Module 8: Tiling a Rectangle by Congruent and Non-congruent Squares

1. **Introduction**

We first tile a rectangle by congruent squares and then develop a more complex method to tile a rectangle by non-congruent squares.

2. Tiling a rectangle with congruent squares:

A rectangle with sides in proportion 3:2 is shown in Fig. 1. Clearly, the fewest number of congruent squares that are needed to tile it is six squares of side 1 unit. On the other hand the fewest number of congruent squares needed to tile a 15:9 rectangle is fifteen squares measuring 3 units on a side (see Fig. 2).

We now ask the following question: For a rectangle of proportion b:a, what are the fewest congruent squares needed to tile the square and what is the side of the square?

The answer is that the side of the smallest triangle is the greatest common divisor of a and b, i.e., gcd(a,b). By definition, the greatest common divisor of integers a and b, gcd(a,b), is the largest integer that divides evenly into both a and b. The number of squares required to tile the square is then,

$$N= \frac{a}{\gcd(a,b)} \times \frac{b}{\gcd(a,b)}. \qquad (1)$$

Clearly, for 3 and 2, gcd(3,2) = 1 whereas gcd(15,9) = 3 so that 3 and 2 are relatively prime but 15 and 9 is not.

Remark: In Module 4 we found that gcd(n,k) =1 , i.e., n and k should be relatively prime was the condition that guarantees that a star polygon can be drawn in a single stroke without taking the pencil off of the paper. Two integers are said to be *relatively prime* if the gcd is 1.

Another quantity that is important to the theory of number is the least common multiple of a and b, i.e., lcm(a,b) where lcm (a,b) is the smallest integer which can be divided evenly by both a and b. It can be shown that

$$a \times b = lcm(a,b) \times \gcd(a,b). \qquad (2)$$

From Eq. 1 and 2 it is easy to show that the minimum number of tiles N can be elegantly expressed by the following formula, :

$$N = \frac{lcm(a,b)}{\gcd(a,b)} \qquad (3)$$

For example, lcm(3,2) = 6 and lcm(15,9) = 45. Since gcd(3,2) = 1 and gcd(15,9) = 3 we see from Eq. 1 that N = 6 for the 3:2 rectangle and N = 15 for the 15:9 rectangle.

What if you have a pair of large numbers such as the 60x27 rectangle shown in Fig. 3. How do you find the gcd? You can find the gcd by extracting squares. First you can extract two 27x27 squares with 6 units left over. Next extract four 6x6 squares, and finally two 3x3 squares can be extracted. The length of the side of the smallest square is the gcd{60,27}. This procedure is equivalent to what is called the *Euclidean algorithm* in the subject of Discrete Mathematics. If the side of the smallest square is 1 unit, then a and b are relatively prime.

Problem: Use this method of extraction to find the gcd of the following rectangles: a) 240:72 b) 55:34 Find the least number of congruent squares needed to tile the rectangles.

3. Tiling a Rectangle with Non-congruent squares

It was quite easy but uninteresting to tile a rectangle by congruent squares when the sides were integers. It is more interesting to now consider the tiling of a rectangle by a finite set of squares no two of which have the same edge length, i.e., non-congruent squares.

Since the set of squares is finite, there is a smallest square. The first problem we must confront in our tiling is where to place this smallest square. Can you see why it would be impossible to place the smallest square in a corner or along one of the edges of the rectangle as shown in Fig. 4a and b? The only other possibility is to place it within the interior of the rectangle as shown in Fig. 4c.

Next we must decide how to surround the smallest square with other squares. Fig. 4d shows how this square must be surrounded. (why?)

Now that we have decided what to do with the smallest square we can show how the rest of the tiling may be determined. Consider a rectangle cut into smaller rectangles in such a way that there is a chance of distorting all the small rectangles into squares. In doing this we must be sure that at least one rectangle (the one we distort into the smallest square) is surrounded by four rectangles as in Fig. 4d. One candidate with nine rectangles is shown in Fig 5. Either D or H could become the smallest square in this tiling. Let us choose H to be the smallest square and assign it an edge length of y units while assigning x units to square E. Fig. 6 shows a step by step assignment of side lengths to all of the other squares of the tiling in terms of x and y.

