CHAPTER 1

LINES

1.1 LINES

A line is the easiest mathematical structure to describe. You need to know only two things about a line to describe it. For example:

i) the y-intercept, b, and the slope, m;
ii) one point on the line and the slope;
iii) two points on the line.

i) The equation of a line takes the form : y = mx + b as shown in Fig. 1 below where the *slope* m is defined as,

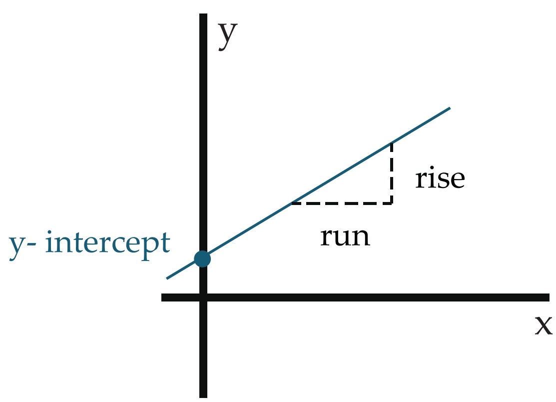
m = rise/run

ii) Here we use the *point-slope* form of the line: $y - y_1 = m (x - x_1)$
Where the point on the line is (x_1,y_1) and m is the slope.

iii) Here we first use the two points: (x_1,y_1) and (x_2,y_2) to find the slope:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Then we use the point-slope formula with either of the two points to find the equation of the line.



$$m = \frac{\text{rise}}{\text{run}}$$

Fig. 1

Example 1: For the line, 3x + 2y = 6, find the slope m and the y-intercept b.

Solution: Solve for y in terms of x: y = -3/2 x + 3 so that m = -3/2 and b = 3.

Example 2: Find the equation of the line that goes through the point (2, -5) and has a slope of m = 3.
Solution: By point slope formula: y – (-5) = 3 (x – 2) or y + 5 = 3(x-2). Can you find the y-intercept?

Example 3: Find the equation that goes through the points: (1,3) and (-2,5).

Solution: First find the slope: $m = \dfrac{5-3}{(-2)-1}$ = -2/3

Taking the first point: y – 3 = $-\dfrac{2}{3}$(x-1)

Problems:

1. Determine the slope and y-intercept of:
   a. 7y + 12x -2 = 0, b. 12x = 6y + 4, c. -4y + 2x + 8 = 0
2. Find the equation of the line that passes through the two points:

   a. (0,2) and (2,3) , b. (4,5) and (2,-1), c. (0,0) and (1,2)

1.2 LINES AND CALCULUS

What makes calculus work is that any curve, however complicated, locally is approximated by a line with the approximation getting better and better the closer to the point you get provided that the curves are for the most part smooth (they have only a finite number of kinks). For example Fig. 2a is a smooth curve and notice that the nearer you get to any point on it the more the curve looks like a line. So calculus reduces the study of curves to the study of lines, the simplest mathematical entity.

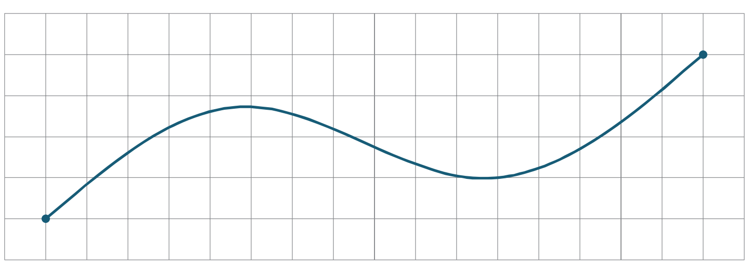


Fig. 2a

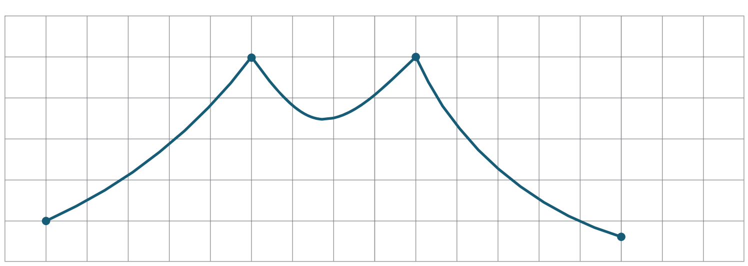


Fig. 2b

Fig. 2b is the graph of a curve that is smooth everywhere but at two points, but in the neighborhood of each of the other points, the curve looks locally like a line. Calculus deals only with mostly smooth curves. However, there are very important curves which are nowhere smooth. Many of these curves are called “fractals.” Fractals better describe forms in nature such as the shape of coastlines, cloud formations, and lightening bolts. On the next page we exhibit one such fractal, the Koch curve. It is nowhere smooth. In fact, the length of the curve between any two points, however close, is infinity. Also notice the self-similarity; each segment of the curve looks like the whole.

