CHAPTER 3

THE AREA UNDER A CURVE

3.1 FINDING THE AREA UNDER A CURVE

We wish to find the area above the x-axis and beneath the graph of a function y = f(x) between x = a and x = b, where $f(x) \geq 0$. In the study of calculus the notation that is used for this area is: $A = \int_a^b f(x)dx$ which is referred to as "*the integral of f(x) between x = a and x = b.*" The reason for this strange notation will soon become clear.

If we consider the linear function, $f(x) = -2x + 4$ between x = 0 and x = 2 as in Fig. 1, then the area under the graph of the y = -2x + 4 is

$$A = \int_0^2 (-2x+4)dx = 4,$$

the area of the triangle. But what if the curve is not a line such as the graph of the function $f(x) = 0.25x^2 + 1$ in Fig. 2a? Again we seek the area under the graph of $y = 0.25x^2 + 1$ and above the x-axis between a = 1 and b = 5. However, first we must consider what we mean by the "area under a curve." One way to define the area is to first draw the curve on a piece of graph paper as we have done in Fig. 2 and count the number of squares entirely within the region being considered (24 squares in this case), and then multiply the this number by the area of each square (0.5x1=0.5 square units). This will give an underestimate of the "area." (12 square units). If we do the same for the squares within the region and those squares pierced by the curve, we will get an overestimate of the "area" (32 squares = 16 square units [check this]). This is similar to what we did in Lab 4 of Lesson 0 for the area of a circle.

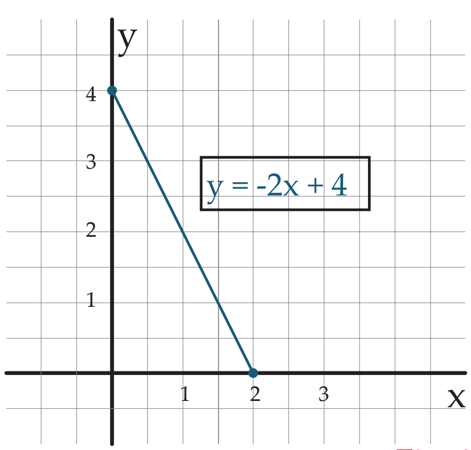

Fig. 1

A = 4

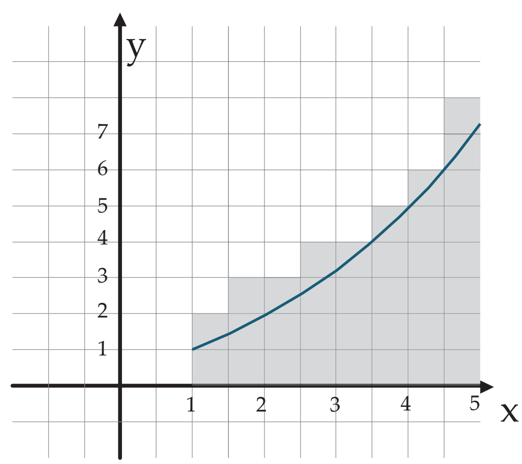

Fig. 2

$y = 0.25x^2 + 1$

As we refine the graph paper by taking smaller squares (we could cut each square into four equal squares), these approximations will give better and better estimates of the "area." If we keep doing this procedure indefinitely, and the estimate of the area approaches a limiting value, we define this limiting value as the area and, for the function $y = 0.25x^2 + 1$, give it the symbol,

$$A = \int_1^5 (0.25x^2 + 1)dx$$

So we see that the computation of area is an infinite process. As a result, we will have to be satisfied with finding only an approximation to the actual area. Instead of adding squares, we will add a sequence of rectangles that approximate the area. In Chapters 15 and 16 we will be able to compute the exact value of this area, and it will turn out to be A = 14.333… But for the time being let's see how well we can approximate this area.

3.2 APPROXIMATING AN AREA BY RECTANGLES

We illustrate the approximation of areas by rectangles for the graph of the function, $f(x) = 0.25x^2 + 1$ over the interval [1,5], shown in Fig. 3. We use the following step by step procedure.