We now have an array of rectangles whose interiors have been labeled in terms of x and y so as to indicate the lengths of the sides of the squares when the rectangles are distorted to become squares. Since the left and right sides, and top and bottom of the rectangle must be equal this leads to two equations :

$$(x + 11y) + (x + 7y) = (2x + y) + (x + 2y) + (x + y) \qquad (4a)$$

$$(x + 11y) + (2x + y) = (x + 7y) + (x + 3y) + (x + 2y) \quad (4b)$$

Solving Eq 4a for x and y yields $x = 7y$, while solving Eq.4b yields an identity. Therefore we may choose a value for y to determine the dimensions of all squares in this arrangement. If $y = 1$, then $x = 7$ and the sides of the other squares can be determined to be: A = 18, B = 15, C = 14, D = 4, E = 7, F = 8, G = 10, H = 1, and I = 9 We have therefore determined the solution now drawn to scale in Fig.7.

This is a beautiful arrangement that was not achieved by guesswork. It is the result of mathematical reasoning.

How can you determine new configurations without having to create detailed breakdowns of the rectangle? This can be done by noticing that tilings can be turned into graphs. We do this by assigning letters to different heights within the tiling. For example in Fig. 7 there are six different heights labeled from a to f. Each of these heights becomes a vertex in a graph. The edge of the graph then becomes the side length of the square between two heights (vertices). In other words, there are as many vertices as there are heights in the tiling and as many edges as there are squares in the tiling. The graph corresponding to Fig. 7 is shown in Fig. 8. There is an electrical analogy here. If the side lengths are thought of as voltages, then voltages can be assigned to each vertex (in the circles) where edges in the graph are changes of voltage from one vertex to another. In this way vertex a is has a voltage equal to the length of the left and right sides of the rectangle, or 32. The voltage at level b is then 17 and level c is 14, etc. while the voltage at the bottom of the square is 0 or ground.

Fig. 9a shows another arrangement of distorted squares. Fig. 9b shows the algebraic solution, and Fig 9c shows a scale model of the tiling of this rectangle by non-congruent squares. The algebraic solution yields: $y = 5/2\ x$ where we assign $x = 2$ and $y = 5$. The other square lengths follow from this.

Problem: Draw the graph for the tiling in Fig. 9c.