I have drawn the first three stages of a simple fractal in Fig. 3. I begin with a straight line in stage 0. In stage 1, I place a triangular roof on top of the line. In stage 2 and 3 wherever I see a straight line segment, I place a triangular roof atop it. Try your hand at drawing several more stages to this fractal. Continue as far as your eye is able to discriminate lines segments. There is no need to redraw the curve for each stage. You can just add each successive stage to your original line. If you were able to continue this ad infinitum you would get a fractal curve that is nowhere smooth.

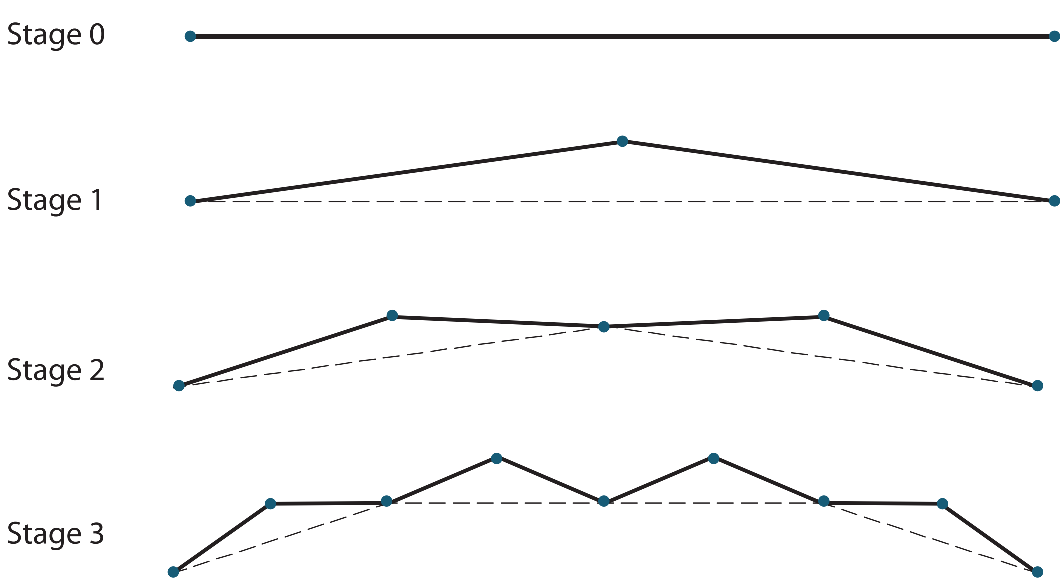


Fig. 3

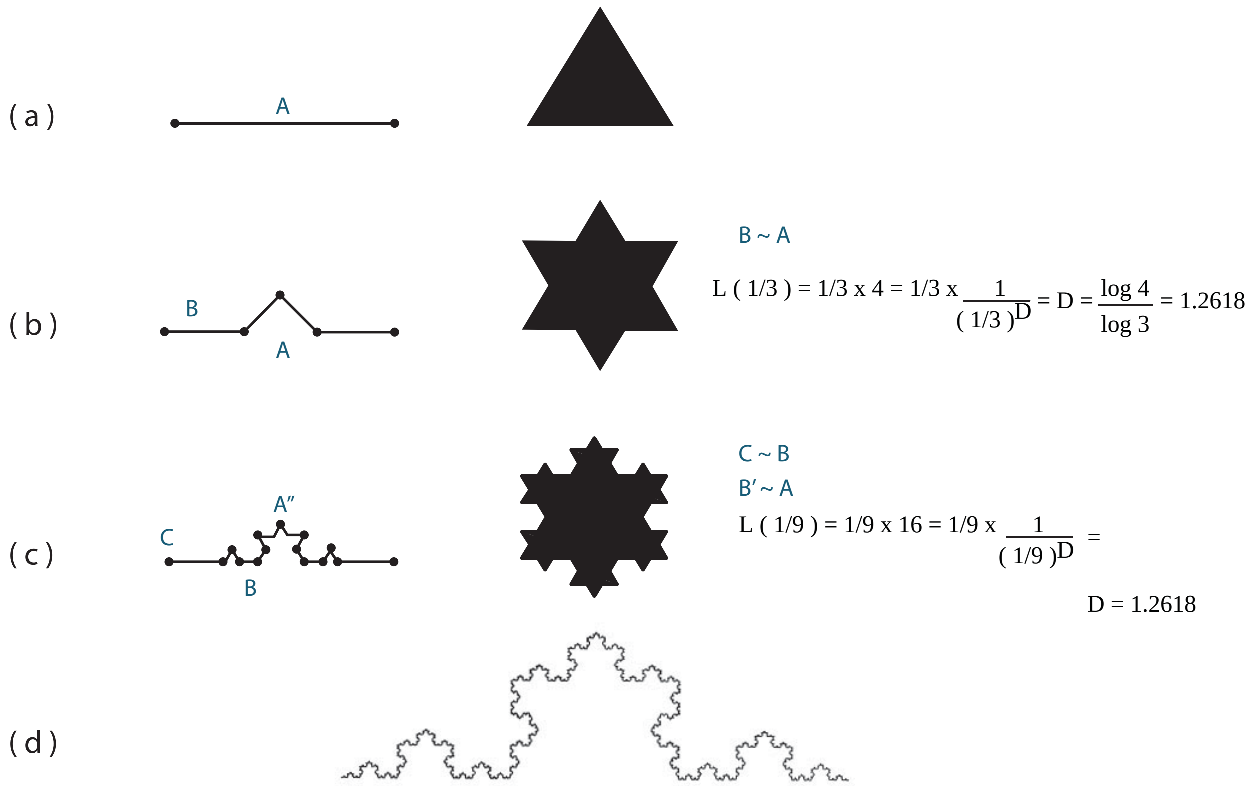


Fig. 4 The Koch snowflake, a nontrivial example of a self - similar curve with dimension D = 1.2618. (a) Koch snowflake at scale of r = 1 ; (b) Koch snowflakeat scale of r = 1/3 ; (c) Koch snowflake at scale of r = 1/9 ; (d) Koch snowflake at an advanced stage in its generation.

Fig. 4

1.3 PROBLEMS

The following problems are meant to help you become familiar with the mathematics of lines. The first group of problems deal with the mechanics of lines, the second incorporate lines into the solution of realistic problems often referred to as “*word problems*.” The solution of word problems presents the student with difficulties that go beyond the mechanical. I will solve the first word problem for you and in the process develop a procedure for solving all word problems.

Gas Tank Problem

Suppose that you get your car’s gas tank filled up, then drive off down the highway. As you drive, the number of minutes, t, since you left the gas station and the number of liters, g, of gas left in the tank are related by a linear function.

a. Which variable should be dependent and which independent?
b. After 40 minutes you have 52 liters left, and after an hour you have 40 liters left. Find the slope o the function, and tell what it represents in the real world. Explain the significance of the slope.
c. Write the equation for this function. Place the independent variable on the abscissa (x-axis) and the dependent variable on the ordinate (y-axis).
d. From the equation, find the two intercepts and tell what each represents in the real world.
e. Plot the graph of this function.
f. What are the domain and range of the linear function?

First read the problem. The procedure for the solution of all word problems follows these three phases of problem solving:

1. **Entry phase**:

Reread the problem and extract the given information and display it in figures, graphs, and tables as in Fig. 5 .State clearly what the problem asks you to find. List any assumptions. Define the symbols and state their units.

Let g be the amount in liters of gas remaining in tank
t is time in minutes since last time you filled up.