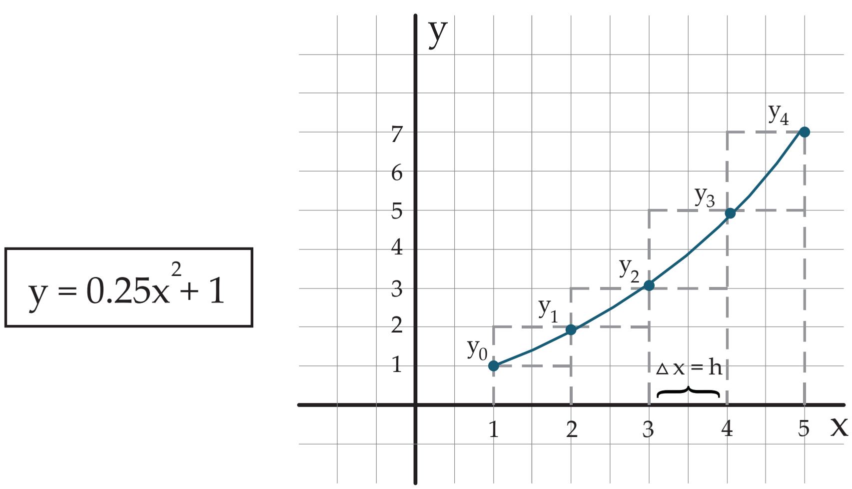

Fig. 3

Example 1:

a. Divide the interval from a to b into any number n of equal parts, Δx, of length $h = \Delta x = \frac{b-a}{n}$. For the curve in Fig. 2, where the interval from 1 to 5 is divided into four equal parts, h= $\Delta x = \frac{5-1}{4} = 1$ and the division points result in a sequence of five x values : $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $x_3 = 4$, $x_4 = 5$ spanning the interval from x = 1 to x = 5. The y values at the division points form the sequence: y_0, y_1, y_2, y_3, y_4 shown in Fig. 3.

b. Make a table in which $y = f(x) = 0.25x^2 + 1$ is tabulated for the five x values as shown in Table 1.

Table 1

x	1	2	3	4	5
y=f(x)	1.25	2	3.25	5	7.25

c. Draw four rectangles using the y values corresponding to the x values at the left side of the four segments as the height of the rectangles and $\Delta x = 1$ as the length of the base. Add up the areas of the four rectangles, and call this sum $A_L^{(4)}$, the *left area* where,

$$A_L^{(4)} = (1.25)(1) + 2(1) + (3.25)(1) + (5)(1) = 11.5$$

Notice that this sum is an underestimate of the actual area (why?).

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d. Repeat step c using the y values above the right side of the segments as the height of the rectangles. This will give you another approximation to the area, the *right area* approximation with four rectangles, $A_R^{(4)}$. Again, the length of the base of each rectangle is $\Delta x = 1$ while the heights are the last four values of f(x) in Table 1,

$$A_R^{(4)} = (2)(1) + (3.25)(1) + (5)(1) + (7.25)(1) = 17.5$$

Notice that this sum is an over-estimate of the actual area (why?).

e. Take the average of the "left" and "right" areas:

$$T^{(4)} = \frac{A_L^{(4)} + A_R^{(4)}}{2} = \frac{11.5 + 17.5}{4} = 14.5$$

The symbol T is used here because this average gives the sum of four trapezoids, the areas of the four trapezoids gotten by connecting the division points on the curve by straight lines as in Fig. 4. Also notice that this value is a much better approximation to the actual area (A = 15.333…). The following formula will give you the sum of the *trapezoids areas* directly:

$$T^{(4)} = \frac{h}{2}(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4)$$
$$= \frac{1}{2}(1.25 + 2(2) + 2(3.25) + 2(5) + 7.25) = 14.5$$

where $y_k = f(x_k)$ and h = Δx .

f. Approximate the area under the curve by dividing the interval from 1 to 5 in two parts where the length of each segment is now h= $\Delta x = 2$ and taking the heights of each rectangle at the midpoint of each interval, the values x = 2 and x = 4, as shown in Fig. 5. The sum of these two rectangles is denoted by the *mid-point area*, $A_M^{(2)}$, where,

$$A_M^{(2)} = (2)(2) + (2)(5) = 14$$

This approximation is easy to compute and is usually about as accurate as the trapezoids.

