CHAPTER 11

SKETCHING CURVES I

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11.1 INTRODUCTION

In the early grades you drew the graph of a function y = f(x) by making a table of x and y. To get an idea of how the function looked you needed to plot many points. Even doing this, it is possible that you may not have chosen points where the function was most interesting. We will show in this lesson how use derivatives to get a complete view of the function by graphing only a few key points.

11.2 SKETCHING A PARABOLA

Example 1

a) Sketching a parabola

Consider the parabola,

$$y = x^2 - 2x - 2\,. \qquad (1)$$

We will sketch the graph of this function in two ways.

a) In Section 2.5 we graphed this equation by completing the square,

$$y = (x-1)^2 - 3$$

Rewriting this as ,

$$y + 3 = (x-1)^2$$

Since the coefficient of x is positive, the parabola is concave up with its vertex (minimum point) at (1, -3). We can also anchor the curve by computing an easy value such as (0,-2). The graph of the parabola is shown in Fig. 1.

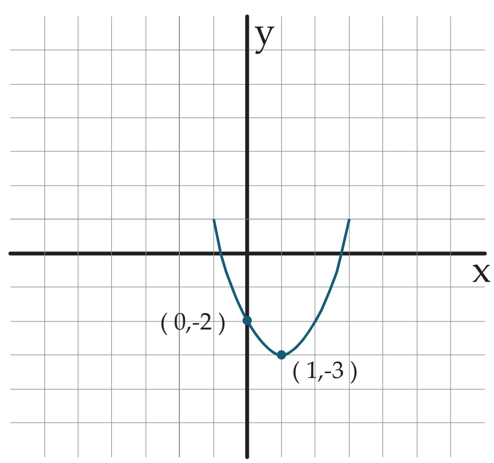

Fig. 1

a) Now that we know how to differentiate functions, we can graph the parabola in another way.

First find a point that is easily obtainable, e.g., from Eq. 1 when x = 0, y = 2. Place this point on the graph in Fig. 2.

Next find out where the function is heading when x approaches infinity, in other words find what happens to the y values when the x values get either very large or very small. To do this notice that when x gets very large or small only the leading term of the function in Eq. 1 matters. The other terms are insignificant, i.e.,

$$x^2 - 2x - 2 \approx x^2 .$$

For large or small values of x, the y values will be large, i.e.,

$$y \to \infty \quad \text{as} \quad x \to \pm\infty$$

Indicate this by placing marks on the graph for large and small values of x as shown in Fig. 2.

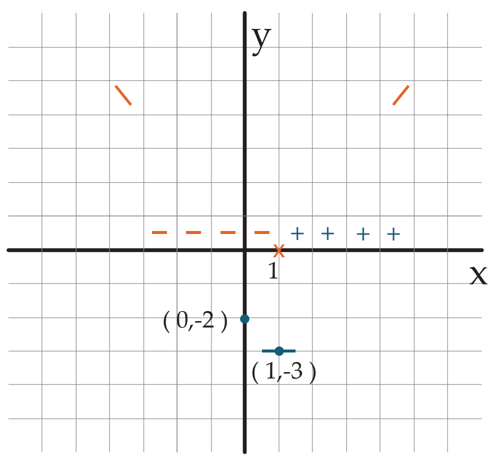

Fig. 2

Next find the critical points of the function, i.e. , the values of x where $\frac{dy}{dx} = 0$. We must first compute the derivative of the function in Eq. 1,

$$\frac{dy}{dx} = 2x - 2 = 2(x-1) \qquad (2)$$

Therefore x = 1 is a critical point ($\frac{dy}{dx} = 0$). Determine the value of y from Eq. 1 at the critical point, y = -3 when x = 1, and place a small horizontal line segment through this horizon line point in Fig. 2. The curve must come through this point tangent to the horizontal line segment. Next place a mark on the x-axis at x = 1 in Fig. 2 . From Eq. 2, $\frac{dy}{dx} > 0$ for $x > 1$ while $\frac{dy}{dx} < 0$

$x > 1$ while $\frac{dy}{dx} < 0$ for $x < 1$. To signify this, place + signs to the right of x = 1 and – signs to the left in Fig. 2 as we have done below..

$$\begin{array}{ccccc} & \text{---------------------------} & & \text{+++++++++++++++} & \\ -\infty & \underline{\hspace{6cm}} & 1 & \underline{\hspace{6cm}} & \infty \quad \text{x-axis} \end{array}$$

The curve decreases over the x values with ‘ –‘signs until it hits the small horizontal line segment at (1,-3), after which it increases for the x-values with ‘+’ signs. At this point we can get a good sketch of the curve as shown in Fig. 1 with the critical point (1,-3) identified as a relative minimum (m)... .

Sometimes we are able to get more details about the graph of the function by computing the second derivative,

$$\frac{d^2y}{dx^2} = 2$$

We see that the 2nd derivative is never zero so there are no points of inflection. On the other hand, since $\frac{d^2y}{dx^2} > 0$ the curve is concave up for all values of x. as Fig. 1 shows. .

