in Fig. 5b intersecting the original square. The eight points where these arcs intersect the square are the vertices of a regular octagon. Each division point divides the side of the square in what is called the sacred cut, two line segments whose ratio is $\theta:1$ where $\theta = 1+\sqrt{2}$ is known as the silver mean. Therefore two diagonals of the octagon shown in Fig. 5c, intersect each other in the ratio of the sacred cut.

Remark: The golden mean ϕ is a root of the equation $x-\frac{1}{x}=1$; the silver mean θ is a root of $x-\frac{1}{x}=2$.

Remark: The golden mean ϕ and silver mean θ can be expanded as the following infinite compound fractions.

$$x=\phi=\frac{1+\sqrt{5}}{2}=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1\ddots}}} \qquad x=\theta=1+\sqrt{2}=2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2\ddots}}}$$

We will show in Module 7 how golden triangles are able to create interesting designs, and in Module 9 and 11 we will show how the golden mean was used by the architect LeCorbusier in a system known as the Modulor from which he created much of his architecture. We will also show in Module 5 how the silver mean was used to create interesting designs with a system that was the basis of the architecture of the Roman Empire. We will further explore these connections to the sacred cut in Module 11.

4. The star dodecagon

The dodecagon {12,7} is shown in Fig. 6. It is depicted as a tone circle in which all twelve tones that make up one *octave* of the *chromatic scale* of music encircle the star with the key or *fundamental* tone of the scale sitting at 12 o'clock. For this scale D is the fundamental tone and it is connected to every seventh tone going clockwise around the circle. This interval of seven tones is called a musical fifth, e.g., from D to A is a fifth because there are five letters separating them: DEFGA. You will notice that since 12 and 7 are relatively prime, the star can be drawn without taking your pencil off of the paper. In other words every tone is visited in a trip around the star. In music terminology this is called the circle of fifths. We will have more to say about music and the scale in Module 11.

5. The Brunes Star

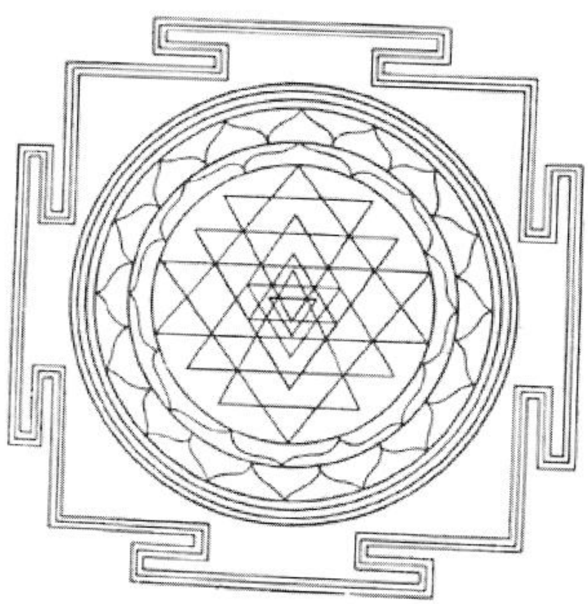
The Danish Engineer, Tons Brunes, studied the great structures of antiquity and hypothesized that two geometrical tools lay at the root of their proportional systems: 1) the sacred cut; and 2) a star-like template which I refer to as the Brunes star shown in Fig. 7 and 8.

To create a Brunes star take a square and divide it into two half squares by a vertical line that bisects the square. Place the two diagonals into each half-square. Then divide the square into two half-squares by a horizontal line that bisects the square and place the two diagonals into each half square.

The result of this construction is a remarkable structure made up, almost exclusively, of 3,4,5-right triangles at four different scales or fragments of 3,4,5-right triangles. Do you see them? In Fig. 7 Triangles ABC, ADJ, DGQ, and IDH are all 3,4,5-right triangles. You can create a 3,4,5-right triangle from a string with 12 equally spaced knots. The Brunes star can be formed by properly these four 3,4,5-right triangles into a square.