When t = 40 , g = 52
t = 60, g = 40

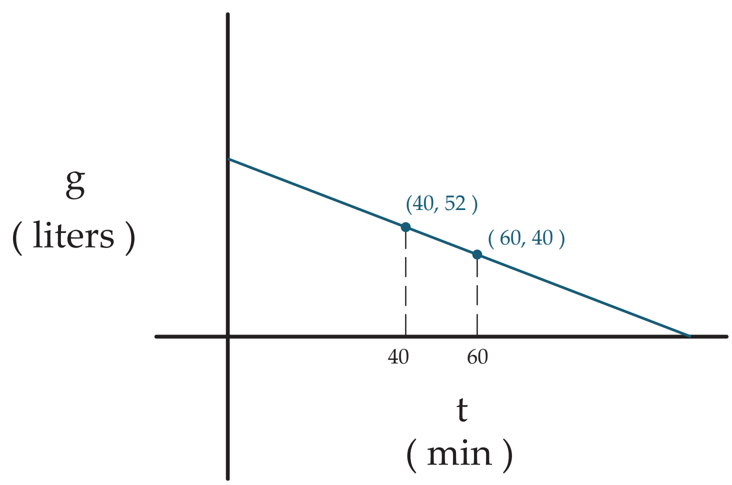


Fig. 5

Find: g when t = 0 (g intercept)
t when g = 0 (t intercept)

2. **Attack phase**

State all equations involving the variables of the problem, i.e., t and g. Decide which is the independent variable (t in this problem) and which is the dependent variable (g in this problem).

Point slope formula for lines:

$$g - 40 = m\,(t - 60)$$

$$m = \frac{52 - 40}{40 - 60} = -0.6 \text{ liters per minute}$$

The slope represents the rate at which gs is being used up. The negative sign means that the tank is emptying.

Therefore,

$$g - 40 = -0.6\,(t - 60) \qquad (1)$$

This equation is known as *the mathematical model*. All questions concerning this problem can be answered from this equation. For example,

Find g intercept. Set t = 0 in Eq. 1.

$$g - 40 = 36 \quad \text{or } g = 76 \text{ liters.}$$

This is the total number of liters in the tank after it was filled at t = 0.

Find the t intercept. Set g = 0 in Eq. 1.

$$-40 = -0.6t + 36 \quad \text{or} \quad t = 126.65 \text{ minutes.}$$

This is how long the car is able to travel before the gas tank is empty.

3. **Review Phase:**

Check the results of the problem to see if they make sense. In this case the results seem reasonable.

Problems

For Problems 1–4, determine the slope and the y-intercept of the line whose equation is given.

1. $7y + 12x - 2 = 0$

2. $3x + 2y = 8$

3. $12x = 6y + 4$

4. $-4y + 2x + 8 = 0$

For Problems 5–8, find the equation of the line that passes through the given points.

5. $(0, 2)$ and $(2, 3)$

6. $(0, 0)$ and $(1, 1)$

7. $(-2, 1)$ and $(2, 3)$

8. $(4, 5)$ and $(2, -1)$

9. Figure 1.19 shows four lines given by equation $y = b + mx$. Match the lines to the conditions on the parameters m and b.

(a) $m > 0, b > 0$ **(b)** $m < 0, b > 0$
(c) $m > 0, b < 0$ **(d)** $m < 0, b < 0$

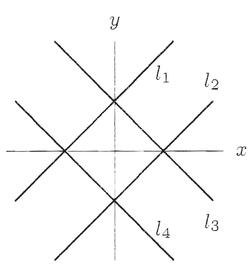


Figure 1.19

10. **(a)** Which two lines in Figure 1.20 have the same slope? Of these two lines, which has the larger y-intercept?
(b) Which two lines have the same y-intercept? Of these two lines, which has the larger slope?

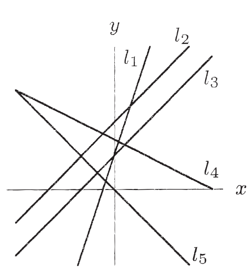


Figure 1.20

11. Match the graphs in Figure 1.21 with the following equations. (Note that the x and y scales may be unequal.)

(a) $y = x - 5$ **(b)** $-3x + 4 = y$
(c) $5 = y$ **(d)** $y = -4x - 5$
(e) $y = x + 6$ **(f)** $y = x/2$

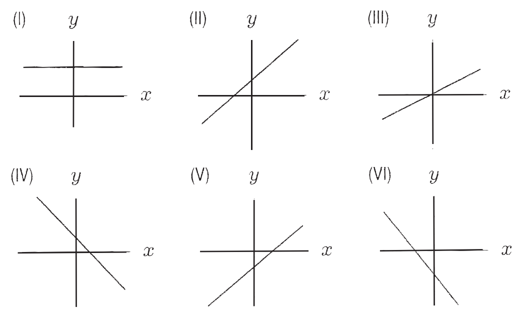


Figure 1.21

12. A city's population was 30,700 in the year 2000 and is growing by 850 people a year.

(a) Give a formula for the city's population, P, as a function of the number of years, t, since 2000.
(b) What is the population predicted to be in 2010?
(c) When is the population expected to reach 45,000?

