Module 6: Wythoff's Game

My own interest in the fascinating world of the golden mean began as the result of playing Wythoff's game. The game is played as follows:

> Begin with two stacks of tokens (pennies). A proper move is to remove any number of tokens from one stack or an equal number from both stacks. The winner is the person removing the last token.

The winning strategy is based on the following theorem due to S. Beatty:

Theorem: If $\frac{1}{x}+\frac{1}{y}=1$ where x and y are irrational numbers, then the sequences [x], [2x], [3x], … and [y], [2y], [3y], …together include every positive integer taken once ([ ] means "integer part of " for example [3.4] = 3).

Since $\frac{1}{\phi}+\frac{1}{\phi^2}=1$ for the golden mean , Beatty's theorem shows that $[n\phi]$ and $[n\phi^2]$ exhausts all of the natural numbers with no repetitions, as n takes on the values , n = 1,2,3,… Table 1 shows results for n = 1,2,…,6. Do you notice a pattern in these numbers that enables you to continue the table without computation? These Beatty pairs are also the winning combinations for Wythoff's game. At any move, a player can reduce the number of counters in each stack to one of the pairs of numbers in Table 1. The player who does this at each turn is assured victory.

Table 1. Winning combinations of Wythoff's game

n	$[n\phi]$	$[n\phi^2]$
1	1	2
2	3	5
3	4	7
4	6	10
5	8	13
6	9	15
.		