The star is also a device for equipartitioning line segments into integer lengths. Fig. 9 shows the how the Brunes star creates a nine square grid by trisecting while Fig. 10a shows the positions at which the edge length is partitioned into 2,3,4,5,6,7,8, equal parts. The sacred cut marks the division point for seven equal lines segments in Fig. 10b. Brunes found it significant that he could then subdivide a square into approximately 28 equal squares equivalent to four 7 day weeks connected with the lunar calendar as in Fig. 10c.

Construction 2: Construct a Brunes star and color the regions in such a way as to bring out its power.



Geometry through Art

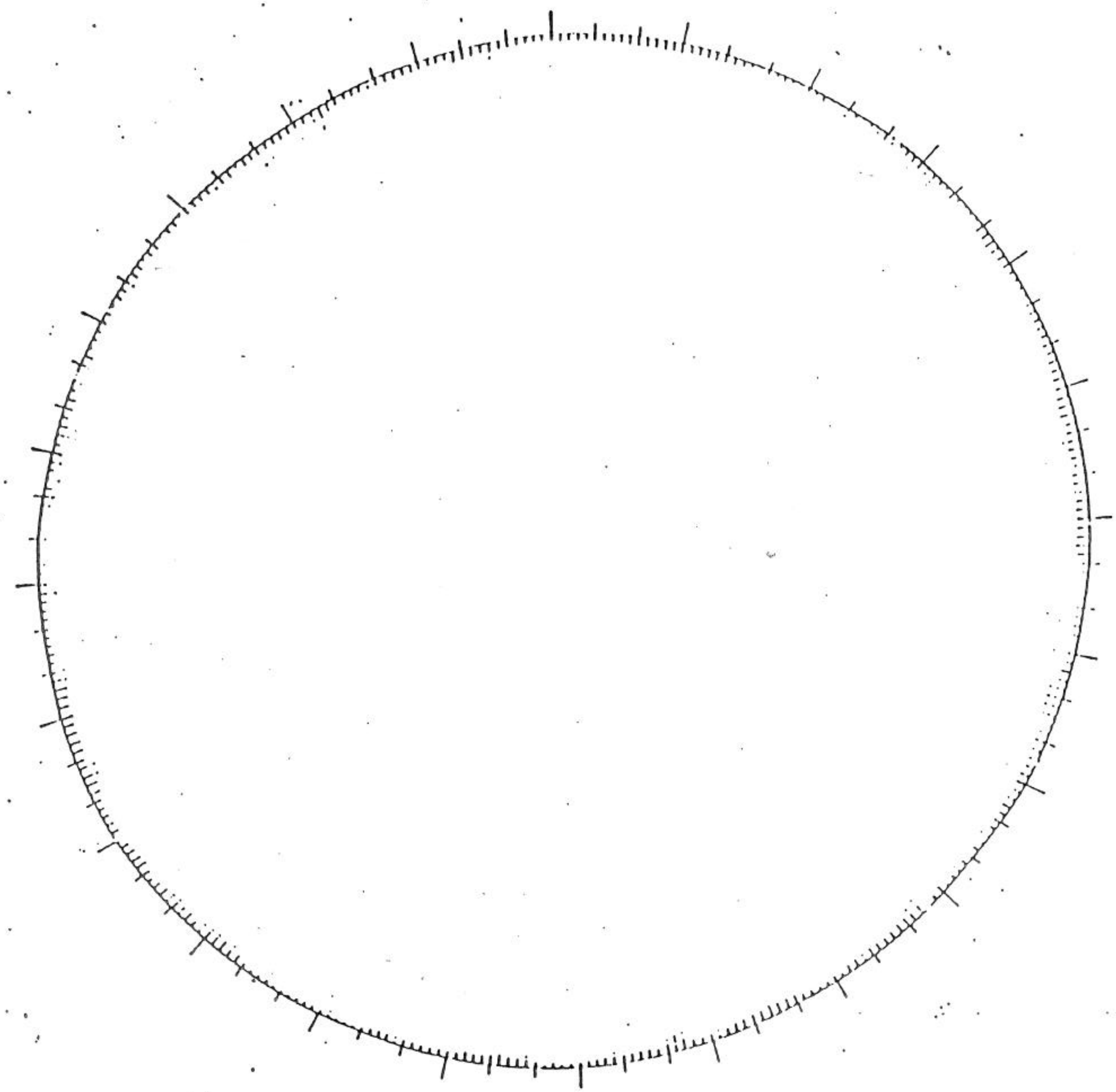
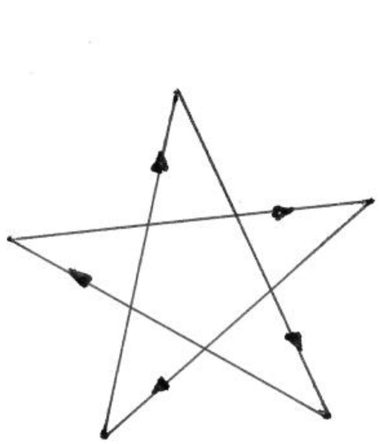
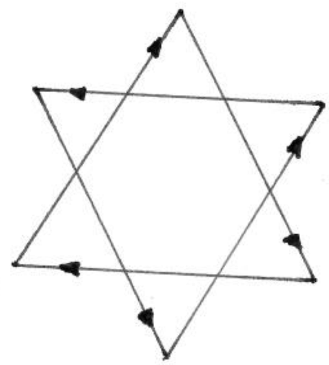