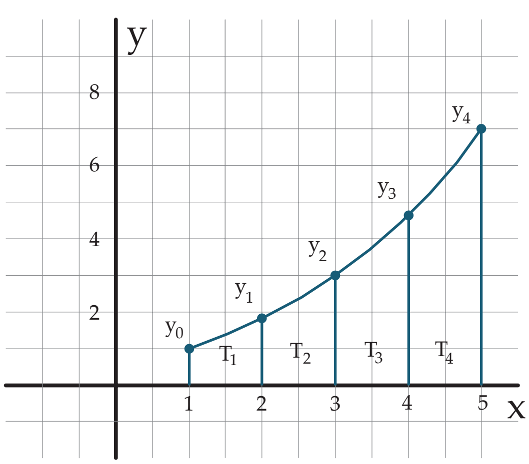


Fig. 4

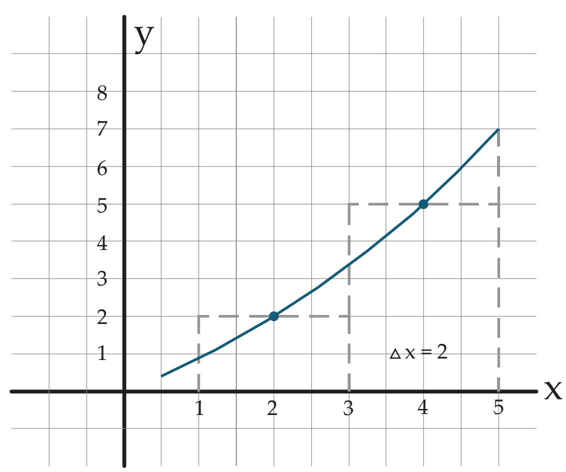


Fig. 5

3.3 THE GENERAL CASE

We can carry out this procedure for any general function $f(x) \geq 0$ on any interval of the x-axis. We will again divide the interval from a to b into four parts, as in Fig.6, although it can be divided into any number of parts. If we had divided it into 10 parts or 100 parts the approximation to the actual area would have been better. In fact if we divide the interval into n parts and let n approach ∞ the approximation will approach closer and closer to the actual area.

Remark 1: To find the area under a curve you do not have to draw the graph. Just follow the procedure outlined below.

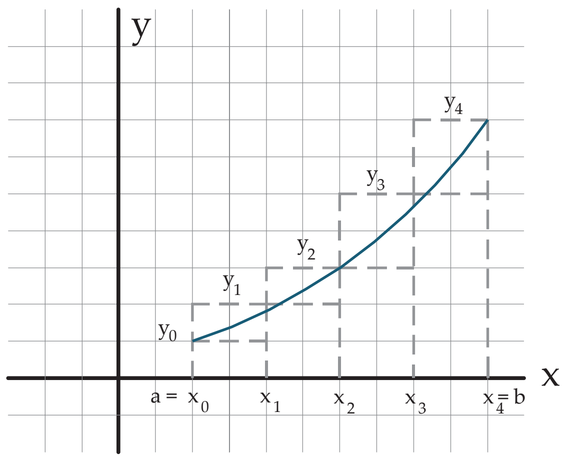


Fig. 6

$y = f(x)$

a. Divide the interval from a to b into n equal parts, each of length, $h = \Delta x = \dfrac{b-a}{n}$. In our example, $n = 4$ and $\Delta x = 1$, where the division points are at : $x_0 = a$, x_1, x_2, x_3, $x_4 = b$ spanning the interval from $x = a$ to $x = b$ with corresponding y values: y_0, y_1, y_2, y_3, y_4 shown in Fig. 5.

b. Make a table of x and y = f(x).

Table 2

x	$a = x_0$	x_1	x_2	x_3	$b = x_4$
y=f(x)	$f(x_0) = y_0$	$f(x_1) = y_1$	$f(x_2) = y_2$	$f(x_3) = y_3$	$f(x_4) = y_4$

c. Compute the area by taking the left point of each interval (all values of x in Table 2 except the last) that I refer to as the 'left area' $A_L^{(4)}$

$$A_L^{(4)} = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x = \sum_{k=0}^{k=3} f(x_k)\Delta x \approx \int_a^b f(x)dx \qquad (1)$$

Note that we have used summation notation. If we change from $\sum_{k=0}^{k=3}$ to $\int_a^b$ and replace Δx by dx we can see

why $\int_a^b f(x)dx$ is a good notation for the area under a curve and above the x-axis. Also note that we divided the interval from a to b again into four segments.

d. Compute the area with the right endpoints (at all values of x except the first) or the "right area," $A_R^{(4)}$:

$$A_R^{(4)} = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x = \sum_{k=1}^{k=4} f(x_k)\Delta x \qquad (2)$$

e. Compute the trapezoids, $T^{(4)}$:

$$T^{(4)} = \frac{h}{2}(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4) \qquad (3)$$

f. Compute $A_M^{(2)}$:

$$A_M^{(2)} = f(x_1)\Delta x + f(x_3)\Delta x \qquad (4)$$

Problem 1:

Here are four functions and four intervals to apply this method to. For each curve compute: $A_L^{(4)}, A_R^{(4)}, T^{(4)}$, and $A_M^{(2)}$.

a. = f(x) = x^2, [1,3]; b. f(x) = $3/x^2$, [1,3]; c. f(x) = x [1,3]; f(x) = 4x (1-x), [0,1]

The following are the exact areas for these functions: a) A = 8.667; b) A = 2; c) A = 4; d) A = 0.667. See how closely these areas are approximated by your values.