11.3 SKETCHING HIGHER DEGREE POLYNOMIALS

We placed a great deal of effort into sketching the parabola in the last section. However, all polynomials and, for that matter most other continuous curves, can be sketched by using the following step by step procedure.

Example 2: Consider the third degree polynomial,

$$y = x^3 - 3x + 1 \qquad (3)$$

a) Evaluate the function at an easy point, e.g., x = 0, y = 1. Place this point on Fig. 4.
b) Find the behavior of the function for large and small values of x. For these values,

$$x^3 - 3x + 1 \approx x^3$$

Therefore if x is a large positive number, then y is a large positive number; when x is a large negative number, y is a large negative number, i.e.,

$$y \to \infty \quad \text{as} \quad x \to \infty, \quad \text{and} \quad y \to -\infty \quad \text{as} \quad x \to -\infty$$

. Place marks in Fig. 3 to illustrate this behavior.

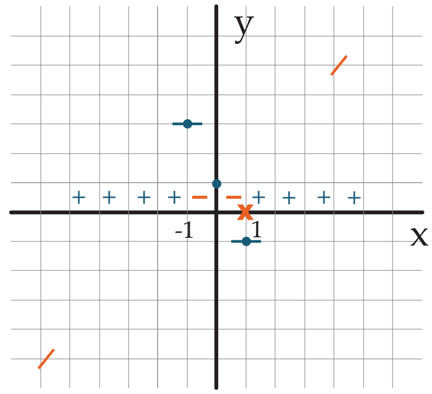

Fig. 3

c). Compute the derivative and factor it. .

$$\frac{dy}{dx} = 3x^2 - 3 \quad (4)$$

$$= 3(x^2 - 1)$$

$$\frac{dy}{dx} = 3(x-1)(x+1) \quad (5)$$

d) Find the critical points and their corresponding y values..

$\frac{dy}{dx} = 0$ at x = 1, y = -1 and x = -1, y = 3.

Graph the (x,y) coordinates of the critical points i.e., (-1,3) and (1,-1) and place them in Fig. 4. Draw small horizontal line segments through these points.

e) Place marks on the x-axis at the locations, of the critical points (x = -1 and x = 1).. This divides the x-axis into a set of intervals.

The intervals are : $-\infty$ _____________ -1 ________ 1 ________________ ∞ x-axis

f) If the function increases (decreases) at one point of one of these intervals it increases (decreases) at all the points of that interval. To determine whether the function increases or decreases choose one point from the interval and insert it in Eq. 5.

Choose x = 2 from [1, ∞). We see from Eq. 5 that $\frac{dy}{dx}$ is the product of three numbers, (+3)(+1)(+3) or positive times x positive x positive is a positive (i.e., + ++ = +). Therefore $\frac{dy}{dx} > 0$ and the function is increasing over this interval. Place a series of +'s above that segment..

Choose x = 0 from [-1,1]. $\frac{dy}{dx}$ is the production of (+3)(-1)(+1) or positive x negative x positive or + - + = -. Therefore $\frac{dy}{dx} < 0$ and the function decreases over this interval. Place - 's above that segment in e).

Choose x = -2 from $(-\infty, -1]$. $\frac{dy}{dx} = + - - = +$ so that $\frac{dy}{dx} > 0$. Place + 's above that interval in Fig. 4 as we have done below.

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     +++++++++++     -----------   ++++++++++++
-∞ _____________ -1 ________ 1 ________________ ∞
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So we see that the curve increases until it hits a "ceiling" (line segment) at (-1, 3). Then it decreases until it hits the "floor" (line segment) at (1, -1) then it increases for all x values beyond x = 1. We now have a good picture of how the graph looks as shown in Fig. 4 the critical point (1,-1) now identified as a relative minimum (m) and (-1,3) as a relative maximum (M)..

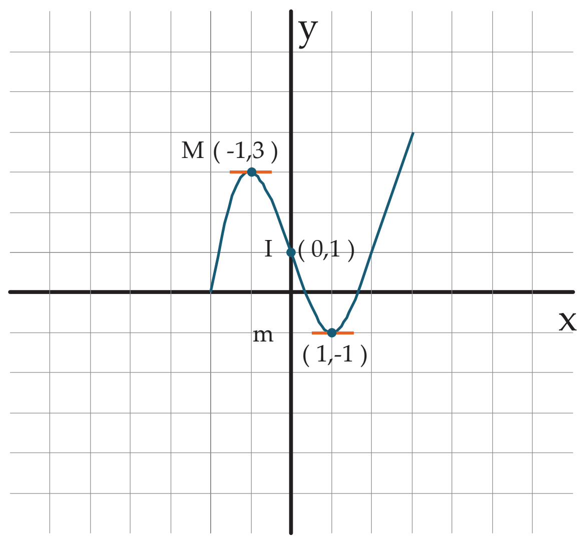

Fig. 4

e) We can sometimes get some additional information by computing the second derivative. Compute $\frac{d^2y}{dx^2}$ by differentiating $\frac{dy}{dx}$ in Eq. 4.

$$\frac{d^2y}{dx^2} = 6x$$

Locate possible inflection points by solving the equation, $\frac{d^2y}{dx^2} = 0$

6x = 0 or x = 0, y = 1.