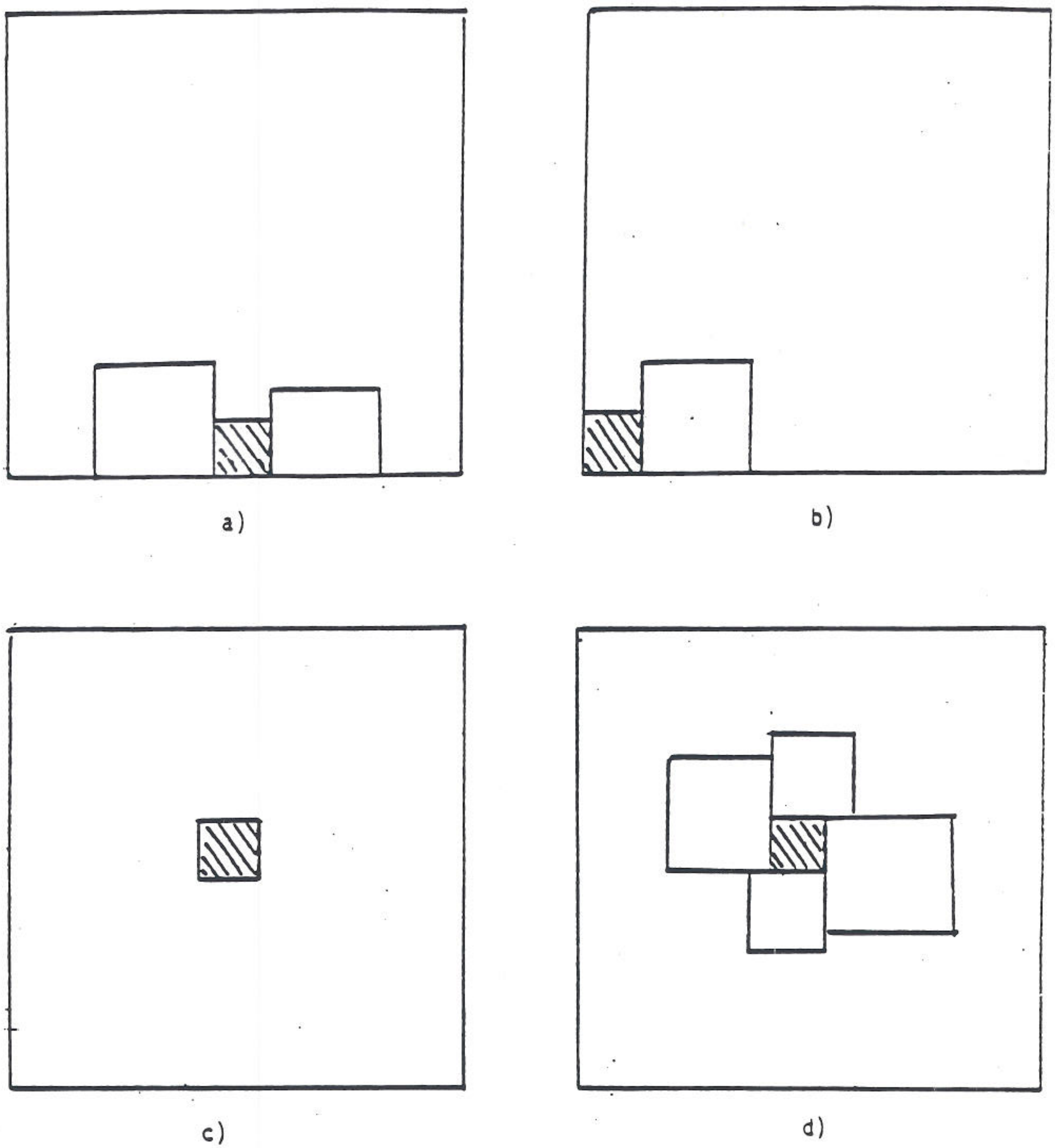
Fig. 10a and 10b and 11a and 11b are tilings by 13 non-congruent squares and their corresponding graphs for two different tilings with the same set of squares.

Construction: Distort the configuration of 9 rectangles shown in Fig. 12 so that the outer rectangle is tiled by non-congruent squares. First decide which rectangle will be transformed into the smallest square, then follow the procedure outlined above. After determining the size of the squares, carry out the tiling with the care of an artist or designer. Color your tilings using the fewest number of colors possible so that no two squares sharing the same edge are the same color. Draw the graph corresponding to this tiling. Can you rearrange the squares in your tiling so as to tile the outer rectangle in a different way?

Remark: There are no solutions to the tiling of a rectangle by non-congruent squares with fewer than 9 squares. There are two solutions with exactly 9 squares, six with 10 squares (four of which can be obtained by annexing a square to one side of a 9-square solution); and 22 with 11 squares (12 of which are a direct result of 10-square solutions)