13. A cell phone company charges a monthly fee of \$25 plus \$0.05 per minute. Find a formula for the monthly charge, C, in dollars, as a function of the number of minutes, m, the phone is used during the month.

14. A company rents cars at \$40 a day and 15 cents a mile. Its competitor's cars are \$50 a day and 10 cents a mile.

(a) For each company, give a formula for the cost of renting a car for a day as a function of the distance traveled.
(b) On the same axes, graph both functions.
(c) How should you decide which company is cheaper?

15. Which of the following tables could represent linear functions?

(a)

x	0	1	2	3
y	27	25	23	21

(b)

t	15	20	25	30
s	62	72	82	92

(c)

u	1	2	3	4
w	5	10	18	28

16. For each table in Problem 15 that could represent a linear function find a formula for that function

17. A company's pricing schedule in Table 1.3 is designed to encourage large orders. (A gross is 12 dozen.) Find a formula for:

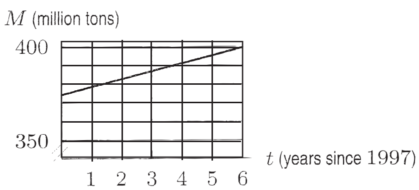
(a) q as a linear function of p.
(b) p as a linear function of q.

Table 1.3

q (order size, gross)	3	4	5	6
p (price/dozen)	15	12	9	6

18. World milk production rose at an approximately constant rate between 1997 and 2003.[7] See Figure 1.22.

(a) Estimate the vertical intercept and interpret it in terms of milk production.
(b) Estimate the slope and interpret it in terms of milk production.
(c) Give an approximate formula for milk production, M, as a function of t.



19. Figure 1.23 shows the distance from home, in miles, of a person on a 5-hour trip.

(a) Estimate the vertical intercept. Give units and interpret it in terms of distance from home.
(b) Estimate the slope of this linear function. Give units, and interpret it in terms of distance from home.
(c) Give a formula for distance, D, from home as a function of time, t in hours.

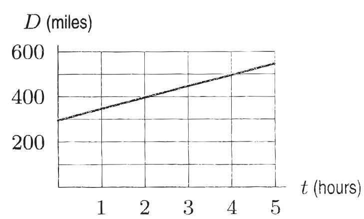


Figure 1 23

Figure 1.22

20. Search and rescue teams work to find lost hikers. Members of the search team separate and walk parallel to one another through the area to be searched. Table 1.4 shows the percent, P, of lost individuals found for various separation distances, d, of the searchers.[8]

Table 1.4

Separation distance d (ft)	20	40	60	80	100
Approximate percent found, P	90	80	70	60	50

(a) Explain how you know that the percent found, P, could be a linear function of separation distance, d.
(b) Find P as a linear function of d.
(c) What is the slope of the function? Give units and interpret the answer.
(d) What are the vertical and horizontal intercepts of the function? Give units and interpret the answers.

21. Table 1.5 gives the average weight, w, in pounds, of American men in their sixties for various heights, h, in inches.[9]

(a) How do you know that the data in this table could represent a linear function?
(b) Find weight, w, as a linear function of height, h. What is the slope of the line? What are the units for the slope?
(c) Find height, h, as a linear function of weight, w. What is the slope of the line? What are the units for the slope?

Table 1.5

h (inches)	68	69	70	71	72	73	74	75
w (pounds)	166	171	176	181	186	191	196	201

[7] *Statistical Abstracts of the US 2004–2005*, Table 1355.
[8] From *An Experimental Analysis of Grid Sweep Searching*, by J. Wartes (Explorer Search and Rescue, Western Region, 1974).
[9] Adapted from "Average Weight of Americans by Height and Age," *The World Almanac* (New Jersey: Funk and Wagnalls, 1992), p. 956.