3.4 SIGNED AREAS

What if f(x) is not ≥ 0 for all x such as the linear function f(x) = x on the interval [-2,2] shown in Figure 7?

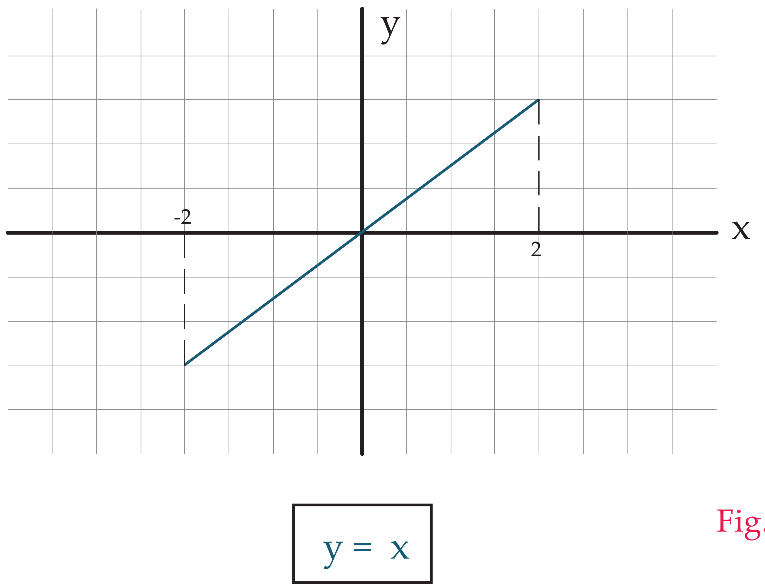


Fig. 7

If we apply our method to this function, we see that it has a positive value when $f(x) \geq 0$ and a negative value when $f(x) < 0$. This means that when the function is positive the area is positive, but when the function is negative the area is negative. So we can say that the integral, $\int_a^b f(x)dx$, will always represent a *signed area*, SA. Since the positive area in the example is 2 units while the negative area is – 2 units, SA =2 + (-2) = 0.

Example 2:

Consider the curve, y + 2 = | x | shown in Fig. 8. In other words, the absolute value curve that we introduced in Section 2.9 has been translated down 2 units.

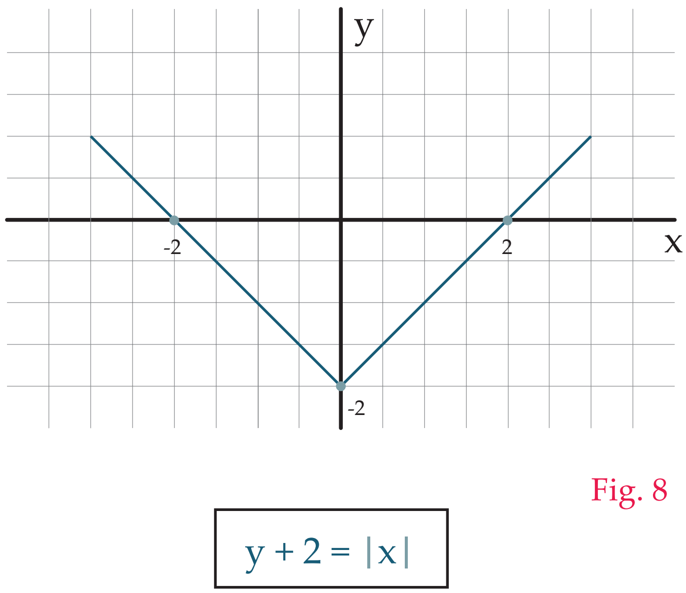


Fig. 8

Since the curve is entirely below the x-axis, the area caught between the x-axis and the curve has area - 4 units as you can easily see.