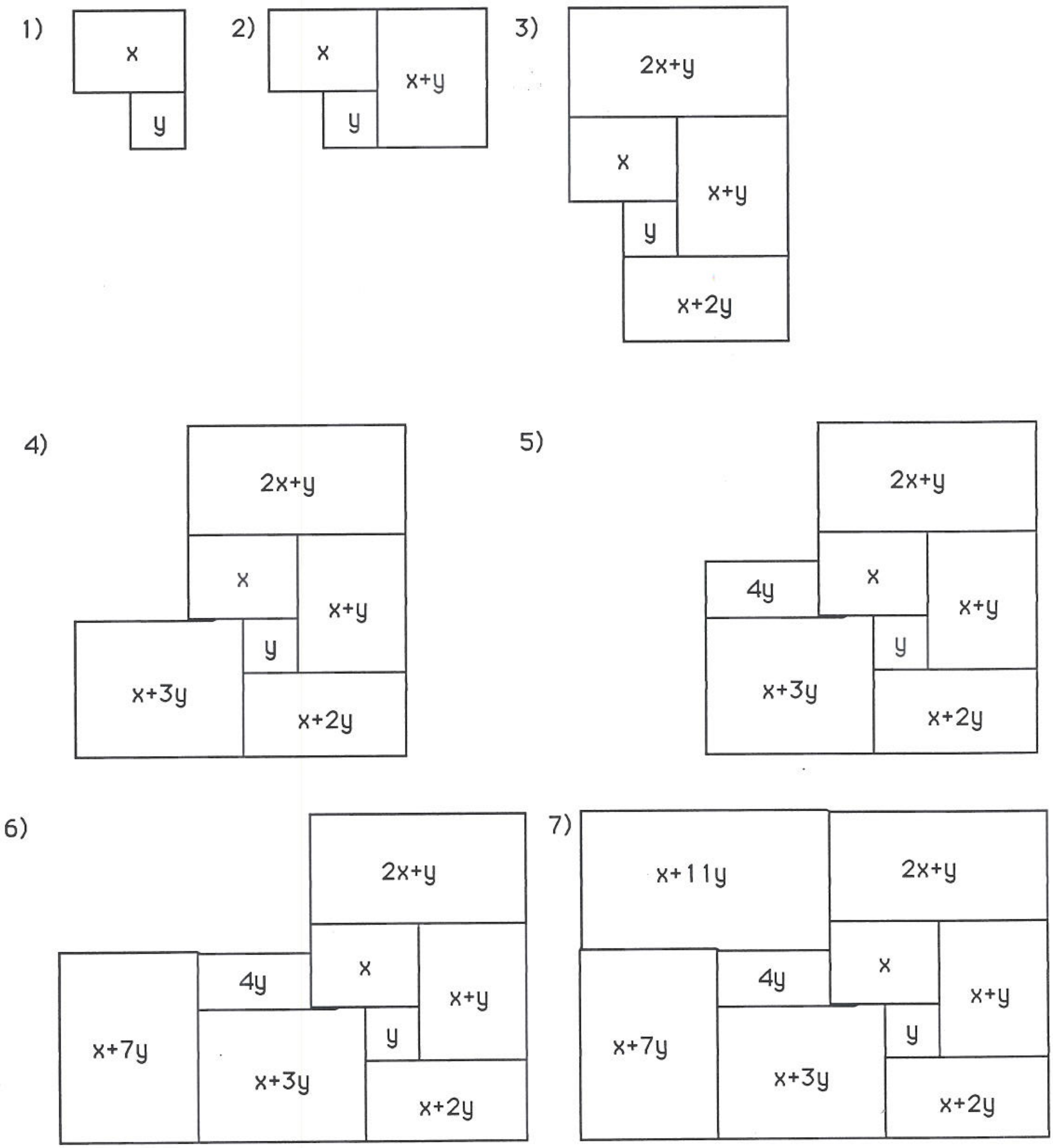
Try your hand at discovering a tiling not represented here. Be forewarned that many diagrams will result in impossible results; both variables will be equal to 0, or one variable or variable expression will be negative.

Problems: Three additional possibilities are given in Fig. 13. Find the edge lengths of the squares in Fig. 13 a and b. Draw the graph for the tiling in Fig. 13c.