FIG. 3



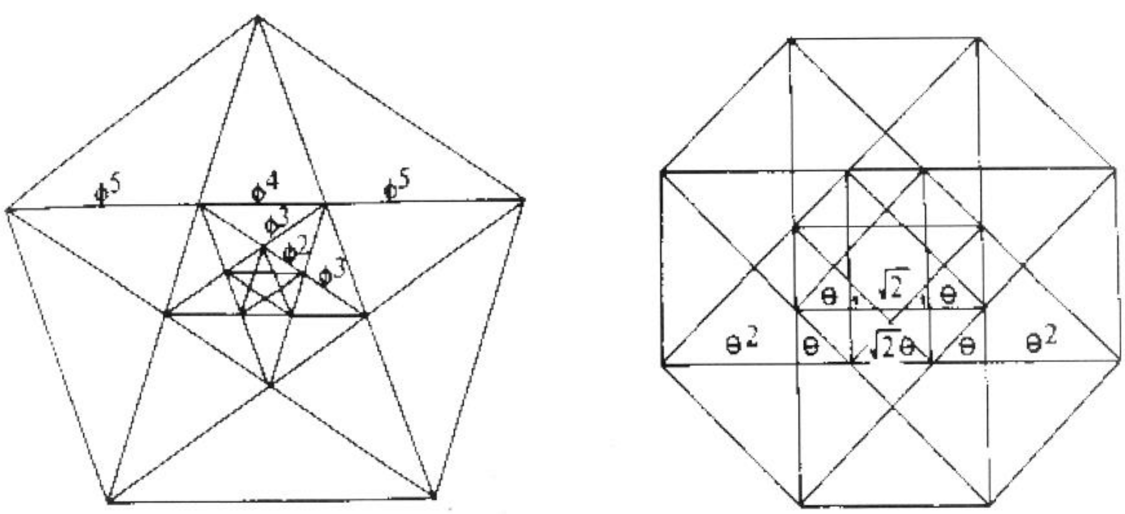
{5, 2}

FIG. 2a



{6, 2}

FIG. 2b



(a) (b)