Example 3:

Fig. 9 shows the graph of $f(x) = x^3 - 7x^2 + 11x$ crossing below the x-axis at about x = 2.38. The integral is the signed area or the area above the x-axis, A_1, minus the area below the x-axis, A_2. Some of you may have a calculator that enables you to compute the value of $\int_a^b f(x)dx$ if you input f(x), a and b. Computing the integral with a calculator shows that,

$$\int_0^4 (x^3 - 7x^2 + 11x)dx = 2.67$$

Breaking the integral into two parts and computing each one separately gives,

$$\int_0^{2.38} (x^3 - 7x^2 + 11x)dx = 7.72 \quad \text{and} \quad \int_{2.38}^4 (x^3 - 7x^2 + 11x)dx = -5.05$$

so $A_1 = 7.72$ and $A_2 = 5.05$. Then as we would expect,

$$\int_0^4 (x^3 - 7x^2 + 11x)dx = A_1 - A_2 = 7.72 - 5.05 = 2.67$$

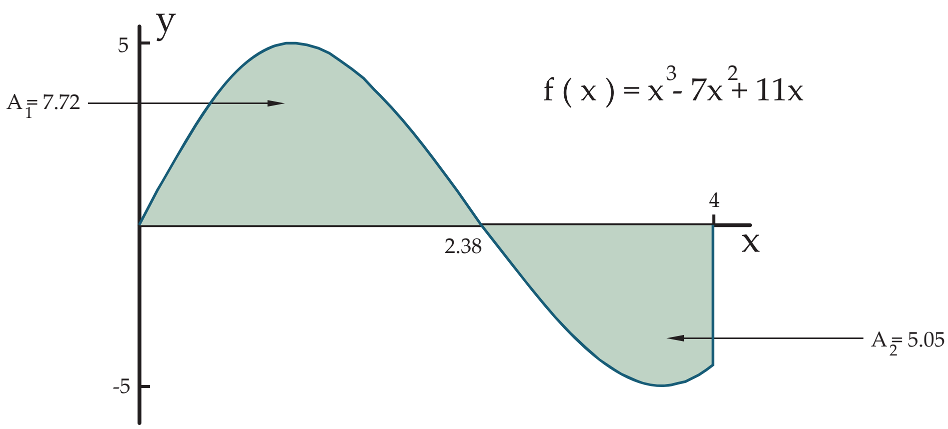


Fig. 9

3.5 TOTAL AREA

To find the total area, reckoned as a positive number, caught between the x-axis and the curve we are effectively first finding the absolute value of f(x), i.e.,

$$|f(x)| = \begin{cases} x^3 - 7x^2 + 11x \ , & x \leq 0 \\ -(x^3 - 7x^2 + 11x)\, , & x > 0 \end{cases}$$

and then finding the signed area under the y = |f(x)| curve.

In general, the area A caught between the curve of y = f(x) and the x-axis on the interval [a,b] is,

$$A = \int_a^b |f(x)|\, dx \tag{5}$$

If we wish to find the actual area caught between the curve and the x-axis we must draw a graph of the function, and those intervals at which the function becomes negative we have to make them positive. Therefore in the last example,

Total shaded area = 7.72 + 5.05 = 12.77.

In other words, where f(x) is negative we make it positive after which,

$$12.77 = \int_0^4 |f(x)|\, dx .$$

Remark 2: To find the signed area, SA, you can fly blind. There is no need to use a graph. On the other hand, to find the actual area caught between the curve and x-axis, you will need to graph the function so that you can tell where it is positive and where it is negative.

3.6 AVERAGE VALUE OF A FUNCTION

A student takes three exams in M113 and scores: 75% on exam 1, 60% on exam 2 and 96% on exam 3. The average grade on these exams is:

$$G_{avg} = \frac{70 + 50 + 96}{3} = 72$$

The scores are plotted on Fig. 10.

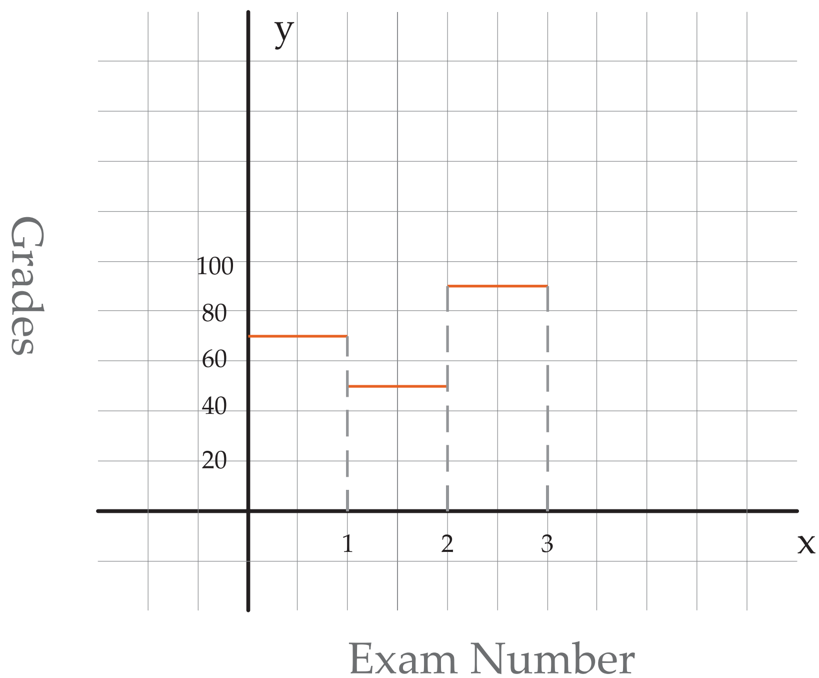


Fig. 10

Notice that,

G_{avg} = area under the curve/length of the interval on the x-axis.