The 2nd derivative changes from negative to positive at this point, so that (0,1) is an inflection point (I). You can see from Fig. 4 that this is the point where the curve goes from being concave down to being concave up.

Example 3:

We demonstrate this procedure without comment for the polynomial,

$$y = 3x^5 - 5x^3 \tag{6}$$

Easy point: (0,0)

$$3x^5 - 5x^3 \approx -5x^3$$

Therefore $y \to \infty$ as $x \to \infty$, and $y \to -\infty$ as $x \to -\infty$

The easy point and the behavior of the function at infinity is placed in Fig. 5.

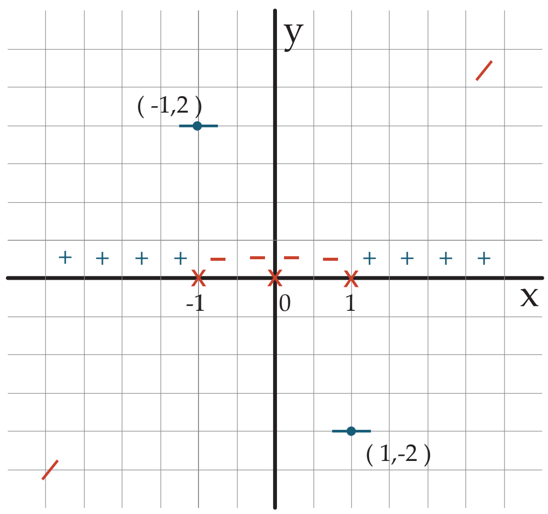

Fig. 5

Find the derivative of Eq. 6.

$$\frac{dy}{dx} = 15x^4 - 15x^2$$

$$= 15x^2(x^2 - 1)$$

$$\frac{dy}{dx} = 15x^2(x-1)(x+1)$$

Critical points: x = 0, (0,0)
x = 1, (1,-2)
x = -1 (-1,2)

The critical points are placed in Fig. 5.

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Intervals of the x-axis:	________________-1	_____0	______1	______________
with signs	choose : x = -2	x= -1/2	x = ½	x = 2

$$\frac{d^2y}{dx^2} = 60x^3 - 30x$$
$$= 30x(2x^2 - 1) = 0$$

Inflection points (I): x = 0, $x = \pm\frac{1}{\sqrt{2}}$. At these points the graph changes from concave down to concave up or concave up to concave down. Putting it all together we show the graph in Fig. 6 with (-1,2) identified as a relative max (M) and (1,-2) as a relative min (m).

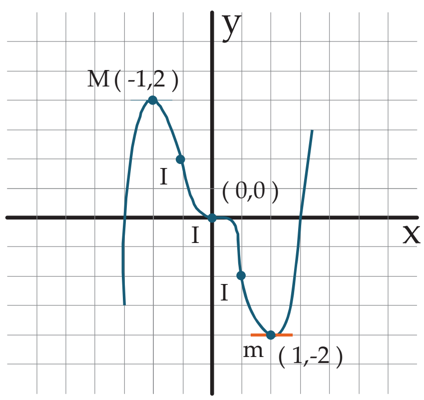

Fig. 6

The sketch of the function in Example 3 also exhibits central symmetry since $f(x) = 3x^5 - 5x^3$ is an odd function (see Section 2.4) , i.e., f(-x) = -f(x). This means that you need only sketch the function for $x \geq 0$ and then reflect this portion of the curve first in the y-axis and then in the x-axis (check this for Fig. 7)

Example 4:

$$y = x^4 - 2x^2 + 1 \qquad (7)$$

Notice that this function is even, i.e., f(-x) = f(x) so that you need only sketch the function for $x \geq 0$ and then reflect the curve in the y-axis (see Section 2.4).

Use the curve sketching ideas of this lesson to find the graph of Eq. 7.

Problems: Sketch the graphs of the following functions. Indicate on your graph relative maxima, minima, and inflection points.

1. $f(x) = 2x^3 - 3x^2 - 12x + 3$
2. $f(x) = 3x^4 - 4x^3 - 5$
3. $f(x) = 3x^5 - 5x^3$
4. $f(x) = 6 + 8x^2 - x^4$
5. $f(x) = (x+3)\sqrt{x}$

11.4 CONCLUSION

In this Chapter we have focused on the graphs of polynomials. However, the same procedure can be applied to other functions as we shall do in Chapter 14..