FIG. 4

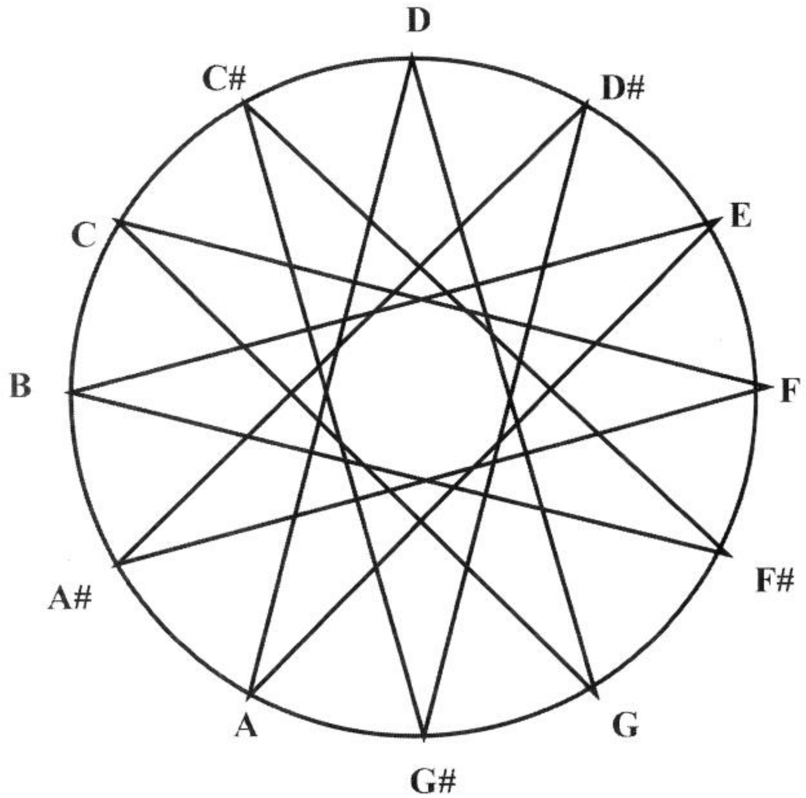
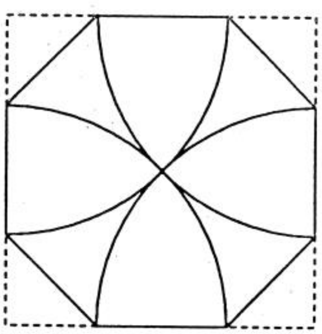


FIG. 6 {12,7} star dodecagon



Construction of the "sacred cut".

FIG. 5b

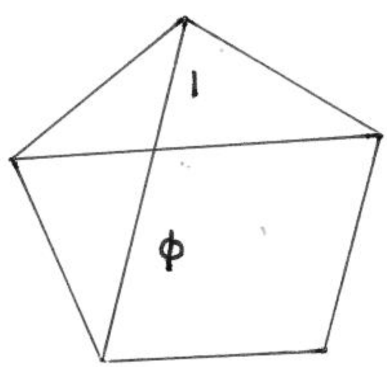


FIG. 5a

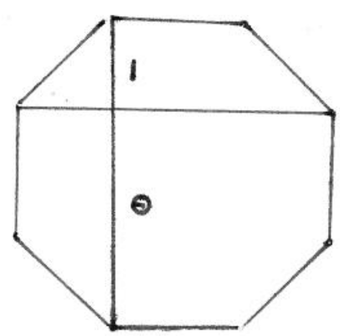
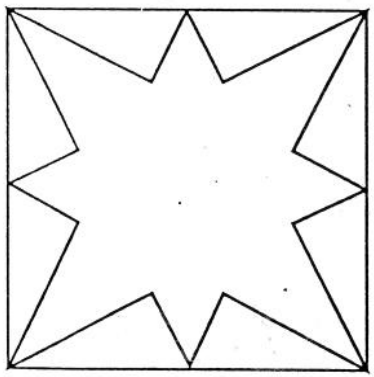
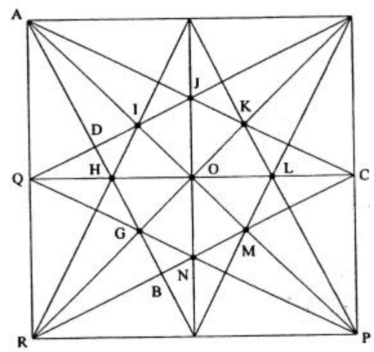


FIG. 5c



. The eight pointed star formed by the "sacred cut".

FIG. 7



The construction lines for the Brunes star.