This will always be the case. If you want to find the average value of f(x) over the interval [a,b], the result will be,

$$f_{avg} = \frac{\int_a^b f(x)dx}{b-a} \quad (6)$$

where $\int_a^b f(x)dx$ is the signed area between the curve and the x-axis over the interval [a,b].

Problem 2: Find the average values of the functions in Problem 1 over the given intervals.

3.7 THE MEAN VALUE THEOREM FOR INTEGRALS

A very important result from the theory of calculus is suggested by Eq. 6. There exists a value of x = x* in the interval [a,b] such that, $f_{avg} = f(x^*)$. This is called the M*ean ValueTheorem of Integrals* and generally stated as follows:

Theorem 1 (Mean Value Theorem): There exists, $a \le x^* \le b$ such that $f(x^*) = \frac{\int_a^b f(x)dx}{b-a}$.

The proof of this theorem follows from two other fundamental theorems of calculus.

Theorem 2: If f(x) is continuous on a closed interval [a,b] (the interval includes its endpoints) then f(x) takes on its maximum and minimum values on [a,b].

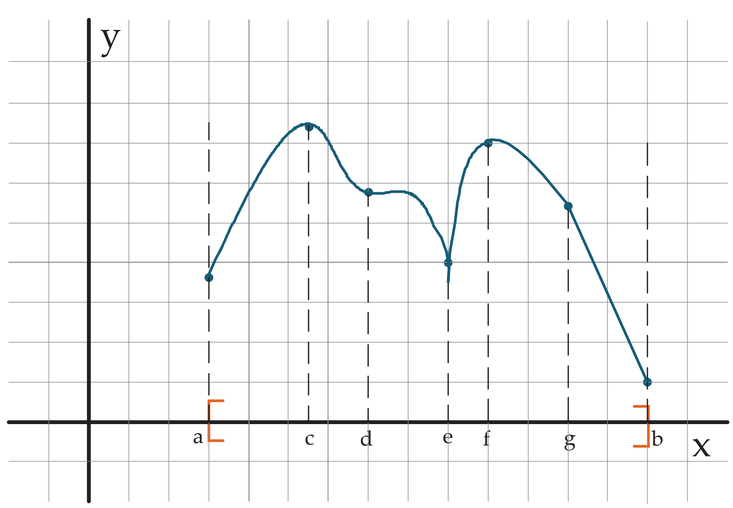


Fig. 11

No proof of this theorem is given, however, it appears evident by looking at Fig. 11. From

Fig. 11 we can see that the maximum and minimum values of f(x) can occur only at places where the slope is horizontal (points c and d, or f), where the slope is not defined (at cusps or kinks, point e or g) or endpoints (a or b).

Remark 5: If the interval is not closed, the theorem might be false, e.g., consider f(x) = 1/x on the interval (0, 1]. This interval does not include x = 0. Notice that as x → 0 the values of the function get bigger and bigger so that there is no maximum value on the given interval.

Remark 6: This theorem will play a major role in Chapter 12 where we seek to find maxima and minima of functions.

The next theorem is known as the Intermediate Value Theorem

Theorem 3 (Intermediate Value theorem): If f(x) is a continuous function on the interval [a,b], f(x) takes on every value between its maximum and minimum values.

A look at Fig. 11 makes this theorem seem obvious. Clearly f(x) takes on every value between its maximum and minimum values. However, this important theorem has variants that are not so obvious. Consider the following problem about a traveling monk.

The Traveling Monk Problem:

A monk wishes to climb the holy mountain from the base camp at the bottom of the mountain to the holy temple at the top. There is a single path up the mountain, and during his trip up the mountain he may stop to meditate or sometimes retrace some of his steps to observe wildlife along the path. He starts his climb at 6 AM and reaches the top at 12 noon. He stays at the temple until the following morning when he begins his descent down the path at 6 AM reaching base camp at 12 noon and again randomly stopping or retracing steps.

Question: Is there always, sometimes, or never a time and place during the descent that the monk find himself at the identical time and place as on his ascent the previous day?