Smallest square a) cannot be placed along an edge
b) cannot be placed in a corner
c) must be placed within the parallelogram
d) a possible configuration of non-congruent squares surrounding the smallest square

FIG. 4


1)
x
y
2)
x
x+y
y
3)
2x+y
x
x+y
y
x+2y
4)
2x+y
x
x+y
x+3y
y
x+2y
5)
2x+y
4y
x
x+y
y
x+3y
x+2y
6)
2x+y
4y
x
x+y
x+7y
y
x+3y
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7)
x+11y
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FIG. 6

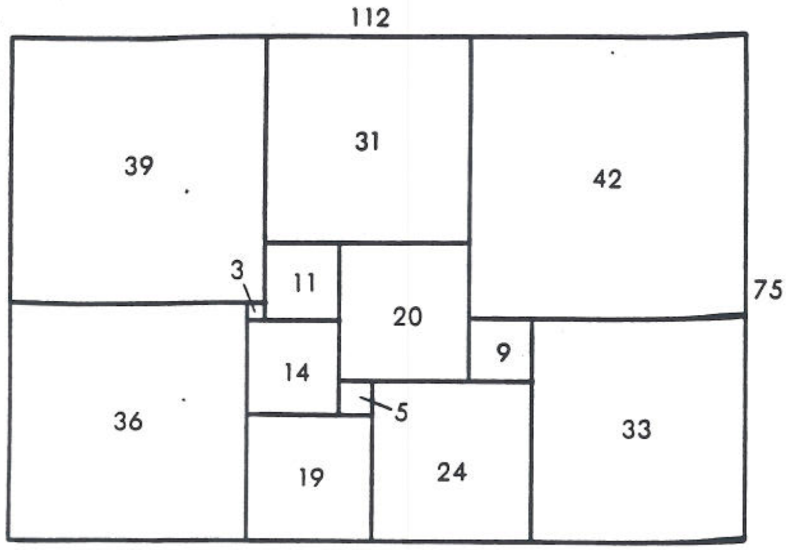

FIG. 10a

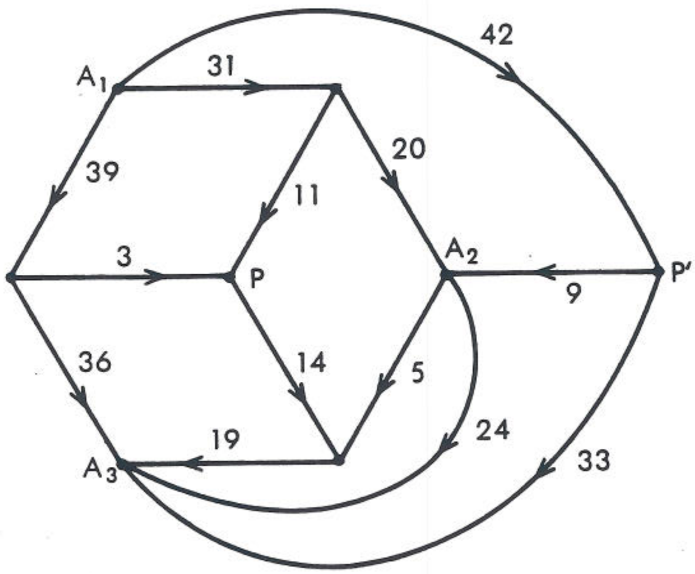

FIG. 10b