FIG. 8

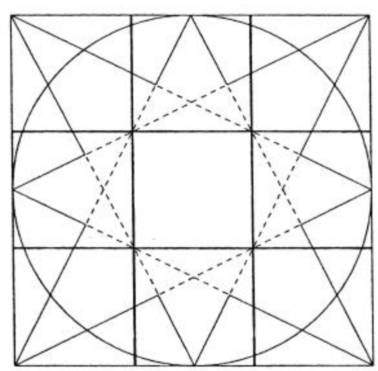
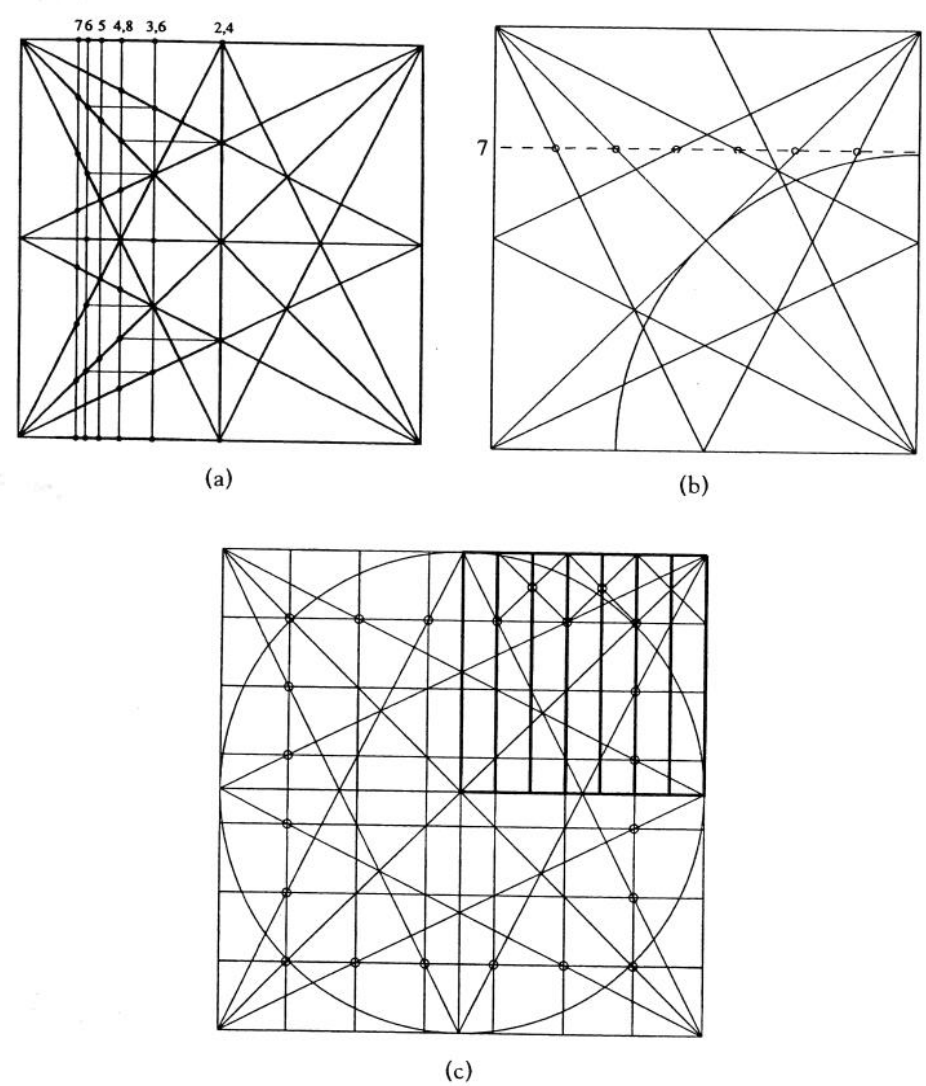


FIG. 9 The relationship between the Brunes star and the inscribed circle within a square showing how the star divides the square into a nine-square grid.



The Brunes star equipartitions a line segment into (a) 3, 4, 5, 6, 7, and 8 equal parts; (b) approximate equipartition into 7 parts; (c) Brunes's division of a square into approximately 28 equal parts.

FIG. 10