Word problems

1. ***John and Mary run a race*** . John runs at a rate of 5 feet/sec. while Mary runs at 3 ft./sec. If Mary gets a 40 foot head start and the finish line is a t 95 feet from the starting line, who wins the race? What if the finish line is at 110 feet? Who wins? Next consider the finish line at 150 feet. How fast should the loser go in order to just win that race?
   a. Make a table
   b. Draw graphs of John and Mary's progress
   c. Use algebra

Make a table:

time (sec)	Johns Distance (ft.)	Mary's distance (ft.)
0	0	40
1	5	43
	ect.	

2. ***Speed on a hill problem***. Assume that the maximum speed your car will go is a linear function of the steepness of the hill it is going up or down. Suppose that the car can go a maximum of 55 mph up a 5 degree hill, and a maximum of 104 mph down a 2 degree hill. (Going downhill can be thought of as going up a hill of – 2degree.)
   a. Write the particular equation expressing maximum speed in terms of steepness.
   b. How fast could you go down a 7 deg. Hill?
   c. If your top speed is 83 mph, how steep is the hill? Is it up or down? Justify your answer.
   d. What does the steepness-intercept, and what does it represent?
   e. Sketch the graph of this function, using a reasonable domain.

3. ***Gas Tank Problem***. Suppose that you get your car's gas tank filled up, then drive off down the highway. As you drive, the number of minutes, t, since you left the gas station and the number of liters, g, of gas left in the tank are related by a linear function.
   a. Which variable should be dependent and which independent?
   b. After 40 minutes you have 52 liters left, and after an hour you have 40 liters left. Find the slope of the function, and tell what it represents in the real world. Explain the significance of the sign of the slope.
   c. Write the particular equation for this function,
   d. From the equation, find the two intercepts and tell what each represents in the real world.
   e. Plot the graph of this function.
   f. What are the domain and range of the linear function?

4. ***Terminal Velocity Problem***. If you jump out of an airplane at high altitude but do not open your parachute, you will soon fall at a constant velocity called your "terminal velocity." Suppose that at time $t = 0$ you jump. When $t = 15$ seconds, your wrist altimeter shows that your distance from the ground, d, is *3600* meters. When $t = 35$, you have dropped to $d = 2400$ meters. Assume that you are at your terminal velocity by time $t = 15$.
   a. Explain why d varies linearly with t after you have reached your terminal velocity.
   b. Write the particular equation expressing d in terms of t.
   c. If you neglect to open your parachute, when will you hit the ground?
   d. According to your mathematical model, how high was the airplane when you jumped?
   e. The place was actually at 4200 meters when you jumped. How do you reconcile this fact with your answer to part d?
   f. Plot the graph of d versus t. Show what the actual graph looks like for values of t less than 15. Also, show what the graph would look like for values of t after you open your parachute.
   g. What is your terminal velocity in meters per second? In kilometers per hour?

5. ***Linear Depreciation Problem***. Suppose that you own a car which is presently 40 months old. From automobile dealer's "Blue Book" you find that its present trade-in value is \$1650. From an old Blue Book, you find that it had a trade-in value of \$2350 ten months ago. Assume that its value decreases linearly with time.
   a. Write the particular equation expressing the trade in value of your car as a function of its age in months.
   b. You plan to get rid of the car before its trade-in value drops below \$500. How longer can you keep the car?
   c. By how many dollars does the car "depreciate" (decrease in value) each month? What part of the mathematical model tells you this?
   d. When do you predict your car will be worthless? What part of the mathematical model tells you this?
   e. According to your model, what was the car's trade-in value when it was new?
   f. If the car actually cost \$5280 when it was new how would you explain the difference between this number and your answer to part e?

6. ***Shoe Size Problem.*** The size of shoe a person needs varies linearly with the length of his or her foot. The smallest adult shoe is Size 5, and it fits a 9 inch long foot. An 11 inch foot requires a Size 11 shoe.
   a. Write the particular equation which expresses shoe size in terms of foot length.
   b. If your foot is a foot long, what size shoe do you need?
   c. Lebron James of the Cleveland Cavaliers wears a Size 11 shoe. How long is his foot?
   d. Plot a graph of adult shoe size versus foot length uses the given points and calculated points. Be sure that the domain is consistent with the information in this problem.