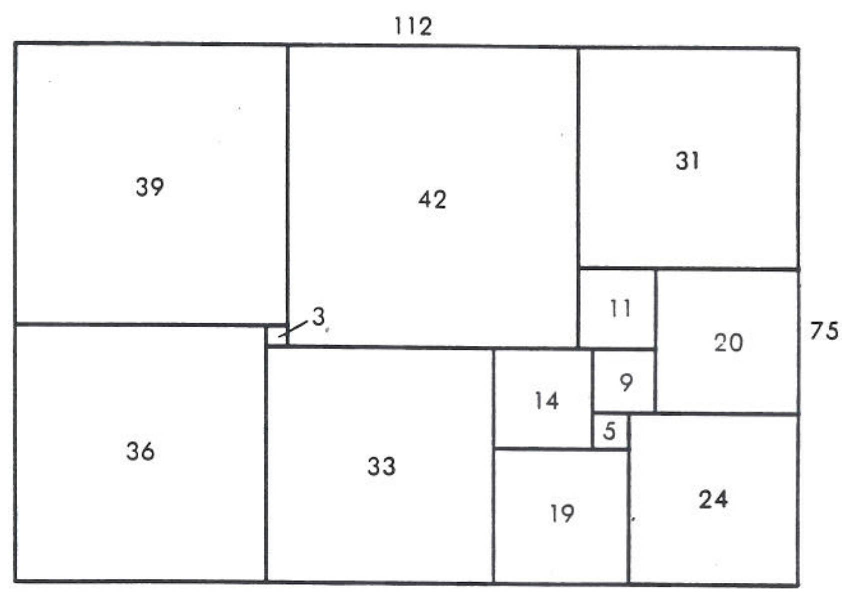

FIG. 11a

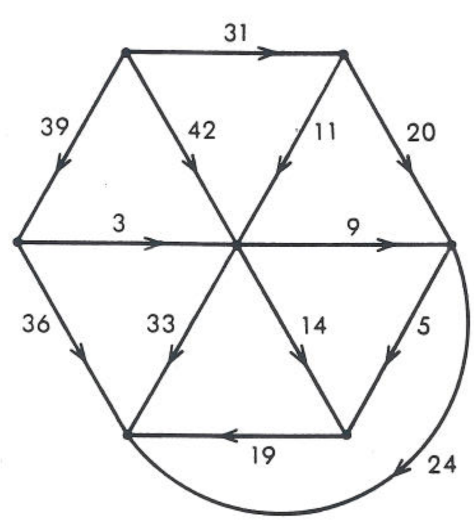

FIG. 11b

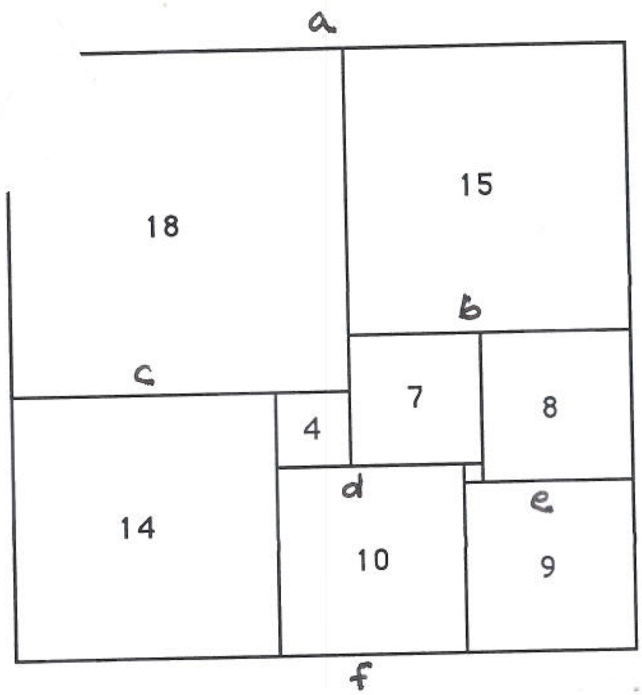

FIG. 7

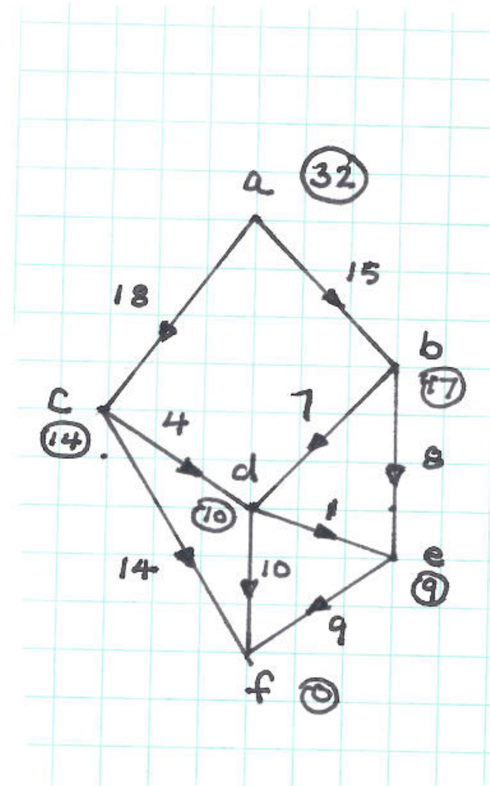

FIG. 8

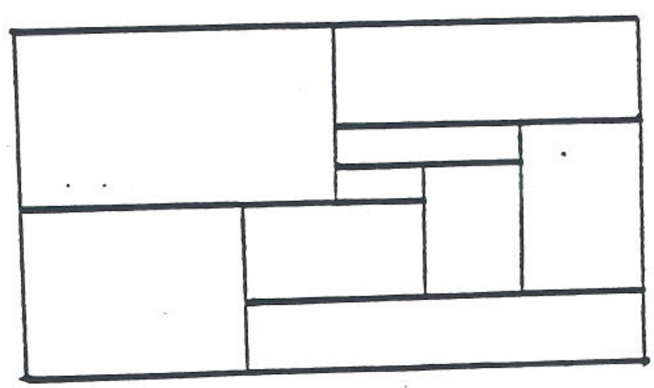

FIG. 12

a)

b)

c)

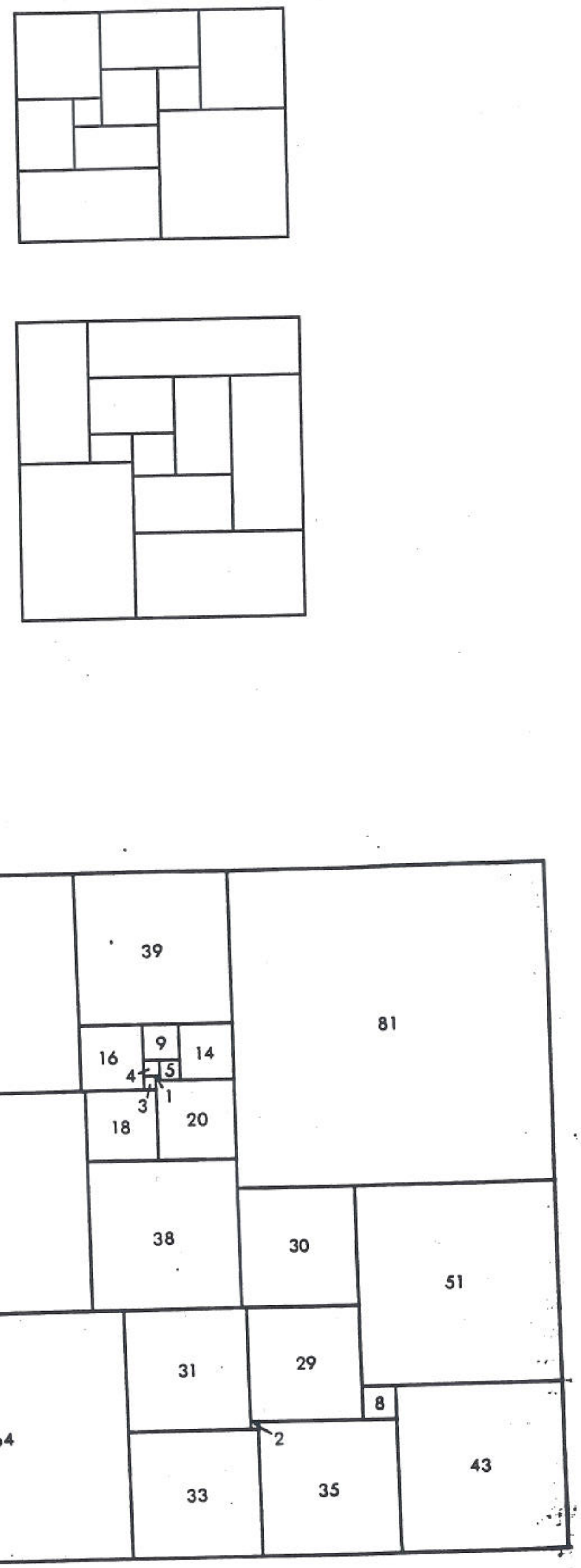

FIG. 13