Hint: The solution to this problem involves the Intermediate Value Theorem

Now we can prove the Mean Values Theorem for Integrals.

Proof of Mean Value Theorem for Integrals:

Consider function y = f(x) on closed interval [a,b] where by Theorem 2, f(x) takes on its largest and smallest values $y_{\max}$ and $y_{\min}$ on [a,b],

Estimate the signed area $A = \int_a^b f(x)dx$ by enclosing the area in a single rectangle of height $y_{\max}$

and width (b-a) and another one of height y_{min} and width (b-a) as shown in Fig. 12. In other words,

$$y_{min}(b-a) \le \int_a^b f(x)dx \le y_{max}(b-a) \qquad (7)$$

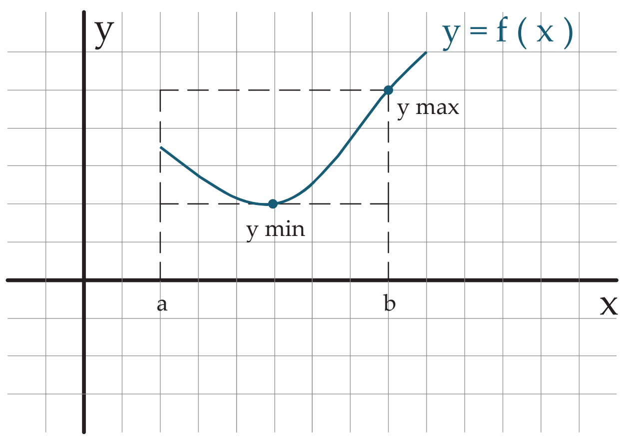


Fig. 12

Dividing Eq. 7 by (b – a),

$$y_{min} \le \frac{\int_a^b f(x)dx}{b-a} \le y_{max}$$

But since f(x) takes on every value between y_{max} and y_{min} on [a,b] and $\frac{\int_a^b f(x)dx}{b-a}$ is a number that lies between these two limiting values, there must be a value of x = x* on the interval [a,b] such that,

$$f(x^*) = \frac{\int_a^b f(x)dx}{b-a}$$

This Theorem will be important in proving some connections between calculus and structures as we shall do in Lesson 17.

Problems

1. Find the area under $P = 100(0.6)^t$ between $t = 0$ and $t = 8$.

2. Find the area under $y = x^3 + 2$ between $x = 0$ and $x = 2$. Sketch this area.

3. **(a)** What is the area between the graph of $f(x)$ in Figure 5.34 and the x-axis, between $x = 0$ and $x = 5$?
   (b) What is $\int_0^5 f(x)\,dx$?

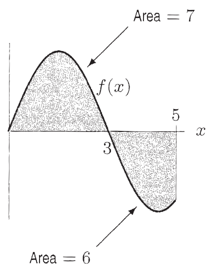
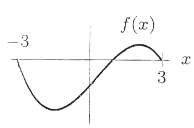


Figure 5.34

For the functions in Problems 4–7, decide whether $\int_{-3}^{3} f(x)dx$ is positive, negative, or approximately zero.

4.



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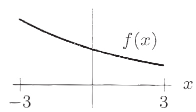

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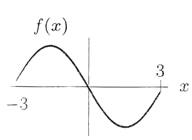

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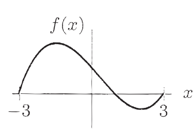


8. **(a)** Estimate (by counting the squares) the total area shaded in Figure 5.35.
   (b) Using Figure 5.35, estimate $\int_0^8 f(x)\,dx$.
   (c) Why are your answers to parts (a) and (b) different?

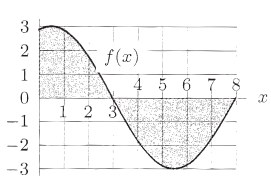


Figure 5.35

9. Using Figure 5.36, estimate $\int_{-3}^{5} f(x)dx$.

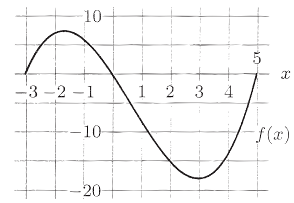


Figure 5.36

10. Given $\int_{-1}^{0} f(x)\,dx = 0.25$ and Figure 5.37, estimate:

(a) $\int_0^1 f(x)\,dx$ **(b)** $\int_{-1}^{1} f(x)\,dx$
(c) The total shaded area.

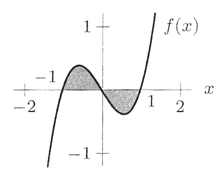


Figure 5.37

11. Given $\int_{-2}^{0} f(x)dx = 4$ and Figure 5.38, estimate:

(a) $\int_0^2 f(x)dx$ **(b)** $\int_{-2}^{2} f(x)dx$
(c) The total shaded area.

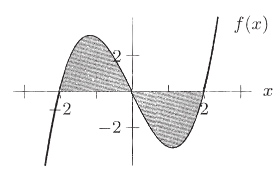
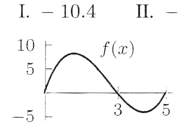


Figure 5.38

In Problems 12–15, match the graph with one of the following possible values for the integral $\int_0^5 f(x)\,dx$:

I. -10.4 II. -2.1 III. 5.2 IV. 10.4

12.



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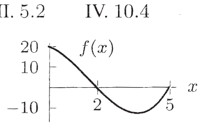

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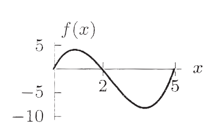

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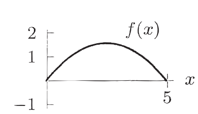

16. Use Figure 5.39 to find the values of

(a) $\int_a^b f(x)\,dx$ (b) $\int_b^c f(x)\,dx$
(c) $\int_a^c f(x)\,dx$ (d) $\int_a^c |f(x)|\,dx$

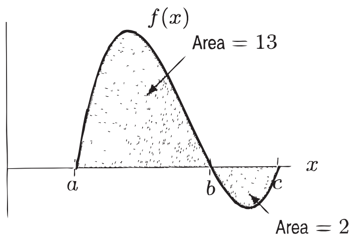

Figure 5.39

17. Using Figure 5.40, list the following integrals in increasing order (from smallest to largest). Which integrals are negative, which are positive? Give reasons.

I. $\int_a^b f(x)\,dx$ II. $\int_a^c f(x)\,dx$ III. $\int_a^e f(x)\,dx$
IV. $\int_b^e f(x)\,dx$ V. $\int_b^c f(x)\,dx$

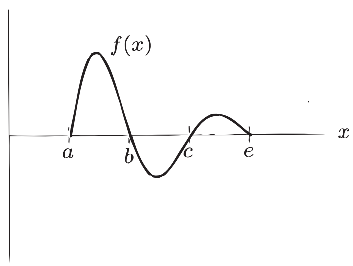

Figure 5.40

18. (a) Graph $f(x) = x(x+2)(x-1)$.
(b) Find the total area between the graph and the x-axis between $x = -2$ and $x = 1$.
(c) Find $\int_{-2}^{1} f(x)\,dx$ and interpret it in terms of areas.

19. Find the area between the graph of $y = x^2 - 2$ and the x-axis, between $x = 0$ and $x = 3$.

20. Compute the definite integral $\int_0^4 \cos\sqrt{x}\,dx$ and interpret the result in terms of areas.

21. (a) Using Figure 5.41, find $\int_{-3}^{0} f(x)\,dx$.
(b) If the area of the shaded region is A, estimate $\int_{-3}^{4} f(x)\,dx$.

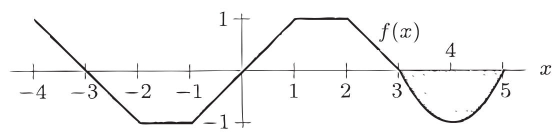

Figure 5.41

22. Use the following table to estimate the area between $f(x)$ and the x-axis on the interval $0 \le x \le 20$.

x	0	5	10	15	20
$f(x)$	15	18	20	16	12

For Problems 23–24, compute the definite integral and interpret the result in terms of areas.

23. $\displaystyle\int_1^4 \frac{x^2-3}{x}\,dx.$ **24.** $\displaystyle\int_1^4 (x - 3\ln x)\,dx.$

In Problems 25–32, use an integral to find the specified area.

25. Under $y = 6x^3 - 2$ for $5 \le x \le 10$.

26. Under $y = 2\cos(t/10)$ for $1 \le t \le 2$.

27. Under $y = 5\ln(2x)$ and above $y = 3$ for $3 \le x \le 5$.

28. Between $y = \sin x + 2$ and $y = 0.5$ for $6 \le x \le 10$.

29. Between $y = \cos x + 7$ and $y = \ln(x-3)$, $5 \le x \le 7$.

30. Above the curve $y = x^4 - 8$ and below the x-axis.

31. Above the curve $y = -e^x + e^{2(x-1)}$ and below the x-axis, for $x \ge 0$.

32. Under $y = \cos t$ and above $y = \sin t$ for $0 \le t \le \pi$.